MAT 540 : Homological algebra

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Contents

<u>Conventions</u> :

- "$\mathbb{N}$" denotes the set of natural numbers $0, 1, 2, \ldots$.
- If x is a set, then we write $\mathfrak{P}(x)$ for the powerset of x.
- If x and y are sets, a *correpondence* between x and y is a triple (x, y, Γ), where $\Gamma \subset x \times y$. This correspondence is called a *function (or map) from x to y* if the first projection $\Gamma \to x$ is bijective.

I Some category theory

The goal of this chapter is to introduce some basics of category theory before we start the study of additive and abelian categories in the next chapter. Category theory is not just a convenient language in which to formulate constructions and results from other parts of mathematics, it is its own field with deep nontrivial results, and we will not have time to do it justice. If you want to learn more, a good place to start is Riehl's book [12].

I.1 Set-theoretical preliminaries

This is just a quick overview. For a much more detailed discussion, see Shulman's paper [13].

The foundations of category theory is one of the topics where we have to be careful about set-theoretical questions. As we will see in the next section, a category is the data of a "collection" of objects and of morphisms between these objects (satisfying some axioms). We would like to talk about things like "the category of all sets" or "the category of all groups" (where the objects are sets resp. groups), but there is no set of all sets or of all groups. So the question is, do we impose the condition that the objects of a category form a set or do we allow bigger "collections" of objects ? Both are possible, and so we have to choose between the following two options :

(1) Work with a version of set theory that allows us to manipulate classes, such as Bernays-Gödel-von Neumann set theory. Intuitively, classes are collections that may bigger than sets; for example, there exists a class of all sets.

(2) Only allow sets and keep track of the sizes of the sets that we are manipulating, for example using Grothendieck universes.

A drawback of solution (1) is that we cannot quantify over classes, so it becomes impossible to state some theorems. The drawback of solution (2) is that we need an extra axiom to ensure the existence of suitable Grothendieck universes, and this axiom is known to be independent of the axioms of ZFC set theory; however, this axiom follows from large cardinal axioms that are well-accepted in modern set theory, so we will choose solution (2) for simplicity.

We work with Zermelo-Fraenkel set theory with teh axiom of choice (abbreviated to "ZFC set theory"); see for example the beginning of chapter 1 of [7] for the list of axioms of this theory.

Definition I.1.1. A *universe* is a set $\mathscr{U}$ satisfying the following properties :

(i). $\varnothing \in \mathscr{U}$;

(ii). if $x \in \mathscr{U}$ and $y \in x$, then $y \in \mathscr{U}$;

(iii). if $x \in \mathscr{U}$, then $\{x\} \in \mathscr{U}$;

(iv). if $x \in \mathscr{U}$, then $\mathfrak{P}(x) \in \mathscr{U}$;

(v). if $(x_i)_{i\in I}$ is a family of sets such that $I \in \mathscr{U}$ and $x_i \in \mathscr{U}$ for every $i \in I$, then $\bigcup_{i\in I} x_i \in \mathscr{U}$;

(vi). $\mathbb{N} \in \mathscr{U}$.

Remark I.1.2. Properties (i) and (vi) were not required in the original definition of universes in [1], appendice de l'exposé I, définition 1.1, but it is usually required in modern expositions. The only difference is that $\varnothing$ and $\mathbb{N}$ are universes in the sense of [1] and not in the sense of definition I.1.1.

The standard reference for universes is [1], appendice de l'exposé I. It contains proofs of the following two propositions.

Proposition I.1.3. *Let $\mathscr{U}$ be a universe. Then :*

(i). If $x \in \mathscr{U}$, then $\bigcup_{y\in x} y \in \mathscr{U}$.

(ii). If $x, y \in \mathscr{U}$, then $x \times y \in \mathscr{U}$.

(iii). If $x \in \mathscr{U}$ and $y \subset x$, then $y \in \mathscr{U}$.

(iv). If $x \in \mathscr{U}$, then every quotient set of x is an element of $\mathscr{U}$.

(v). If $(x_i)_{i\in I}$ is a family of sets such that $I \in \mathscr{U}$ and $x_i \in \mathscr{U}$ for every $i \in I$, then $\coprod_{i\in I} x_i \in \mathscr{U}$;

(vi). If $x, y \in \mathscr{U}$, then every correpondence between x and y is also an element of $\mathscr{U}$.

(vii). If $x, y \in \mathscr{U}$, then every set of correpondences between x and y is also an element of $\mathscr{U}$.

(viii). If $(x_i)_{i\in I}$ is a family of sets such that $I \in \mathscr{U}$ and $x_i \in \mathscr{U}$ for every $i \in I$, then $\prod_{i\in I} x_i \in \mathscr{U}$;

(ix). If $x \subset \mathscr{U}$ and if there exists an element y of $\mathscr{U}$ such that $\operatorname{card}(x) \leq \operatorname{card}(y)$, then $x \in \mathscr{U}$.

(x). For every $n \in \mathbb{N}$, the universe $\mathscr{U}$ contains a finite set of cardinality n.

Proposition I.1.4. *Any nonempty intersection of a family of universes is a universe.*

We will now see that universes are closely related to inaccessible cardinals. Remember that cardinals are a special type of ordinals, defined for example on page 29 of [7], and such that any set is in bijection with exactly one cardinal.

Definition I.1.5. Let $\mathfrak{c}$ be a cardinal. We say that $\mathfrak{c}$ is *inaccessible* (or *strongly inaccessible*) if it satisfies the following properties :

(i). $\mathfrak{c}$ is a *strong limit cardinal*, that is, for every cardinal $\mathfrak{d}$ such that $\mathfrak{d} < \mathfrak{c}$, we have $2^{\mathfrak{d}} < \mathfrak{c}$ (where $2^{\mathfrak{d}} = \operatorname{card}(\mathfrak{P}(\mathfrak{d}))$);

(ii). $\mathfrak{c}$ is *regular*, that is, if $(\mathfrak{d}_i)_{i\in I}$ is a family of cardinals such that $\mathfrak{d}_i < \mathfrak{c}$ for every $i \in I$ and that $\operatorname{card}(I) < \mathfrak{c}$, then $\sum_{i\in I} \mathfrak{d}_i < \mathfrak{c}$.

(iii). $\operatorname{card}(\mathbb{N}) < \mathfrak{c}$.

Definition I.1.6. Let $\mathfrak{c}$ be a cardinal and x be a set. We say that x is *strictly of type* $\mathfrak{c}$ if, for every finite sequence $x_0, \ldots, x_n$ such that $x_n \in x_{n-1} \in \ldots \in x_1 \in x_) = x$, we have $\operatorname{card}(x_n) < \mathfrak{c}$.

Theorem I.1.7. *(i). If $\mathscr{U}$ is a universe, then $\operatorname{card}(\mathscr{U})$ is inaccessible, and $\mathscr{U}$ is exactly the set of sets that are stricly of type $\operatorname{card}(\mathscr{U})$.*

(ii). Conversely, if $\mathfrak{c}$ is an inaccessible cardinal, then the set $\mathscr{U}_\mathfrak{c}$ of sets that are strictly of type $\mathfrak{c}$ is a universe, and we have $\operatorname{card}(\mathscr{U}_\mathfrak{c}) = \mathfrak{c}$.

This is [1], appendice de l'exposé I, théorème 6.2. Note that, as we are assuming the axiom of regularity (or axiom of foundation), every set is artinian in the sense of [1], so the theorem becomes simpler.

We will add the following axiom to set theory :

(AU) For every set x, there exists a universe containing x.

This is called the *axiom of universes*. It is equivalent to the fact that every cardinal is bounded above by an inaccessible cardinal.

Definition I.1.8. Let $\mathscr{U}$ be a universe. We say that a set is a *$\mathscr{U}$-set* if it is an element of $\mathscr{U}$, and that it is *$\mathscr{U}$-small* if it is in bijection with an element of $\mathscr{U}$.

I.2 Vocabulary

I.2.1 Categories

Definition I.2.1.1. A *category* $\mathscr{C}$ is the data of :

(1) a set $\operatorname{Ob}(\mathscr{C})$ whose elements are called the *objects of* $\mathscr{C}$;

(2) for all $X, Y \in \operatorname{Ob}(\mathscr{C})$, a set $\operatorname{Hom}_\mathscr{C}(X, Y)$ whose elements are called the *morphisms from* X *to* Y;

(3) for all $X, Y, Z \in \operatorname{Ob}(\mathscr{C})$, a function $\operatorname{Hom}_\mathscr{C}(X,Y) \times \operatorname{Hom}_\mathscr{C}(Y,Z) \to \operatorname{Hom}_\mathscr{C}(X,Z)$, called the *composition* and denoted by $(f,g) \longmapsto g \circ f$;

such that the following conditions hold :

(a) the composition is associative, that is, for all $X, Y, Z, T \in \operatorname{Ob}(\mathscr{C})$ and all $f \in \operatorname{Hom}_\mathscr{C}(X,Y)$, $g \in \operatorname{Hom}_\mathscr{C}(Y,Z)$ and $h \in \operatorname{Hom}_\mathscr{C}(Z,T)$, we have $h\circ(g\circ f) = (h\circ g)\circ f$;

(b) for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$, called the *identity of* X, such that, for every $Y \in \mathrm{Ob}(\mathscr{C})$ and all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, we have $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

If $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, we also write $f : X \to Y$ and we call X the *source* and Y the *target* of Y.

We sometimes write gf instead of $g \circ f$ for the composition law.

Remark I.2.1.2. The identity morphisms are uniquely determined by condition (b). In particular, each object of $\mathscr{C}$ corresponds to a unique identity morphism, so we could define a category just as a set of morphisms with a partially defined composition law satisfying some obvious conditions.

If $X \in \mathrm{Ob}(\mathscr{C})$, we also write $\mathrm{End}_{\mathscr{C}}(X)$ for $\mathrm{Hom}_{\mathscr{C}}(X, X)$, and we call its elements *endomorphisms of* X.

Definition I.2.1.3. Let $\mathscr{U}$ be a universe and $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is a *$\mathscr{U}$-category* if $\mathrm{Hom}_{\mathscr{C}}(X, Y) \in \mathscr{U}$ for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, and that $\mathscr{C}$ is *$\mathscr{U}$-small* is it is a $\mathscr{U}$-category and if $\mathrm{Ob}(\mathscr{C})$ is $\mathscr{U}$-small.

Definition I.2.1.4. Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism of $\mathscr{C}$.

(i). We say that f is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In that case, the morphism g is uniquely determined by f and we denote it by f^{-1}; if $X = Y$, and isomorphism from X to Y is also called an *automorphism* of X. If f is an isomorphism, we sometimes write $f : X \xrightarrow{\sim} Y$. We say that X and Y are *isomorphic* if there exists an isomorphism from X to Y, and we write $X \simeq Y$.

(ii). We say that f is a *monomorphism* (or a *monic* morphism) if, for every $Z \in \mathrm{Ob}(\mathscr{C})$, the function $f \circ \cdot : \mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y)$, $g \longmapsto f \circ g$ is injective, that is, for all $g_1, g_2 \in \mathrm{Hom}_{\mathscr{C}}(Z, X)$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. In that case, we sometimes write $f : X \hookrightarrow Y$.

(iii). We say that f is a *epimorphism* (or an *epic* morphism) if, for every $Z \in \mathrm{Ob}(\mathscr{C})$, the function $\cdot \circ f : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$, $g \longmapsto g \circ f$ is injective, that is, for all $g_1, g_2 \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. In that case, we sometimes write $f : X \twoheadrightarrow Y$.

Remark I.2.1.5. Note that an isomorphism is both a monomorphism and an epimorphism, but that the converse is false in general. For example, if X is $\mathbb{R}$ with the discrete topology and Y is $\mathbb{R}$ with the usual topology, then the identity is a morphism from X to Y in the category $\mathbf{Top}$ of topology spaces (see Example I.2.1.7(3)), and it is easy to see that it is a monomorphism and an epimorphism, but it is not an isomorphism (because an isomorphism in $\mathbf{Top}$ is a homeomorphism).

Definition I.2.1.6. Let $\mathscr{C}$ be a category.

(i). The *opposite category* $\mathscr{C}^{\mathrm{op}}$ is defined by :

(1) $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$;

(2) for all $X, Y \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$, $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$;

(3) for all $X, Y, Z \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$, and all $f \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(Y, Z)$, the composition $g \circ f$ in $\mathscr{C}^{\mathrm{op}}$ is the composition $f \circ g$ when f and g are viewed as morphism of $\mathscr{C}$.

For every $X \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}})$, the identity morphisms of X in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are the same.

(ii). If $\mathscr{C}'$ is another category, we say that $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$ and write $\mathscr{C}' \subset \mathscr{C}$ if :

(1) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$;

(2) for all $X, Y \in \mathrm{Ob}(\mathscr{C}')$, we have $\mathrm{Hom}_{\mathscr{C}'}(X, Y) \subset \mathrm{Hom}_{\mathscr{C}}(X, Y)$;

(3) the composition of $\mathscr{C}'$ is the restriction of the composition of $\mathscr{C}$.

We say that $\mathscr{C}'$ is a *full subcategory* of $\mathscr{C}$ if it is a subcategory and if, for all $X, Y \in \mathrm{Ob}(\mathscr{C}')$, we have $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$; note that a full subcategory of $\mathscr{C}$ is determined by its set of objects.

(iii). We say that $\mathscr{C}$ is *discrete* if its only morphisms are the identity morphisms.

(iv). We say that $\mathscr{C}$ is *finite* if $\bigcup_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is finite (so in particular, $\mathrm{Ob}(\mathscr{C})$ is finite).

(v). We say that $\mathscr{C}$ is *connected* if, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, we have $\mathrm{Hom}_{\mathscr{C}}(X, Y) \neq \varnothing$.

(vi). We say that $\mathscr{C}$ is a *groupoid* if every morphism of $\mathscr{C}$ is an isomorphism.

(vii). Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. Their *product* $\mathscr{C} \times \mathscr{C}'$ is the category with set of objects $\mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}')$, such that

$$\mathrm{Hom}_{\mathscr{C} \times \mathscr{C}'}((X, X'), (Y, Y')) = \mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}'}(X', Y')$$

for all $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $X', Y' \in \mathrm{Ob}(\mathscr{C}')$, and whose composition law is the product of the composition laws of $\mathscr{C}$ and $\mathscr{C}'$.

Example I.2.1.7. We fix a universe $\mathscr{U}$.

(1) The category of ($\mathscr{U}$-)sets, denoted by $\mathbf{Set}_{\mathscr{U}}$ or just $\mathbf{Set}$ if $\mathscr{U}$ is clear from the context, is the category whose objects are the $\mathscr{U}$-sets and whose morphisms are the maps between sets. It is a $\mathscr{U}$-category but is not $\mathscr{U}$-small (indeed, $\mathrm{Ob}(\mathbf{Set}_{\mathscr{U}}) = \mathscr{U}$).

(2) The full subcategory of $\mathbf{Set}_{\mathscr{U}}$ whose objects are finite $\mathscr{U}$-sets is called the category of finite ($\mathscr{U}$-)sets and denoted by $\mathbf{Set^f}_{\mathscr{U}}$ or $\mathbf{Set^f}$.

(3) The category $\mathbf{Top}_{\mathscr{U}}$ (or $\mathbf{Top}$) of ($\mathscr{U}$-)topological spaces is the category whose objects are topological space whose underlying set is in $\mathscr{U}$ and whose morphisms are continuous maps.

(4) We can define pointed versions $\mathbf{Set}_*$ and $\mathbf{Top}_*$ of the categories of sets and of topological spaces : their objects are sets X (resp. topological spaces) with a fixed point x, and morphisms from (X, x) to (Y, y) are maps (resp. continuous maps) $f : X \to Y$ such that $f(x) = y$.

(5) The category $\mathbf{Grp}_{\mathscr{U}}$ (or $\mathbf{Grp}$) of ($\mathscr{U}$-)groups is the category whose objects are groups whose underlying set is in $\mathscr{U}$ and whose morphisms are morphisms of groups. It has full a subcategory $\mathbf{Ab}_{\mathscr{U}}$ or $\mathbf{Ab}$ whose objects are abelian groups. We could also define the category $\mathbf{Mon}_{\mathscr{U}}$ or $\mathbf{Mon}$ of ($\mathscr{U}$-)monoids; note that $\mathbf{Grp}$ is a full subcategory of $\mathbf{Mon}$.

(6) The category of unitary ($\mathscr{U}$-)rings is the category whose objects are rings whose underlying set is in $\mathscr{U}$ and whose morphisms are morphisms of rings. We denote by $\mathbf{CRing}$ the full subcategory of $\mathbf{Ring}$ whose objects are commutative rings, and by $\mathbf{Field}$ the full subcategory of $\mathbf{CRing}$ whose objects are fields.

(7) Let R be a ring (unitary but not necessarily commutative). The category ${}_R\mathbf{Mod}_{\mathscr{U}}$ or ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_{R\mathscr{U}}$ or $\mathbf{Mod}_R$) of ($\mathscr{U}$)-left (resp. right) R-modules is the category whose objects are left (right) R-modules whose underlying set is in $\mathscr{U}$ and whose morphisms are R-linear maps. We often write $\mathrm{Hom}_R(X, Y)$ instead of $\mathrm{Hom}_{{}_R\mathbf{Mod}}(X, Y)$ or $\mathrm{Hom}_{\mathbf{Mod}_R}(X, Y)$. If R is commutative, then $\mathbf{Mod}_R = {}_R\mathbf{Mod}$ and if $R = \mathbb{Z}$, then ${}_R\mathbf{Mod} = \mathbf{Ab}$.

(8) Let R be a commutative ring. The category $R - \mathbf{Alg}_{\mathscr{U}}$ or $R - \mathbf{Alg}$ (resp. $R - \mathbf{Lie}_{\mathscr{U}}$ or $R - \mathbf{Lie}$) of ($\mathscr{U}$)-R-algebras (resp. Lie algebras) is the category whose objects are R-algebras (resp. R-Lie algebras) whose underlying set is in $\mathscr{U}$ and whose morphisms are morphisms of R-algebras (resp. R-linear morphisms of Lie algebras). We denote by $R - \mathbf{CAlg}$ the full subcategory of $R - \mathbf{Alg}$ whose objects the commutative R-algebras.

(9) The category $\mathbf{Rel}_{\mathscr{U}}$ or $\mathbf{Rel}$ of ($\mathscr{U}$-)relations is defined by $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and, for all $X, Y \in \mathrm{Ob}(\mathbf{Rel}_{\mathscr{U}})$, $\mathrm{Hom}_{\mathbf{Rel}}(X, Y) = \mathfrak{P}(X \times Y)$. Its composition law is defined as follows : if $X, Y, Z \in \mathrm{Ob}(\mathbf{Rel})$, $f \in \mathrm{Hom}_{\mathbf{Rel}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathbf{Rel}}(Y, Z)$, then

$$g \circ f = \{(x, z) \in X \times Z \mid \exists y \in Y,\ (x, y) \in f \text{ and } (y, z) \in g\}.$$

Note that $\mathbf{Set}$ is a subcategory of $\mathbf{Rel}$.

(10) Here is a subcategory $\mathscr{C}$ of $\mathbf{Set}$: take $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathbf{Set})$ and, for all $X, Y \in \mathrm{Ob}(\mathbf{Set})$, take $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to be the set of injective maps from X to Y. (In other words, $\mathscr{C}$ is the category of sets and injective maps.)

Example I.2.1.8. (1) Let X be a set. We can see X as a discrete category with set of objects X.

(2) Let I be a poset. We see I as a category with set of objects I and such that, for $a, b \in I$, $\mathrm{Hom}_I(a, b)$ is a singleton if $a \leq b$ and empty otherwise; if $a, b, c \in I$, the composition law is the only map from $\mathrm{Hom}_I(a, b) \times \mathrm{Hom}_I(b, c)$ to $\mathrm{Hom}_I(a, c)$. The opposite category is the category corresponding to the poset with the opposite order.

(3) Let X be a topological set. The *fundemental groupoid* $\Pi_1(X)$ of X is the category whose set of objects is X and such that, for $x, y \in X$, the set of morphisms from x to y is the set of continuous maps $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(y) = 1$ modulo homotopies fixing the endpoints (i.e. continuous paths from x to y modulo homotopy); the composition law of $\Pi_1(X)$ is giving by concatenation of paths. This is a groupoid, and it is connected if and only if X is path-connected.

(4) Let G be a monoid. The category BG is the category whose set of objects is a singleton $\{*\}$ and such that $\mathrm{End}_{BG}(*) = G$; the composition law is given by the multiplication of G. Note that BG is a groupoid if and only if G is a group.

Conversely, if $\mathscr{C}$ is a category with only one object X, then it is isomorphic (see Definition I.2.2.5) to $B\,\mathrm{End}_{\mathscr{C}}(X)$.

(5) For every $n \in \mathbb{N}$, we denote by $[n]$ the set $\{0, 1, \dots, n\}$ with the usual ordering. The *simplicial category* Δ is the subcategory of $\mathbf{Set}$ whose objects are the $[n]$ for all $n \in \mathbb{N}$ and whose morphisms are the nondecreasing maps. This is a very nice category : it is $\mathscr{U}$-small for every universe $\mathscr{U}$.

From now on, we fix a universe $\mathscr{U}$. All categories will be assumed to be $\mathscr{U}$-categories, and the categories Set, Top etc will be their $\mathscr{U}$-versions. We will omit the $\mathscr{U}$ from notation and never mention it again, unless we need to introduce another universe.

Definition I.2.1.9. Let $\mathscr{C}$ be a category and X be an object of $\mathscr{C}$.

(i). We say that X is *initial (in $\mathscr{C}$)* if, for every $Y \in \mathrm{Ob}(\mathscr{C})$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a singleton.

(ii). We say that X is *final or terminal (in $\mathscr{C}$)* if it is initial as an object of $\mathscr{C}^{\mathrm{op}}$, that is, for every $Y \in \mathrm{Ob}(\mathscr{C})$, the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ is a singleton.

(iii). We say that X is a *zero object (in $\mathscr{C}$)* if it is both initial and final. In that case, we often denote the object X by 0; if Y is an object of $\mathscr{C}$, then the endomorphism of Y obtained by compposing the unique morphisms $Y \to X$ and $X \to Y$ is also denoted by 0.

Remark I.2.1.10. If an initial (resp. final) object exists in a category $\mathscr{C}$, then it is unique up to unique isomorphism.

Example I.2.1.11. (1) In the categories $\mathbf{Set}$, $\mathbf{Set^f}$ and $\mathbf{Top}$, the empty set is initial and any singleton is final.

(2) A singleton is a zero object in the categories $\mathbf{Set}_*$ and $\mathbf{Top}_*$.

(3) The trivial group (or monoid) is a zero object in the categories $\mathbf{Grp}$, $\mathbf{Ab}$ and $\mathbf{Mon}$, and the zero R-module is a zero object in the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$.

(4) The ring $\mathbb{Z}$ is an initial object in $\mathbf{Ring}$ and $\mathbf{CRing}$, and the zero ring is a final object in these categories. The category of fields has neither an initial object nor a final object.

(5) Let R be a commutative ring. Then R is an initial object of $R-\mathbf{Alg}$, and the zero R-algebra is a final object of this category. In the category $R-\mathbf{Lie}$, the zero Lie algebra is a zero object.

(6) The empty set is a zero object of the category $\mathbf{Rel}$.

(7) Let $\mathscr{C}$ be a discrete category. If $\mathscr{C}$ has at least two objects, then it has neither an initial nor a final object. If $\mathscr{C}$ has only one object, then this object is a zero object.

(8) Let G be a monoid. If $G \neq \{1\}$, then BG has no initial or final object.

(9) Let I be a poset, seen as a category. Then an initial (resp. final) object of I is the same as a smallest (resp. biggest) element of I.

(10) The simplicial category Δ does not have an initial object. It has a final object, which is $[0]$.

I.2.2 Functors

Definition I.2.2.1. Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A *functor from $\mathscr{C}$ to $\mathscr{C}'$*, denoted by $F : \mathscr{C} \to \mathscr{C}'$, is the data of a map $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$, and, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, of a map $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X),F(Y))$, still denoted by F, such that :

(a) for every $X \in \mathrm{Ob}(\mathscr{C})$, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$;

(b) for all $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ and all morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y,Z)$, we have $F(g \circ f) = F(g) \circ F(f)$.

A functor $F : \mathscr{C} \to \mathscr{C}'$ is also called a *covariant* from $\mathscr{C}$ to $\mathscr{C}'$. A *contravariant functor* from $\mathscr{C}$ to $\mathscr{C}'$ is a (covariant) functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}'$.

If $F : \mathscr{C} \to \mathscr{C}''$ and $G : \mathscr{C}' \to \mathscr{C}''$, their composition $G \circ F : \mathscr{C} \to \mathscr{C}''$ is defined in the obvious way.

A functor from $\mathscr{C}$ to itself is also called an *endofunctor* of $\mathscr{C}$.

Remark I.2.2.2. If $\mathscr{C}$ and $\mathscr{C}'$ are two categories, a map $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$ is often called *functorial* if it can be upgraded "naturally" to a functor. [1] For example, we might say that the construction of the abelianization G^{ab} of a group G is functorial, because it is pretty clear that any morphism of groups $G \to H$ will define a morphism of groups $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ in a way that is compatible with composition. In other words, we have upgraded the abelianization to a functor $\mathbf{Grp} \to \mathbf{Ab}$.

Sometimes, if the effect of a functor on morphisms is obvious, we define it only by the map $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$. (See for example Example I.2.2.7.)

Example I.2.2.3. (1) Let $\mathscr{C}$ be a category. We have the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$. The identity also defines a contravariant functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}$.

[1] Of course, the meaning of "naturally" depends on the speaker...

More generally, if $\mathscr{C}'$ is a subcategory of $\mathscr{C}$, then the inclusion if a functor from $\mathscr{C}'$ to $\mathscr{C}$.

(2) Let $\mathscr{C}$ be a category and $X \in \mathrm{Ob}(\mathscr{C})$.

We define a functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \to \mathbf{Set}$ by :

(a) for every $Y \in \mathrm{Ob}(\mathscr{C})$, the image of Y by this functor is the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$;

(b) for every morphism $f : Y \to Z$ of $\mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X, f) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ sends $g : X \to Y$ to $f \circ g$; we also denote this map by f_*.

We define a contravariant functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ from $\mathscr{C}$ to $\mathbf{Set}$ by :

(a) for every $Y \in \mathrm{Ob}(\mathscr{C})$, the image of Y by this functor is the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$;

(b) for every morphism $f : Y \to Z$ of $\mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(f, X) : \mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Hom}_{\mathscr{C}}(Z, X)$ sends $g : Y \to X$ to $g \circ f$; we also denote this map by f^*.

The functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is sometimes also denoted by h_X or $\underline{X}$.

(3) Let k be a field. Note that $\mathbf{Mod}_k$ is the category of k-vector spaces. The *duality functor* is the contravariant endofunctor of $\mathbf{Mod}_k$ that sends a k-vector space V to its dual $V^* := \mathrm{Hom}_k(V, k)$ and a k-linear map $u : V \to W$ to its transpose $u^T : W^* \to V^*$, $\alpha \longmapsto \alpha \circ u$.

(4) Let R be a ring and M be a right (resp. left) R-module. Then we get a functor $M \otimes_R (\cdot)$ (resp. $(\cdot) \otimes_R M$) from ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$) to $\mathbf{Ab}$.

There are many variants of this. For example, if M is a R-bimodule (which is automatic if R is commutative), then $M \otimes_R (\cdot)$ (resp. $(\cdot) \otimes_R M$) upgrades to an endofunctor of ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$).

(5) The construction of the fundamental group defines a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$.

(6) If $\mathscr{C}$ is a category of "sets with extra structure" (such as $\mathbf{Set}_*$, $\mathbf{Top}$, $\mathbf{Top}_*$, $\mathbf{Mon}$, $\mathbf{Grp}$, $\mathbf{Ring}$, $\mathbf{CRing}$, $\mathbf{Field}$, $\mathbf{Ab}$, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, $R-\mathbf{Alg}$, $R-\mathbf{Lie}$), [2] then we have a *forgetful functor* $\mathrm{For} : \mathscr{C} \to \mathbf{Set}$ that forgets that extra structure. More generally, we have forgetful functors between various categories of sets with extra structures; for example, there are forgetful functors $R - \mathbf{Alg} \to {}_R\mathbf{Mod}$, ${}_R\mathbf{Mod} \to \mathbf{Ab}$, $\mathbf{Grp} \to \mathbf{Set}_*$ etc.

(7) In the other direction, the construction of a "free foo on bar" are also functorial. For example, the free monoid (resp. group, abelian group, left R-module, R-algebra, commutative R-algebra, R-Lie algebra) on a set defines a functor from $\mathbf{Set}$ to $\mathbf{Mon}$ (resp. $\mathbf{Grp}$, $\mathbf{Ab}$, ${}_R\mathbf{Mod}$, $R - \mathbf{Alg}$, $R - \mathbf{CAlg}$, $R - \mathbf{Lie}$). If R is a commutative ring, we have a functor "free (commutative) R-algebra on a R-module" from ${}_R\mathbf{Mod}$ to $R - \mathbf{Alg}$ (or $R - \mathbf{CAlg}$). Etc.

[2] Such a category is called a *concrete category*.

(8) We have a functor from the category of Lie groups over $\mathbb{R}$ (resp. $\mathbb{C}$) to the category of Lie algebras over $\mathbb{R}$ (resp. $\mathbb{C}$) that sends a Lie group to its Lie algebra and a morphism of Lie groups to its differential at the identity.

(9) Spec is a contravariant functor from the category of commutative rings to the category of affine schemes.

(10) We have a contravariant functor from **Top** to the category of $\mathbb{C}$-algebras that sends a topological space X to the algebra $\mathscr{C}(X)$ of continuous functions from X to $\mathbb{C}$ and a continuous map $u : X \to Y$ to the morphism of algebras $u^* : \mathscr{C}(Y) \to \mathscr{C}(X)$, $f \longmapsto f \circ u$. In fact, if we restrict it to the full subcategory of compact Hausdorff spaces, this functor lands in the subcategory of commutative Banach algebras over $\mathbb{C}$, and even in the subcategory of commutative C^*-algebras over $\mathbb{C}$. The various versions of this functor are called the *Gelfand transform*.

If $\mathscr{C}$, $\mathscr{C}'$ and $\mathscr{C}''$ are categories, a functor from $\mathscr{C} \times \mathscr{C}'$ to $\mathscr{C}''$ is also called a *bifunctor*.

Example I.2.2.4. (1) Let R be a commutative ring. Then $\otimes_R$ is a bifunctor from $\mathbf{Mod}_R$ to itself.

(2) Let $\mathscr{C}$ be a category. Then $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$ is a bifunctor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$. We also say that it is a bifunctor from $\mathscr{C}$ to $\mathbf{Set}$, contravariant in the first variable and covariant in the second variable.

If $\mathscr{U}$ is a universe, then $\mathscr{U}$-categories and functors between them form a category, which we denote by $\mathscr{U} - \mathbf{Cat}$. Note that $\mathscr{U} - \mathbf{Cat}$ is not a $\mathscr{U}$-category; if $\mathscr{U}'$ is another universe such that $\mathscr{U} \in \mathscr{U}'$, then $\mathscr{U} - \mathbf{Cat}$ is a $\mathscr{U}'$-category.

In particular, we cam define isomorphisms of categories.

Definition I.2.2.5. A functor $F : \mathscr{C} \to \mathscr{C}'$ is called an *isomorphism of categories* if there exists a functor $G : \mathscr{C}' \to \mathscr{C}$ such that $F \circ G = \mathrm{id}_{\mathscr{C}}$ and $G \circ F = \mathrm{id}_{\mathscr{C}}$.

This is not a very useful notion, because it is too rigid. Still, we give some example of isomorphisms of categories.

Definition I.2.2.6. Let $\mathscr{C}$ be a category and X be an object of $\mathscr{C}$. We define the *slice categories of $\mathscr{C}$ over and under X*, denoted by $\mathscr{C}/X$ and $X\backslash\mathscr{C}$, in the following way :

(a) $\mathscr{C}/X$ is the category whose objects are morphisms $f : Y \to X$ with $Y \in \mathrm{Ob}(\mathscr{C})$, and in which a morphism from $f : Y \to X$ to $g : Z \to X$ is a morphism $h : Y \to Z$ such that $h \circ f = g$, ie that the triangle

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes. The composition law of $\mathscr{C}/X$ is given by the composition law of $\mathscr{C}$.

(b) $X\backslash\mathscr{C}$ is the category whose objects are morphisms $f : X \to Y$ with $Y \in \mathrm{Ob}(\mathscr{C})$, and in which a morphism from $f : X \to Y$ to $g : X \to Z$ is a morphism $h : Y \to Z$ such that $g \circ h = f$, ie that the triangle

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"'] & & Z \end{tikzcd}$$

commutes. The composition law of $X\backslash\mathscr{C}$ is given by the composition law of $\mathscr{C}$.

The category $\mathscr{C}/X$ is also called the *category of objects of $\mathscr{C}$ over X*. The category $X\backslash\mathscr{C}$ is also called the *coslice category of $\mathscr{C}$ with respect to X*, or the *category of objects of $\mathscr{C}$ under X*.

Example I.2.2.7. (1) Let $*$ be a singleton. Then the slice categories $*\backslash\mathbf{Set}$ and $*\backslash\mathbf{Top}$ are isomorphic to the categories of pointed sets and of pointed topological spaces by the functor that sends a map $f : * \to X$ to the pointed set (or topological space) $(X, f(*))$. The inverse functor sends a pointed set or topological space (X, x) to the map $* \simeq \{x\} \subset X$.

(2) The category $\mathbb{Q}\backslash\mathbf{Field}$ (resp $\mathbb{F}_p\backslash\mathbf{Field}$, if p is a prime number) is isomorphic the full subcategory of $\mathbf{Field}$ whose objects are field of characteristic 0 (resp. p) by the functor sending a morphism $\mathbb{Q} \to K$ (resp. $\mathbb{F}_p \to K$) to K.

(3) If X is an initial (resp. final) object of $\mathscr{C}$, then $X\backslash\mathscr{C}$ (resp. $\mathscr{C}/X$) is isomorphic to $\mathscr{C}$ by the functor sending a morphism $X \to Y$ (resp. $Y \to X$) to Y. The fact that this is an isomorphism of categories is a direct translation of the definition of an initial (resp. final) object.

Definition I.2.2.8. Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor. The functor F is called :

(i). *faithful* if the map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is injective for all $X, Y \in \mathrm{Ob}(\mathscr{C})$;

(ii). *full* if the map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is suejective for all $X, Y \in \mathrm{Ob}(\mathscr{C})$;

(iii). *fully faithful* if it is full and faithful;

(iv). *essentially surjective* if, for every object X' of $\mathscr{C}'$, there exists an object X of $\mathscr{C}$ such that $F(X)$ and X' are isomorphic;

(v). *conservative* if, for every morphism $f : X \to Y$ of $\mathscr{C}$, the morphism f is an isomorphism if and only $F(f)$ is an isomorphism. [3]

Example I.2.2.9. (1) The forgetful functors of Example I.2.2.3(6) are faithful (but usually not full).

[3]Note that one direction of this equivalence is automatic : if f is an isomorphism, then $F(f)$ is automatically an isomorphism, and $F(f)^{-1} = F(f^{-1})$.

(2) The forgetful functors from $\mathbf{Mon}$ (resp. $\mathbf{Grp}$, $\mathbf{Ab}$, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, $R-\mathbf{Alg}$, $\mathbf{Ring}$, $R-\mathbf{Lie}$) to $\mathbf{Set}$ are conservative, but the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is not conservative. In other words, if a morphism of monoids (resp. groups, rings, R-modules etc) is a bijection, then its inverse is also a morphism of monoids (resp. ...); but if a continuous map is a bijection, then its inverse is not always continuous.

(3) If $\mathscr{C}' \subset \mathscr{C}$, then the inclusion functor from $\mathscr{C}'$ to $\mathscr{C}$ is faithful, and it is fully faithful if and only if $\mathscr{C}'$ is a full subcategory of $\mathscr{C}$.

(4) A fully faithful fonctor is conservative.

(5) Let k be a field, and let $\mathscr{C}$ be the full subcategory of $\mathbf{Mod}_k$ whose objects are the vector spaces $k^{(I)}$, for I a set. Then the inclusion $\mathscr{C} \subset \mathbf{Mod}_k$ is essentially surjective; in other words, every vector space has a basis.

(6) Let k be a field. Then the forgetful functor $\mathbf{Mod}_k \to k-\mathbf{Lie}$ is faithful and essentially surjective, but not full. The essential surjectivity means that every k-vector space has a Lie algebra structure (the commutative one for example).

(7) The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is faithful and essentially surjective, but not full.

(8) Let k be a field. The duality functor $\mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Mod}_k$ is faithful and conservative, but not full or essentially surjective. If we restrict it to the full subcategory of finite-dimensional vector spaces, it becomes fully faithful.

(9) The inclusion functor from $\mathbf{Set}$ to $\mathbf{Rel}$ is faithful and essentially surjective (it is even a bijection on objects), but not full.

(10) The functor of Example I.2.2.3(8) (from Lie groups to Lie algebras) is neither full not faithful : a morphism of Lie algebras does not always lift to a morphism of Lie groups, and a morphism of Lie groups is not always determined by its differential at the identity (for example if the source Lie group is not connected).

I.2.3 Morphisms of functors

Definition I.2.3.1. Let $F, G : \mathscr{C} \to \mathscr{C}'$ be tzo functors. A *morphism of functors* $u : F \to G$ from F to G is the data, for every $X \in \mathrm{Ob}(\mathscr{C})$, of a map $u(X) : F(X) \to G(X)$ such that, for every morphism $f : X \to Y$, the following diagram commutes :

$$\begin{tikzcd} F(X) \arrow[r, "u(X)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "u(Y)"'] & G(Y) \end{tikzcd}$$

A morphism of functors is also called a *natural transformation*. We sometimes represent it by a diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow u"] & \mathscr{C}' \end{tikzcd}$$

If $F, G, H : \mathscr{C} \to \mathscr{C}'$ are functors and $u : F \to G$ and $v : G \to H$ are morphisms of functors, then we can define a morphism of functors $v \circ u : F \to H$ by $(v \circ u)(X) = v(X) \circ u(X)$, for every $X \in \mathrm{Ob}(\mathscr{C})$.

Definition I.2.3.2. Let $\mathscr{C}$ et $\mathscr{C}'$ be two categories. We denote by $\mathrm{Func}(\mathscr{C}, \mathscr{C}')$ the category whose objects are functors from $\mathscr{C}$ to $\mathscr{C}'$ and whose morphism are morphisms of functors. The composition law is the one explained just before this definition.

In particular, we can talk about isomorphisms of functors and isomorphic functors.

Remark I.2.3.3. It is easy to show that a morphism of functors $u : F \to G$ is an isomorphism of functors if and only if $u(X) : F(X) \to G(X)$ is an isomorphism for every object X of the source category. (Indeed, in that case the family $(u(X)^{-1})$ automatically satisfies the condition defining a morphism of functors.)

Remark I.2.3.4. Let $\mathscr{U}$ be a universe. If $\mathscr{C}$ and $\mathscr{C}'$ are $\mathscr{U}$-categories, then $\mathrm{Func}(\mathscr{C}, \mathscr{C}')$ is not always a $\mathscr{U}$-category; but it is a $\mathscr{U}$-category if $\mathscr{C}$ is $\mathscr{U}$-small.

Example I.2.3.5. (1) Consider the functor $F : \mathbf{Ab} \to \mathbf{Ab}$ sending an abelian group A to the free abelian group $\mathbb{Z}^{(A)} = \bigoplus_{a \in A} \mathbb{Z}e_a$ on the underlying set of A. Then there is a morphism of functors $u : F \to \mathrm{id}_{\mathbf{Ab}}$ defined in the following way : For every abelian group A, the morphism of abelian groups $u(A) : \mathbb{Z}^{(A)} \to A$ sends the generator e_a to a, for every $a \in A$.

(2) Let k be a field. We denote the duality functor by D. Then $D^2 = D \circ D$ is a (covariant) endofunctor of $\mathbf{Mod}_k$, and we have a morphism of functors $u : \mathrm{id}_{\mathbf{Mod}_k} \to D^2$ defined in the following way : For every k-vector space V, the map $u(V) : V \to V^{**}$ sends $v \in V$ to the element ev_v of $V^{**} = \mathrm{Hom}_k(V^*, k)$ defined by $\mathrm{ev}_v(\alpha) = \alpha(v)$, for every $\alpha \in V^* = \mathrm{Hom}_k(V, k)$.

(3) Let $\mathbf{cHaus}$ (resp. $\mathbf{Ban}$) be the category of locally compact Hausdorff spaces and continuous maps (resp. real Banach spaces and continuous $\mathbb{C}$-linear maps). Let $\mathscr{M} : \mathbf{cHaus} \to \mathbf{Ban}$ be the functor sending a locally compact Hausdorff space X to the Banach space of regular complex measures on the σ-algebra of Borel subsets of X, and a continuous function $f : X \to Y$ to the map $\mathscr{M}(X) \to \mathscr{M}(Y)$, $\mu \longmapsto \mu \circ f^{-1}$. let $\mathscr{C}^* : \mathbf{cHaus} \to \mathbf{Ban}$ be the functor sending a locally compact Hausdorff space X to the bounded dual of the Banach space $\mathscr{C}_0(X)$ of continuous functions $f : X \to \mathbb{C}$ that vanish at infinity.

We have a morphism of functors $u : \mathscr{M} \to \mathscr{C}^*$ defined in the following way : Let X be a locally compact Hausdorff space. If $\mu \in \mathscr{M}(X)$, then the map $f \longmapsto \int_X f d\mu$ is a bounded linear functional on $\mathscr{C}_0(X)$, and we take $u(X)(\mu)$ equal to this linear functional. The Riesz representation theorem says that this morphism of functors is an isomorphism.

The following notion is more useful than the notion of an isomorphism of categories.

Definition I.2.3.6. A functor $F : \mathscr{C} \to \mathscr{C}'$ is called an *equivalence of categories* if there exists a functor $G : \mathscr{C}' \to \mathscr{C}$ such that $F \circ G \simeq \mathrm{id}_{\mathscr{C}'}$ and $G \circ F \simeq \mathrm{id}_{\mathscr{C}}$. In that case, the functor G is called a *quasi-inverse* of F.

The next theorem is very useful when we need to prove that a functor is an equivalence of categories. For a proof, see problem A.1.2.

Theorem I.2.3.7. *A functor $F : \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Definition I.2.3.8. Let $F : \mathscr{C} \to \mathscr{C}'$ be a fully faithful functor. The *essential image* of F is the full subcategory $\mathscr{D}$ of $\mathscr{C}'$ whose objects are the objects Y of $\mathscr{C}'$ such that there exists $X \in \mathrm{Ob}(\mathscr{C})$ with $F(X) \simeq Y$.

Corollary I.2.3.9. *If $F : \mathscr{C} \to \mathscr{C}'$ is a fully faithful functor, then it induces an equivalence from $\mathscr{C}$ to its essential image.*

Example I.2.3.10. (1) Let k be a field, and let $\mathscr{C}$ be the category of finite-dimensional k-vector spaces and k-linear maps. Then the functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ defined by restriction of the duality functor is an equivalence of categories. It is its own quasi-inverse.

(2) Let k be a field. Let $\mathscr{C}$ be the category of finite-dimensional k-vector spaces and k-linear maps. Let $\mathscr{C}'$ be the category with set of objects $\mathbb{N}$, and such that $\mathrm{Hom}_{\mathscr{C}'}(n, m) = M_{mn}(k)$ (the set of $m \times n$ matrices with coefficients in k) for all $n, m \in \mathbb{N}$; the composition law of $\mathscr{C}'$ is given by matrix multiplication. Let $F : \mathscr{C}' \to \mathscr{C}$ be the functor sending $n \in \mathbb{N}$ to k^n and a matrix $A \in M_{mn}(k)$ to the corresponding linear transformation from k^n to k^m. Then F is an equivalence of categories.

(3) The functor Spec is a contravariant equivalence from the category of commutative rings to the category of affine schemes.

(4) Let $\mathscr{C}$ be the category of compact Hausdorff spaces, let $\mathscr{C}'$ be the category of commutative unital complex C^*-algebras, and let $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$ be the functor of Example I.2.2.3(10). The Gelfand-Naimark theorem implies that this functor is an equivalence of categories, with quasi-inverse given by the spectrum functor.

(5) Let $\mathscr{C}$ be the category of finite linearly ordered sets (and nondecreasing maps). Then the simplicial category Δ is a subcategory of $\mathscr{C}$, and the inclusion $\Delta \subset \mathscr{C}$ is an equivalence of categories.

I.3 The Yoneda lemma

I.3.1 Presheaves

Definition I.3.1.1. Let $\mathscr{C}$ be a category. The category $\mathrm{PSh}(\mathscr{C})$ of *presheaves (of sets) on* $\mathscr{C}$ is the catgeory $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$.

The terminology is explained by the following example.

Example I.3.1.2. Let X be a topological set. We denote by $\mathrm{Open}(X)$ the category corresponding the poset of open subsets of X, ordered by inclusion (see Example I.2.1.8(2)). Then a presheaf on $\mathrm{Open}(X)$ is the same thing as a preseheaf on the topological space X.

Remark I.3.1.3. If $\mathscr{U}$ is a universe and the category $\mathscr{C}$ is $\mathscr{U}$-small, then $\mathrm{PSh}(\mathscr{C})$ is a $\mathscr{U}$-category, and it is $\mathscr{U}'$-small for every universe $\mathscr{U}'$ such that $\mathscr{U} \in \mathscr{U}'$.

Example I.3.1.4. If $\mathscr{C}$ is a category and $X \in \mathrm{Ob}(\mathscr{C})$, then the functor $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ of Example I.2.2.3(2) is a presheaf on $\mathscr{C}$. A presheaf that is isomorphic to some h_X is called *representable*. It will follow from Yoneda's lemma (Theorem I.3.2.2) that such a X is unique up to unique isomorphism.

Remark I.3.1.5. A functor $\mathscr{C} \to \mathbf{Set}$ is the same as a presheaf on $\mathscr{C}^{\mathrm{op}}$. For example, for every $X \in \mathrm{Ob}(\mathscr{C})$, the functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$ of Example I.2.2.3(2) is a presheaf on $\mathscr{C}^{\mathrm{op}}$. A presheaf on $\mathscr{C}^{\mathrm{op}}$ that is isomorphic to one of these functors is also called a representable presheaf on $\mathscr{C}^{\mathrm{op}}$, or a representable functor from $\mathscr{C}$ to $\mathbf{Set}$.

See problem A.1.8 for some examples of representable functors.

I.3.2 The Yoneda embedding

Using the fact that $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$ is a bifunctor, we see that the formation of the representable presheaf h_X is functorial in X. More precisely :

Definition I.3.2.1. Let $\mathscr{C}$ be a category. The *Yoneda embedding* is the functor $h_{\mathscr{C}} = h : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ defined in the following way :

(1) for every $X \in \mathrm{Ob}(\mathscr{C})$, the image of X by h is the presheaf $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ of Example I.2.2.3(2);

(2) for every morphism $f : X \to Y$ of $\mathscr{C}$, the morphism $h_f : h_X \to h_Y$ is the morphism of functors such that, for every $Z \in \mathrm{Ob}(\mathscr{C})$, the map $h_f(Z) : \mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ is $\mathrm{Hom}_{\mathscr{C}}(Z, f)$, that is, the map $g \longmapsto f \circ g$.

The *Yoneda lemma* is the following result.

Theorem I.3.2.2. *Let $\mathscr{C}$ be a category, let $X \in \mathrm{Ob}(\mathscr{C})$, and let F be a presheaf on $\mathscr{C}$. Then the map $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F) \to F(X)$ sending a morphism of functors $u : h_X \to F$ to $u(X)(\mathrm{id}_X) \in F(X)$ is a bijection.*

Proof. See problem A.1.5.

□

Corollary I.3.2.3. *The functor $h : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ is fully faithful.*

In other words, the Yoneda embeddings induces an equivalence from $\mathscr{C}$ to the category of representable presheaves on $\mathscr{C}$.

Proof. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$. We want to show that the map $h : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, h_Y)$ is bijective. We denote by $\varphi : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, h_Y) \to h_Y(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$ the map of Theorem I.3.2.2. Then, for every morphism $f : X \to Y$, we have $\varphi(h_f) = h_f(X)(\mathrm{id}_X)$, which is by definition the image of $\mathrm{id}_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ by the map $\mathrm{Hom}_{\mathscr{C}}(X, X) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$, $g \longmapsto f \circ g$, i.e. f. So $\varphi \circ h$ is the identity of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. As φ is bijective by Theorem I.3.2.2, so is h.

□

Definition I.3.2.4. Let $F \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C}))$. We say that a couple (X, x) with $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$ *represents* the presheaf F if the morphism $\alpha : h_X \to F$ corresponding to x by Theorem I.3.2.2 is an isomorphism.

Corollary I.3.2.5. *If $F \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C}))$ is representable, then a couple (X, x) representing F is uniquely determined up to unique isomorphism. That is, if the couples (X, x) and (Y, y) both represent F, then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $F(f)(y) = x$.*

Applying Yoneda's lemma to the opposite category, we get the following result.

Corollary I.3.2.6. *Let $\mathscr{C}$ be a category. We denote by $k_{\mathscr{C}} = k : \mathscr{C}^{\mathrm{op}} \to \mathrm{Func}(\mathscr{C}, \mathbf{Set})$ the functor sending an object X of $\mathscr{C}$ to $k_X = \mathrm{Hom}_{\mathscr{C}}(X, \cdot)$, and defined in the obvious way on the morphisms (see Definition I.3.2.1). Then, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every functor $F : \mathscr{C} \to \mathbf{Set}$, the map $\mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathbf{Set})}(k_X, F)$ sending $u : k_X \to F$ to $u(X)(\mathrm{id}_X) \in F(X)$ is bijective.*

Definition I.3.2.7. Let $F : \mathscr{C} \to \mathbf{Set}$ be a functor. We say that a couple (X, x) with $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$ *represents* the functor F if the morphism $\alpha : k_X \to F$ corresponding to x by Corollary I.3.2.6 is an isomorphism.

Corollary I.3.2.8. *If a functor $F : \mathscr{C} \to \mathbf{Set}$ is representable, then a couple (X, x) representing F is uniquely determined up to unique isomorphism. That is, if the couples (X, x) and (Y, y) both represent F, then there exists a unique isomorphism $f : X \xrightarrow{\sim} Y$ such that $F(f)(x) = y$.*

Corollary I.3.2.9. *Let $\mathscr{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathscr{C}$. The following are equivalent :*

(i). f *is an isomorphism.*

(ii). The map $\mathrm{Hom}_{\mathscr{C}}(Z, f) : \mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ *is bijective for every* $Z \in \mathrm{Ob}(\mathscr{C})$.

(iii). The map $\mathrm{Hom}_{\mathscr{C}}(f, Z) : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ *is bijective for every* $Z \in \mathrm{Ob}(\mathscr{C})$.

Proof. If f is an isomorphism, then the maps of (ii) and (iii) are clearly bijective (with inverses $\mathrm{Hom}_{\mathscr{C}}(Z, f^{-1})$ and $\mathrm{Hom}_{\mathscr{C}}(f^{-1}, Z)$).

Conversely, the fact that (ii) implies (i) follows from the fact that the functor $h : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ is fully faithful (Corollary I.3.2.3), hence conservative. The fact that (iii) implies (i) follows by applying the previous sentence to $\mathscr{C}^{\mathrm{op}}$.

□

Remark I.3.2.10. Many universal constructions are in fact asking whether a functor from a category (or its opposite) to the category of sets is representable. The automatic uniqueness statements in these constructions are actually consequences of the Yoneda lemma (or rather of Corollaries I.3.2.5 and I.3.2.8).

Here are two examples :

(1) Let $\mathfrak{g}$ be a Lie algebra, say over a commutative ring R. Consider the functor $F : R - \mathbf{Alg} \to \mathbf{Set}$ sending a R-algebra A to the set of morphisms $\mathrm{Hom}_{R-\mathbf{Lie}}(\mathfrak{g}, A)$, where we use the Lie algebra structure on A given by the commutator bracket. In other words, we have a "forgetful" functor $L : R - \mathbf{Alg} \to R - \mathbf{Lie}$ that sends a R-algebra A to the Lie algebra A with the commutator bracket, and F is the functor $k_{\mathfrak{g}} \circ L$. A universal envelopping algebra of $\mathfrak{g}$ is just a couple (U, u) (with $u \in F(U) = \mathrm{Hom}_R - \mathbf{Lie}(\mathfrak{g}, L(U))$) representing the functor F. Indeed, saying that (U, u) represents F means that, for every R-algebra A, the map $\mathrm{Hom}_{R-\mathbf{Alg}}(U, A) \to \mathrm{Hom}_{R-\mathbf{Lie}}(\mathfrak{g}, L(A))$ sending $f : U \to A$ to $L(f) \circ u$ is bijective. (In other words, for every morphisms of Lie algebras $v : \mathfrak{g} \to L(A)$, there exists a unique R-algebra morphism $f : U \to A$ such that $v = f \circ u$.)

(2) Let R be a commutative ring, and let M and N be R-modules. We consider the functor $\mathrm{Bil}(M \times N, \cdot) : {}_R\mathbf{Mod} \to \mathbf{Set}$ that sends a R-module P to the set of R-bilinear maps $M \times N \to P$. We know that this functor is representable : a pair representing it is given by the tensor product $M \otimes_R N$ and the canonical bilinear map $b : M \times N \to M \otimes_R N$, $(m, n) \longmapsto m \otimes n$. The universal property of this couple is the following : for every R-module P, the map $\mathrm{Hom}_R(M \otimes_R N, P) \to \mathrm{Bil}(M \times N, P)$ sending $f : M \otimes_R N \to P$ to $f \circ b$ is bijective; that is, every R-bilinear map $B' : M \times N \to P$ can be written in a unique way $b' = f \circ b$, with $f : M \otimes_R N \to P$ a R-linear map.

In fact, as we will see in the next section, many of these constructions upgrade to one half of a pair of adjoint functors.

I.4 Adjoint functors

Definition I.4.1. Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories, and let $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}$ be functors. We say that (F, G) is a *pair of adjoint functors*, or that G is *right adjoint* to F, or that F is *left adjoint* to G, if there exists an isomorphism of bifunctors $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}' \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot)).$$

The following lemma is one half of problem A.1.7.

Lemma I.4.2. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two catgeories, and let $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}$ be functors. Suppose that there exists an isomorphism of bifunctors $\alpha : \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$.*

Then, for every morphism $f : X_1 \to X_2$ in $\mathscr{C}$ and every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, if $u : F(X_1) \to Y_1$ and $v : F(X_2) \to Y_2$ are morphisms in $\mathscr{C}'$, the square

$$\begin{tikzcd}
F(X_1) \arrow[r, "u"] \arrow[d, "F(f)"'] & Y_1 \arrow[d, "g"] \\
F(X_2) \arrow[r, "v"'] & Y_2
\end{tikzcd}$$

is commutative if and only if the square

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha(X_1,Y_1)(u)"] \arrow[d, "f"'] & G(Y_1) \arrow[d, "G(g)"] \\
X_2 \arrow[r, "\alpha(X_2,Y_2)(v)"'] & G(Y_2)
\end{tikzcd}$$

is commutative.

Proposition I.4.3. *If $F : \mathscr{C} \to \mathscr{C}'$ has two right adjoints $G_1, G_2 : \mathscr{C}' \to \mathscr{C}$, then there exists a unique isomorphism $u : G_1 \overset{\sim}{\to} G_2$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}'}(F(X), Y) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Hom}_{\mathscr{C}}(X, G_1(Y)) \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(X, u(Y))"] \\
\mathrm{Hom}_{\mathscr{C}'}(F(X), Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{C}}(X, G_2(Y))
\end{tikzcd}$$

for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $Y \in \mathrm{Ob}(\mathscr{C}')$.

There is a similar statement for left adjoints.

Proof. Let $Y \in \mathrm{Ob}(\mathscr{C}')$. By Corollary I.3.2.3, the isomorphism of functors $\mathrm{Hom}_{\mathscr{C}}(\cdot, G_1(Y)) \simeq \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G_2(Y))$ comes from a unique morphism $u(Y) : G_1(Y) \to G_2(Y)$ of $\mathscr{C}$, and this morphism is an isomorphism by Corollary I.3.2.9.

It remains to show that the family $(u(Y))_{Y \in \mathrm{Ob}(\mathscr{C}')}$ defines a morphism of functors. Let $f : Y \to Y'$ be a morphism of $\mathscr{C}'$. We want to show that the square

$$\begin{tikzcd} G_1(Y) \arrow[r, "u(Y)"] \arrow[d, "G_1(f)"'] & G_2(Y) \arrow[d, "G_2(f)"] \\ G_1(Y') \arrow[r, "u(Y')"'] & G_2(Y') \end{tikzcd}$$

is commutative, that is, that $u(Y') \circ G_1(f) = G_2(f) \circ u(Y)$. If we apply the Yoneda embedding $h : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ to this square, we get by definition of u a diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(\cdot, G_1(Y)) \arrow[r, "\sim"] \arrow[d, "h_{G_1(f)}"'] & \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y) \arrow[r, "\sim"] \arrow[d, "h_f \circ F"'] & \mathrm{Hom}_{\mathscr{C}}(\cdot, G_2(Y)) \arrow[d, "h_{G_2(f)}"'] \\ \mathrm{Hom}_{\mathscr{C}}(\cdot, G_1(Y')) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y') \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{C}}(\cdot, G_2(Y')) \end{tikzcd}$$

and both halves of this diagram are commutative because

$$\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G_{1/2}(\cdot))$$

are morphisms of bifunctors. As the Yoneda embedding is faithful, this implies that the original square is also commutative.

□

Let (F, G) be a pair of adjoint functors. Then we get isomorphisms of bifunctors

$$\mathrm{Hom}_{\mathscr{C}'}(F \circ G(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathscr{C}}(G(\cdot), G(\cdot))$$

and

$$\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), F(\cdot)) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G \circ F(\cdot)).$$

Taking the images of the identity maps by these isomorphisms, we get morphisms of functors $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{C}'}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$, that are called the *counit* and the *unit* of the adjunction.

Proposition I.4.4. *The morphisms of functors*

$$F \overset{F(\eta)}{\to} F \circ G \circ F \overset{\varepsilon(F)}{\to} F$$

and

$$G \overset{\eta(G)}{\to} G \circ F \circ G \overset{G(\varepsilon)}{\to} G$$

are equal to id_F *and* id_G *respectively.*

Proof. We denote by α the isomorphism of bifunctors $\operatorname{Hom}_{\mathscr{C}'}(F(\cdot),\cdot) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$.

Let X be an object of $\mathscr{C}$. As $\eta(X) = \alpha(X, F(X))(\mathrm{id}_{F(X)}) : X \to G(F(X))$, we have a commutative square

$$\begin{tikzcd}
X \arrow[r, "\alpha(X,F(X))(\mathrm{id}_{F(X)})"] \arrow[d, "\eta(X)"'] & G(F(X)) \arrow[d, "\mathrm{id}_{G(F(X))}"] \\
G(F(X)) \arrow[r, "\mathrm{id}_{G(F(X))}"'] & G(F(X))
\end{tikzcd}$$

So, by Lemma I.4.2, the following square also commutes :

$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{id}_{F(X)}"] \arrow[d, "F(\eta(X))"'] & F(X) \arrow[d, "\mathrm{id}_{F(X)}"] \\
F(G(F(X))) \arrow[r, "\alpha(G(F(X)),F(X))^{-1}(\mathrm{id}_{G(F(X))})"'] & F(X)
\end{tikzcd}$$

As $\varepsilon(Y) = \alpha(G(Y), Y)^{-1}(\mathrm{id}_{G(Y)})$ for every $Y \in \operatorname{Ob}(\mathscr{C}')$, we get the first statement.

The proof of the second statement is similar. (In fact, the second statement is the first for the opposite categories.)

□

In fact, the unit and counit of an adjunction determine the isomorphism of functors $\operatorname{Hom}_{\mathscr{C}'}(F(\cdot),\cdot) \simeq \operatorname{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$. We start with the following lemma.

Lemma I.4.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be categories and $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}$ be functors.*

(i). Let $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ be a morphism of functors. For $X \in \operatorname{Ob}(\mathscr{C})$ and $Y \in \operatorname{Ob}(\mathscr{C}')$, we define a map

$$\Phi(X,Y) : \operatorname{Hom}_{\mathscr{C}'}(F(X), Y) \to \operatorname{Hom}_{\mathscr{C}}(X, G(Y)),\ f \longmapsto G(f) \circ \eta(X).$$

Then the family $(\Phi(X,Y))_{X \in \operatorname{Ob}(\mathscr{C}), Y \in \operatorname{Ob}(\mathscr{C}')}$ is a morphism of functors from $\operatorname{Hom}_{\mathscr{C}'}(F(\cdot),\cdot)$ to $\operatorname{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$.

(ii). Let $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{C}'}$ be a morphism of functors. For $X \in \operatorname{Ob}(\mathscr{C})$ and $Y \in \operatorname{Ob}(\mathscr{C}')$, we define a map

$$\Psi(X,Y) : \operatorname{Hom}_{\mathscr{C}}(X, G(Y)) \to \operatorname{Hom}_{\mathscr{C}'}(F(X), Y),\ g \longmapsto \varepsilon(Y) \circ F(g).$$

Then the family $(\Psi(X,Y))_{X \in \operatorname{Ob}(\mathscr{C}), Y \in \operatorname{Ob}(\mathscr{C}')}$ is a morphism of functors from $\operatorname{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$ to $\operatorname{Hom}_{\mathscr{C}'}(F(\cdot),\cdot)$.

Proof. We only prove (i) (the proof of (ii) is similar). Note that the functors $\operatorname{Hom}_{\mathscr{C}'}(F(\cdot),\cdot)$ and $\operatorname{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$ that we are considering are functors from $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}'$ to $\mathbf{Set}$. So let $u : X_2 \to X_1$

be a morphism of $\mathscr{C}$ (that is, u is a morphism from X_1 to X_2 in $\mathscr{C}^{\mathrm{op}}$) and $v : Y_1 \to Y_2$ be a morphism of $\mathscr{C}'$. We want to show that the following square is commutative :

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1) \arrow[r, "\Phi(X_1,Y_1)"] \arrow[d, "F(u)^* \circ v_*"'] & \mathrm{Hom}_{\mathscr{C}}(X_1, G(Y_1)) \arrow[d, "u^* \circ G(v)_*"] \\
\mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2) \arrow[r, "\Phi(X_2,Y_2)"'] & \mathrm{Hom}_{\mathscr{C}}(X_2, G(Y_2))
\end{tikzcd}$$

Let $f \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$. Unwrapping the definitions, we see that $u^* \circ G(v)_* \circ \Phi(X_1, Y_1)(f)$ (resp. $\Phi(X_2, Y_2) \circ F(u)^* \circ v_*(f)$) is the composition of the horizontal morphisms in the first (resp. second) row of the following diagram :

$$\begin{tikzcd}
X_2 \arrow[r, "u"] \arrow[d, equal] & X_1 \arrow[r, "\eta(X_1)"] \arrow[d, phantom, "(***)"] & G \circ F(X_1) \arrow[r, "G(f)"] \arrow[d, equal] & G(Y_1) \arrow[r, "G(v)"] \arrow[d, equal] & G(Y_2) \arrow[d, equal] \\
X_2 \arrow[r, "\eta(X_2)"'] & G \circ F(X_2) \arrow[r, "G \circ F(u)"'] & G \circ F(X_1) \arrow[r, "G(f)"'] & G(Y_1) \arrow[r, "G(v)"'] & G(Y_2)
\end{tikzcd}$$

But rectangle (***) in this diagram commutes because η is a morphism of functors, so the whole diagram commutes, and so $u^* \circ G(v)_* \circ \Phi(X_1, Y_1)(f) = \Phi(X_2, Y_2) \circ F(u)^* \circ v_*(f)$, which is what we wanted.

□

We could also have deduced the lemma from Lemma I.4.2.

Proposition I.4.6. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories, and let $F : \mathscr{C} \to \mathscr{C}'$ and $G : \mathscr{C}' \to \mathscr{C}$ be functors. Suppose that we are given morphisms of functors $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{C}'}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ such that the compositions*

$$F \overset{F(\eta)}{\to} F \circ G \circ F \overset{\varepsilon(F)}{\to} F$$

and

$$G \overset{\eta(G)}{\to} G \circ F \circ G \overset{G(\varepsilon)}{\to} G$$

are equal to id_F and id_G respectively.

Then the morphisms of functors $\Phi : \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot) \to \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$ and $\Psi : \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot)) \to \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot)$ are isomorphisms and inverses of each other. In particular, (F, G) is a pair of adjoint functors.

Proof. Let $f \in \mathrm{Hom}_{\mathscr{C}'}(F(X), Y)$. The morphism $f' := \Psi(X, Y)(\Phi(X, Y)(f))$ is the composition of the horizontal morphisms in the first row of the following diagram :

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta(X))"] & F \circ G \circ F(X) \arrow[r, "F \circ G(f)"] \arrow[dr, "\varepsilon(F(X))"'] & F \circ G(Y) \arrow[r, "\varepsilon(Y)"] \arrow[d, phantom, "(*)"] & Y \\
& & F(X) \arrow[ur, "f"'] &
\end{tikzcd}$$

As ε is a morphism of functors, the quadrilateral (*) in this diagram commutes, so $f' = f \circ \varepsilon(F(X)) \circ F(\eta(X))$. But $\varepsilon(F(X)) \circ F(\eta(X)) = \mathrm{id}_{F(X)}$ by assumption, so $f' = f$.

Let $g \in \mathrm{Hom}_{\mathscr{C}}(X, G(Y))$. The morphism $g' := \Phi(X,Y)(\Psi(X,Y)(g))$ is the composition of the horizontal morphisms in the first row of the following diagram :

$$\begin{tikzcd} X \arrow[r, "\eta(X)"] \arrow[dr, "g"'] & G \circ F(X) \arrow[r, "G\circ F(g)"] \arrow[d, phantom, "(**)"] & G \circ F \circ G(Y) \arrow[r, "G(\varepsilon(Y))"] & G(Y) \\ & G(Y) \arrow[ur, "\eta(G(Y))"'] & & \end{tikzcd}$$

As η is a morphism of functors, the quadrilateral (**) in this diagram commutes, so $g' = G(\varepsilon(Y)) \circ \eta(G(Y)) \circ g$. But $G(\varepsilon(Y)) \circ \eta(G(Y)) = \mathrm{id}_{G(Y)}$ by assumption, so $g' = g$.

□

Proposition I.4.7. *(i). Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor. Then F has a right adjoint if and only if, for every $Y \in \mathrm{Ob}(\mathscr{C}')$, the functor $\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable.*

(ii). Let $G : \mathscr{C}' \to \mathscr{C}$ be a functor. Then G has a left adjoint if and only if, for every $X \in \mathrm{Ob}(\mathscr{C})$, the functor $\mathrm{Hom}_{\mathscr{C}}(X, G(\cdot)) : \mathscr{C}' \to \mathbf{Set}$ is representable.

Proof. We prove (i) (the proof of (ii) is similar). If F has a right adjoint G, then, for every $Y \in \mathrm{Ob}(\mathscr{C}')$, the functor $\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable by $G(Y)$.

Conversely, suppose that the functor $\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable for every $Y \in \mathrm{Ob}(\mathscr{C}')$. For $Y \in \mathrm{Ob}(\mathscr{C}')$, we denote by $(G(Y), \eta(Y))$ a couple representing the functor $\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y)$, where $\eta(Y) \in \mathrm{Hom}_{\mathscr{C}'}(F(G(Y)), Y)$. Let $g : Y_1 \to Y_2$ be a morphism of $\mathscr{C}'$. Then we get a morphism of functors $\mathrm{Hom}_{\mathscr{C}}(\cdot, G(Y_1)) \simeq \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y_1) \stackrel{g_*}{\to} \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y_2) \simeq \mathrm{Hom}_{\mathscr{C}}(\cdot, G(Y_2))$, which comes by Yoneda's lemma from a morphism $G(g) : G(Y_1) \to G(Y_2)$ such that the following diagram commutes :

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y_1) \arrow[r, "g_*"] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), Y_2) \arrow[d, "\wr"'] \\ \mathrm{Hom}_{\mathscr{C}}(\cdot, G(Y_1)) \arrow[r, "G(g)_*"'] & \mathrm{Hom}_{\mathscr{C}}(\cdot, G(Y_2)) \end{tikzcd}$$

It also follows from Yoneda's lemma that the assignment $g \longmapsto G(g)$ respects the composition law and sends identity morphisms to identity morphisms. In other words, G is a functor, and η is a morphism of functors from $F \circ G$ to $\mathrm{id}_{\mathscr{C}'}$ such that the map $\Phi(X,Y) : \mathrm{Hom}_{\mathscr{C}'}(F(X), Y) \to \mathrm{Hom}_{\mathscr{C}}(X, G(Y))$ sending $f : F(X) \to Y$ to $G(f) \circ \eta(X)$ is bijective for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $Y \in \mathrm{Ob}(\mathscr{C}')$. By Lemma I.4.5, the maps $\Phi(X,Y)$ define an isomorphism of functors from $\mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot)$ to $\mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$. This shows that (F, G) is a pair of adjoint functors.

□

Example I.4.8. (1) The "free monoid (resp. group, abelian group, R-module, R-algebra, commutative R-algebra, R-Lie algebra) on a set" functor is left adjoint to the forgetful functor from $\mathbf{Mon}$ (resp. $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Mod}_R$ or ${}_R\mathbf{Mod}$, $R - \mathbf{Alg}$, $R - \mathbf{CAlg}$, $R - \mathbf{Lie}$) to $\mathbf{Set}$.

(2) Let R be a commutative ring. The "universal envelopping algebra" functor is left adjoint to the functor $L : R - \mathbf{Alg} \to R - \mathbf{Lie}$ that sends a R-algebra A to the Lie algebra on A given by the commutator bracket.

(3) Let X be a set. Then the functor $\mathrm{Hom}_{\mathbf{Set}}(X, \cdot) : \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $(.) \times X$. This is a complicated way to say that, if Y and Z are two other sets, then we have a bijection
$$\mathrm{Hom}_{\mathbf{Set}}(Y, \mathrm{Hom}_{\mathbf{Set}}(X, Z)) \simeq \mathrm{Hom}_{\mathbf{Set}}(Y \times X, Z)$$
that is functorial in Y and in Z (and actually also in X).

(4) Let R be a commutative ring and M be a R-module. Then the functor $\mathrm{Hom}_R(M, \cdot) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right adjoint to the functor $(\cdot) \otimes_R M$. In other words, if N and P are R-modules, we have a bijection
$$\mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P)) \simeq \mathrm{Hom}_R(N \otimes_R M, P)$$
that is functorial in N and in P (and actually also in M).

(5) Let $\alpha : R \to S$ be a morphism of commutative rings. Then we get a "restriction of scalars" functor $\alpha^* : \mathbf{Mod}_S \to \mathbf{Mod}_R$, and it has a left adjoint $(\cdot) \otimes_R S$ called "extension of scalars".

(6) The abelianization functor from $\mathbf{Grp}$ to $\mathbf{Ab}$ is left adjoint to the forgetful functor from $\mathbf{Ab}$ to $\mathbf{Grp}$.

(7) Let X be a topological space, and let $\mathrm{PSh}(X)$ and $\mathrm{Sh}(X)$ be the categories of presheaves and sheaves on X. Then the forgetful functor $\mathrm{PSh}(X) \to \mathrm{Sh}(X)$ has a left adjoint, called the sheafification functor.

(8) Let $\mathrm{For} : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. Then For has both left and right adjoints. The left (resp. right) adjoint of For is the functor $\mathbf{Set} \to \mathbf{Top}$ that sends a set X to itself with the discrete topology (resp. with the indiscrete or coarse topology).

Example I.4.9. Let R be a commutative ring and M be a R-module. Consider the pair of adjoint functors (F, G) given by $F = (\cdot) \otimes_R M$ and $G = \mathrm{Hom}_R(M, \cdot)$. (See Example I.4.8(4).) For every R-module N, the unit $\varepsilon(N) : N \to G(F(N))$ is the R-linear morphism $N \to \mathrm{Hom}_R(M, N \otimes_R M)$ sending $n \in N$ to the R-linear map $m \longmapsto n \otimes m$, and the counit $\eta(M) : F(G(N)) \to N$ is the R-linear morphism $\mathrm{Hom}_R(M, N) \otimes_R M \to N$ corresponding to the R-bilinear map $\mathrm{Hom}_R(M, N) \times M \to N$, $(f, m) \longmapsto f(m)$. The statement of Proposition I.4.4 is easy to check in this case.

As another example of an adjoint functor, we construct the free category on a directed graph.

Definition I.4.10. A *directed graph* $\mathscr{C}$ is the data of a set of *vertices* $\mathrm{Ob}(\mathscr{C})$ and, for any $X, Y \in \mathrm{Ob}(\mathscr{C})$, of a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of *edges* from X to Y. If e is an edge from X to Y, we call X (resp. Y) the *source* (resp. *target*) of e.

A *morphism of directed graphs* $F : \mathscr{C} \to \mathscr{D}$ is the data of a map $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, of a map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$.

We denote by $\mathcal{DG}$ the category of directed graphs.

We have made that definition resemble that of a category as much as possible. In fact, with our notation, a directed graph is just a category without a notion of composition. There is an obvious forgetful functor $\mathrm{For} : \mathbf{Cat} \to \mathcal{DG}$.

Definition I.4.11. Let $\mathscr{C}$ be a directed graph. The *free category* (or *path category*) on $\mathscr{C}$ is the category $\mathscr{P}\mathscr{C}$ defined by:

(a) $\mathrm{Ob}(\mathscr{P}\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$.

(b) For all $X, Y \in \mathrm{Ob}(\mathscr{C})$, the set $\mathrm{Hom}_{\mathscr{P}\mathscr{C}}(X, Y)$ is the set of finite sequences $(X; e_1, \dots, e_n; Y)$, where either $n = 0$ and $X = Y$, or $n \geq 1$, the source of e_1 is X, the target of e_n is Y, and the target of e_i is equal to the source of e_{i+1} for $1 \leq i \leq n - 1$.

(c) If $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, and if $(X; e_1, \dots, e_n; Y)$ and $(Y; f_1, \dots, f_m; Z)$ are morphisms of $\mathscr{P}\mathscr{C}$, then their composition is $(X; e_1, \dots, e_n, f_1, \dots, f_m; Z)$.

Note that the identity morphism of $X \in \mathrm{Ob}(\mathscr{P}\mathscr{C})$ is the length 0 sequence $(X; \varnothing; X)$.

If $F : \mathscr{C} \to \mathscr{C}'$ is a morphism of directed graphs, we define a functor $\mathscr{P}F : \mathscr{P}\mathscr{C} \to \mathscr{P}\mathscr{C}'$ by taking $\mathscr{P}F = F$ on objects, and, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$ and every $u = (X; e_1, \dots, e_n; Y) \in \mathrm{Hom}_{\mathscr{P}\mathscr{C}}(X, Y)$, setting $\mathscr{P}F(u) = (F(X); F(e_1), \dots, F(e_n); F(Y))$. (This clearly respects identity morphisms and the composition law.)

Proposition I.4.12. *The functor $\mathscr{P} : \mathcal{DG} \to \mathbf{Cat}$ is left adjoint to the forgetful functor* $\mathrm{For} : \mathbf{Cat} \to \mathcal{DG}$.

Proof. Let $\mathscr{C}$ be a directed graph and $\mathscr{D}$ be a category. We have an obvious morphism of directed graphs $\eta(\mathscr{C}) : \mathscr{C} \to \mathscr{P}\mathscr{C}$, that is the identity on objects and sends an edge $e \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ to the morphism $(X; e; Y) \in \mathrm{Hom}_{\mathscr{P}\mathscr{C}}(X, Y)$. Composition on the right with $\eta(\mathscr{C})$ induces a map $\alpha : \mathrm{Hom}_{\mathbf{Cat}}(\mathscr{P}\mathscr{C}, \mathscr{D}) \to \mathrm{Hom}_{\mathcal{DG}}(\mathscr{C}, \mathscr{D})$, and we want to show that this map is a bijection.

Let $F_1, F_2 : \mathscr{P}\mathscr{C} \to \mathscr{D}$ be two functors such that $\alpha(F_1) = \alpha(F_2) = F$. Then F_1 and F_2 are equal on objects by definition of α. Let $X, Y \in \mathrm{Ob}(\mathscr{P}\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$, and let $u = (X; e_1, \dots, e_n; Y) \in \mathrm{Hom}_{\mathscr{P}\mathscr{C}}(X, Y)$. If $n = 0$, then $X = Y$ and $u = \mathrm{id}_X$, so $F_1(u) = F_2(u) = \mathrm{id}_{F(X)}$. If $n \geq 1$, then $u = u_n \circ \ldots \circ u_1$, where, for $i \in \{1, \dots, n\}$, u_i is the morphism $(X_i; e_i; X_{i+1})$ from the source X_i to the target X_{i+1} of e_i; note that $X_1 = X$ and $X_{n+1} = Y$. For every i, we have $F_1(u_i) = F_2(u_i) = F(e_i)$. So $F_1(u) = F_2(u)$. This shows that $F_1 = F_2$, hence that α is injective.

We show that α is surjective. Let $F : \mathscr{C} \to \mathscr{D}$ be a morphism of directed graphs. We want to extend it to a functor $G : \mathscr{P}\mathscr{C} \to \mathscr{D}$. We take G equal to F on objects. Let $X, Y \in \mathrm{Ob}(\mathscr{P}\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$, and let $u = (X; e_1, \ldots, e_n; Y) \in \mathrm{Hom}_{\mathscr{P}\mathscr{C}}(X, Y)$. If $n = 0$, then $X = Y$ and $u = \mathrm{id}_X$, and we take $G(u) = \mathrm{id}_{F(X)}$. If $n \geq 1$, then we take $G(u) = (F(X); F(e_1), \ldots, F(e_n); F(Y))$. By definition of the composition law in $\mathscr{P}\mathscr{C}$, this does define a functor $G : \mathscr{P}\mathscr{C} \to \mathscr{D}$, and it is obvious that $G \circ \eta(\mathscr{C}) = \mathscr{F}$.

□

I.5 Limits

I.5.1 Definition and first properties

Definition I.5.1.1. Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. A *cone over* F is a couple $(X, (u_i)_{i \in \mathrm{Ob}(\mathscr{I})})$, where $X \in \mathrm{Ob}(\mathscr{C})$ and, for every $i \in \mathrm{Ob}(\mathscr{I})$, $u_i : X \to F(i)$ is a morphism of $\mathscr{C}$, such that, for every morphism $\alpha : i \to j$ of $\mathscr{I}$, the following diagram is commutative :

$$\begin{tikzcd} & X \arrow[dl, "u_i"'] \arrow[dr, "u_j"] & \\ F(i) \arrow[rr, "F(\alpha)"'] & & F(j) \end{tikzcd}$$

The object X is called the *apex* of the cone and the morphisms u_i are its *legs*. A *morphism of cones* from $(X, (u_i)_{i \in \mathrm{Ob}(\mathscr{I})})$ to $(Y, (v_i)_{i \in \mathrm{Ob}(\mathscr{I})})$ is a morphism $f : X \to Y$ of $\mathscr{C}$ such that, for every $i \in \mathrm{Ob}(\mathscr{I})$, the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u_i"'] & Y \arrow[dl, "v_i"] \\ F(i) & \end{tikzcd}$$

We can compose morphisms of cones in the obvious way, so the cones over F form a category.

A *limit* of F is a final object in the category of cones over F. If such a limit exists, its apex is denoted by $\varprojlim F$, or $\varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$. We say that the limit is indexed by the category $\mathscr{I}$.

Limits are also called *inverse limits* or *projective limits*. .

Definition I.5.1.2. Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. A *cone under* F (or a *cocone over* F) is a couple $(X, (u_i)_{i \in \mathrm{Ob}(\mathscr{I})})$, where $X \in \mathrm{Ob}(\mathscr{C})$ and, for every $i \in \mathrm{Ob}(\mathscr{I})$, $u_i : F(i) \to X$ is a morphism of $\mathscr{C}$, such that, for every morphism $\alpha : i \to j$ of $\mathscr{I}$, the following diagram is commutative :

$$\begin{tikzcd} F(i) \arrow[rr, "F(\alpha)"] \arrow[dr, "u_i"'] & & F(j) \arrow[dl, "u_j"] \\ & X & \end{tikzcd}$$

The object X is called the *nadir* of the cone and the morphisms u_i are its *legs*. A *morphism of cones* from $(X, (u_i)_{i\in \mathrm{Ob}(\mathscr{I})})$ to $(Y, (v_i)_{i\in \mathrm{Ob}(\mathscr{I})})$ is a morphism $f : X \to Y$ of $\mathscr{C}$ such that, for every $i \in \mathrm{Ob}(\mathscr{I})$, the following diagram commutes :

$$\begin{tikzcd} F(i) \arrow[d, "u_i"'] \arrow[dr, "v_i"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We can compose morphisms of cones in the obvious way, so the cones under F form a category.

A *colimit* of F is a initial object in the category of cones under F. If such a limit exists, its nadir is denoted by $\varinjlim F$ or $\varinjlim_{\mathscr{I}} F(i)$. We say that the colimit is indexed by the category $\mathscr{I}$.

Colimits are also called *direct limits* or *inductive limits*. .

Example I.5.1.3. (1) Let $F : I \to \mathscr{C}$ be a functor, with I a discrete category (i.e. a set). This means that F is a family $(F(i))_{i\in I}$ of objects of $\mathscr{C}$ indexed by the set I. A limit of F is called a *product* of this family and denoted by $\prod_{i\in I} F(i)$, and a colimit of F is called a *coproduct* of this family and denoted by $\coprod_{i\in I} F(i)$.

A product of $(F(i))_{i\in I}$ is the data an object $\prod_{i\in I} F(i)$ of $\mathscr{C}$, together with *projection morphisms* $p_j : \prod_{i\in I} F(i) \to F(j)$ for every $j \in I$, such that, if X is an object of $\mathscr{C}$ and $p'_j : X \to F(j)$, $j \in J$, are morphisms, then there exists a unique morphism $f : X \to \prod_{i\in I} F(i)$ such that $p'_j = p_j \circ f$. In other words, we have a bijection, functorial in X :

$$\mathrm{Hom}_{\mathscr{C}}(X, \prod_{i\in I} F(i)) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X, F(i)).$$

Dually, a coproduct of $(F(i))_{i\in I}$ is the data an object $\coprod_{i\in I} F(i)$ of $\mathscr{C}$, together with *inclusion morphisms* $q_j : F(j) \to \prod_{i\in I} F(i)$ for every $j \in I$, such that, if X is an object of $\mathscr{C}$ and $p'_j : F(j) \to X$, $j \in J$, are morphisms, then there exists a unique morphism $g : \coprod_{i\in I} F(i) \to X$ such that $q'_j = g \circ p_j$. In other words, we have a bijection, functorial in X :

$$\mathrm{Hom}_{\mathscr{C}}(\coprod_{i\in I} F(i), X) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(F(i), X).$$

(2) Let $\mathscr{I}$ be the category that has two objects 0 and 1 and two non-identity maps from 0 to 1. A functor $F : \mathscr{I} \to \mathscr{C}$ is just the data of two morphisms $f, g : X \to Y$ of $\mathscr{C}$ with the same source and target. A limit (resp. colimit) of F is called a *kernel* or *equalizer* (resp.

cokernel or *coequalizer*) of the couple (f, g) and denoted by $\mathrm{Ker}(f, g)$ (resp. $\mathrm{Coker}(f, g)$).

Note that a cone over F is the data of an object Z of $\mathscr{C}$ and of two morphisms $w : Z \to X$ and $w' : Z \to Y$ such that $w' = f \circ w = g \circ w$; in particular, the morphism w' is determined by w. Hence a kernel of (f, g) is the data of an object $\mathrm{Ker}(f, g)$ of $\mathscr{C}$, together with a morphism $u : \mathrm{Ker}(f, g) \to X$ such that $f \circ u = g \circ u$, such that, for every object Z of $\mathscr{C}$ and every morphism $v : Z \to X$ such that $f \circ v = g \circ v$, there exists a unique $w : Z \to \mathrm{Ker}(f, g)$ such that $v = u \circ w$.

$$\begin{tikzcd} Z \arrow[d, dashed] \arrow[dr, "v"] & & \\ \mathrm{Ker}(f,g) \arrow[r, "u"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

In other words, we have an isomorphism, functorial in Z :

$$\mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f, g)) \xrightarrow{\sim} \mathrm{Ker}(\ \mathrm{Hom}_{\mathscr{C}}(Z, X) \underset{g_*}{\overset{f_*}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(Z, Y)) \ .$$

Dually, a cokernel of (f, g) is the data of an object $\mathrm{Coker}(f, g)$ of $\mathscr{C}$, together with a morphism $u : Y \to \mathrm{Coker}(f, g)$ such that $u \circ f = u \circ g$, such that, for every object Z of $\mathscr{C}$ and every morphism $v : Y \to Y$ such that $v \circ f = v \circ g$, there exists a unique $w : \mathrm{Coker}(f, g) \to Z$ such that $v = w \circ u$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "u"] \arrow[dr, "v"'] & \mathrm{Coker}(f,g) \arrow[d, dashed] \\ & & Z \end{tikzcd}$$

In other words, we have an isomorphism, functorial in Z :

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coker}(f, g), Z) \xrightarrow{\sim} \mathrm{Ker}(\ \mathrm{Hom}_{\mathscr{C}}(Y, Z) \underset{g^*}{\overset{f^*}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(X, Z)\ ).$$

If $\mathscr{C}$ has a zero object (see Definition I.2.1.9) and $f : X \to Y$ is a morphism of $\mathscr{C}$, then we write $\mathrm{Ker}(f) = \mathrm{Ker}(f, 0)$ and $\mathrm{Coker}(f) = \mathrm{Coker}(f, 0)$ and call these the kernel and cokernel of f.

(3) Suppose that the category $\mathscr{I}$ is empty (that is, $\mathrm{Ob}(\mathscr{I})$ is empty). Then there is a unique functor $F : \mathscr{I} \to \mathscr{C}$. A cone over or under F is just an object of $\mathscr{C}$, and a morphism of cones is a morphism in $\mathscr{C}$. So a limit of F is a final object of $\mathscr{C}$, and a colimit of F is an initial object of $\mathscr{C}$.

Functoriality

Let $F, G : \mathscr{I} \to \mathscr{C}$ be two functors, and let $\phi : F \to G$ be a morphism of functors. Then, if $(X, (u_i))$ is a cone over F, the pair $(X, (\phi(i) \circ u_i))$ is a cone over G. If F and G both have limits, this implies that we have a unique morphism $\varprojlim F \to \varprojlim G$ of cones over G; we denote this morphism by $\varprojlim \phi$. If $\phi = \mathrm{id}_F$, then clearly $\varprojlim \phi = \mathrm{id}_{\varprojlim F}$. Also, if $\phi : F \to G$ and $\psi : G \to H$ are two morphisms of functors from $\mathscr{I}$ to $\mathscr{C}$ and F, G and H have limits, then $\varprojlim \psi \circ \varprojlim \phi : \varprojlim F \to \varprojlim H$ is a morphism of cones over H, so it has to be equal to $\varprojlim(\psi \circ \phi)$. In particular :

Proposition I.5.1.4. *If every functor $F : \mathscr{I} \to \mathscr{C}$ has a limit, then the construction above defines a functor $\varprojlim : \mathrm{Func}(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$.*

Similarly, we prove :

Proposition I.5.1.5. *If every functor $F : \mathscr{I} \to \mathscr{C}$ has a colimit, then we have a functor $\varinjlim : \mathrm{Func}(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$ such that, for every morphism $\phi : F \to G$ in $\mathrm{Func}(\mathscr{I}, \mathscr{C})$, the morphism $\varprojlim \phi$ is the unique morphism of cones under F from $\varprojlim F$ to $\varprojlim G$.*

I.5.2 The case of Set

This is one situation where we have to be careful about set-theoretical issues.[4] The upshot is that all limits and colimits exist in then category of sets, if the catgeory $\mathscr{I}$ indexing them is not too big. If we work with catgeories whose objects can form a proper class, this means that $\mathrm{Ob}(\mathscr{I})$ must be a set. If we work with universes, as we do here, it means that we have the following result :

Theorem I.5.2.1. *Let $\mathscr{U}$ be a universe, and let $\mathbf{Set} = \mathbf{Set}_{\mathscr{U}}$ (so the objects of $\mathbf{Set}$ are the sets $X \in \mathscr{U}$). Let $F : \mathscr{I} \to \mathbf{Set}$ be a functor, and assume that the category $\mathscr{I}$ is $\mathscr{U}$-small, i.e. that $\mathrm{Ob}(\mathscr{I}) \in \mathscr{U}$. Then F has a limit and a colimit in $\mathbf{Set}$.*

More precisely, let

$$\varprojlim F = \{(x_i) \in \prod_{i \in \mathrm{Ob}(\mathscr{I})} F(i) \mid \forall \alpha : i \to j,\ F(\alpha)(x_i) = x_j\},$$

and, for each $i \in \mathrm{Ob}(\mathscr{I})$, let $p_i : \varprojlim F \to F(i)$ be the restriction of the canonical projection $\prod_{i' \in \mathrm{Ob}(\mathscr{I})} F(i') \to F(i)$. Then $(\varprojlim F, (p_i))$ is a limit of F.

On the other hand, let $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$ be the disjoint union of the family $(F(i))_{i \in \mathrm{Ob}(\mathscr{I})}$. We write elements of this disjoint union as pairs (i, x), where $i \in \mathrm{Ob}(\mathscr{I})$ and $x \in F(i)$. consider the relation R on $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$ given by $(i, x)R(j, y)$ if there exists a morphism $\alpha : i \to j$ in

[4] But you can ignore this in first approximation. See Remark I.5.2.5 for an example of what can go wrong.

$\mathscr{I}$ such that $F(\alpha)(x) = y$. Let $\sim$ be the equivalence relation generated by R; in other words, we have $(i,x) \sim (j,y)$ if there exists a finite sequence $(i_0,x_0),\dots,(i_{2n},x_{2n})$ of elements of $\coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$ such that $(i,x) = (i_0,x_0)$, $(i_{2n},x_{2n}) = (j,y)$, and, for every $m \in \{1,\dots,n\}$, $(i_{2m-2},x_{2m-2})R(i_{2m-1},x_{2m-1})$ and $(i_{2m-1},x_{2m-1})R(i_{2m}x_{2m})$.

$$\begin{tikzcd}[column sep=small]
 & (i_1,x_1) \arrow[dl, "R"'] \arrow[dr, "R"] & & \cdots \arrow[dl, "R"'] \arrow[dr, "R"] & & (i_{2n-1},x_{2n-1}) \arrow[dl, "R"'] \arrow[dr, "R"] & \\
(i,x)=(i_0,x_0) & & (i_2,x_2) & & \cdots & & (i_{2n},x_{2n})=(j,y)
\end{tikzcd}$$

Let $\varinjlim F = \coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)/\sim$, and, for every $i \in \mathrm{Ob}(\mathscr{I})$, let $q_i : F(i) \to \varinjlim F$ be the composition of the map $F(i) \to \coprod_{i'\in\mathrm{Ob}(\mathscr{I})} F(i')$, $x \longmapsto (i,x)$ and of the canonical projection from $\coprod_{i'\in\mathrm{Ob}(\mathscr{I})} F(i')$ to $\varinjlim F$. Then $(\varinjlim F,(q_i))$ is a colimit of F.

In other words, the category $\mathbf{Set}_{\mathscr{U}}$ has all limits and colimits indexed by $\mathscr{U}$-small categories.

Proof. Let $(X,(u_i))$ be a cone over F. Then the family (u_i) defines a map $g : X \to \prod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$, and the condition that $F(\alpha)\circ u_i = u_j$ for every morphism $\alpha : i \to j$ of $\mathscr{I}$ says exactly that this map factors through the subset $\varprojlim F$ of $\prod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$, hence induces a morphism of cones $f : X \to \varprojlim F$. Conversely, let $f' : X \to \varprojlim F$ be a morphism of cones over F, and let $g : X \to \prod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$ be its composition with the injection of $\varprojlim F$ in $\prod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$. Then, for every $i \in \mathrm{Ob}(\mathscr{I})$, we have $p_i \circ g = g_i$, so the ith component of g is u_i; this implies that $g' = g$, hence that $f' = f$.

Now let $(X,(u_i))$ be a cone under F. We consider the map $g : \coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i) \to X$ defined by $g(i,x) = u_i(x)$, for $i \in \mathrm{Ob}(\mathscr{I})$ and $x \in F(i)$. The condition that $u_i = u_j \circ F(\alpha)$ for every morphism $\alpha : i \to j$ of $\mathscr{I}$ implies that $g(i,x) = (j,y)$ if $(i,x)R(j,y)$, hence that $g(i,x) = (j,y)$ if $(i,x) \sim (j,y)$. So g factors through the canonical quotient map $\coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i) \to \varinjlim F$ and we get a map $f : \varinjlim F \to X$, which is clearly a morphism of cones under F. Conversely, let $f' \varinjlim F \to X$ be a morphism of cones under F, and let g' be its composition with the canonical quotient map $\coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i) \to \varinjlim F$. Then, for every $i \in \mathrm{Ob}(\mathscr{I})$, we have $f \circ q_i = u_i$, hence $g(i,x) = u_i(x)$ for every $x \in F(i)$. This implies that $g = g'$, hence that $f = f'$.

□

Example I.5.2.2. (1) Let $I \in \mathscr{U}$ be a discrete category, i.e. a set. A functor from I to $\mathbf{Set}$ is just a family $(X_i)_{i\in I}$ of sets indexed by I. The limit of such a functor is the product of this family, and its colimit is the coproduct (or disjoint union) of this family.

(2) Let X and Y be sets, and $f, g : X \to Y$ be maps. Then

$$\mathrm{Ker}(f,g) = \{x \in X \mid f(x) = g(x)\}$$

and $\mathrm{Coker}(f,g)$ is the quotient of Y by the equivalence relation generated by the relation R defined by $f(x)Rg(x)$, for every $x \in X$.

Example I.5.2.3. Let p be a prime number. Consider the set $\mathbb{N}$ with its usual order as a category, and let $F : \mathbb{N}^{\mathrm{op}} \to \mathbf{Set}$ be the functor that sends $n \in \mathbb{N}$ to $\mathbb{Z}/p^n\mathbb{Z}$ and that, for $m \leq n$, sends the unique morphism $\alpha : n \to m$ to the canonical projection $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$. Then $\varprojlim F = \mathbb{Z}_p$. We also denote this by $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ (or just $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$).

Remark I.5.2.4. Let $F : \mathscr{I} \to \mathbf{Set}$ be a functor, with $\mathscr{I}$ a $\mathscr{U}$-small category. For every morphism $\alpha : i \to j$ in $\mathscr{I}$, we write $i = \sigma(\alpha)$ and $j = \tau(\alpha)$. Let $\mathrm{Mor}(\mathscr{I}) = \coprod_{i,j\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{I}}(i, j)$.

We define two maps $f, g : \prod_{i\in\mathrm{Ob}(\mathscr{I})} F(i) \to \prod_{\alpha\in\mathrm{Mor}(\mathscr{I})} F(\tau(\alpha))$ in the following way :

- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, the composition of f with the projection on the α component of $\prod_{\alpha\in\mathrm{Mor}(\mathscr{I})} F(\tau(\alpha))$ is the map $(x_i) \longmapsto x_{\tau(\alpha)}$;
- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, the composition of g with the projection on the α component of $\prod_{\alpha\in\mathrm{Mor}(\mathscr{I})} F(\tau(\alpha))$ is the map $(x_i) \longmapsto F(\alpha)(x_{\sigma(\alpha)})$.

Then $\varprojlim F = \mathrm{Ker}(f, g)$.

Similarly, we define two maps $f, g : \coprod_{\alpha\in\mathrm{Mor}(\mathscr{I})} F(\sigma(\alpha)) \to \coprod_{i\in\mathrm{Ob}(\mathscr{I})} F(i)$ in the following way :

- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, if $x \in F(\sigma(\alpha))$, then $f(\alpha, x) = (\sigma(\alpha), x)$;
- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, if $x \in F(\sigma(\alpha))$, then $g(\alpha, x) = (\tau(\alpha), F(\alpha)(x))$.

Then $\varinjlim_F F = \mathrm{Coker}(f, g)$.

In other words, we know how to calculate limits and colimits in $\mathbf{Set}$ as soon as we know how to calculate kernels and cokernels. This generalizes to arbitrary categories, as long as all the limits exist.

Remark I.5.2.5. Here is an example of what can go wrong if we don't bound the size of the indexing category. Let $\mathscr{U}$ be a universe as before, and let $I = \coprod_{X,Y\in\mathrm{Ob}(\mathbf{Set})} \mathrm{Hom}_{\mathbf{Set}}(X, Y)$. Note that $I \notin \mathscr{U}$ (for example because $\mathrm{card}(I) = \mathrm{card}(\mathscr{U})$). Let $X \in \mathscr{U}$ be a set with two elements. Then the direct product X^I has cardinality 2^I, which is strictly bigger than $\mathrm{card}(I) = \mathrm{card}(\mathscr{U})$, so it is not an object of $\mathbf{Set}_{\mathscr{U}}$. So the limit of the functor $F : I \to \mathbf{Set}_{\mathscr{U}}, i \longmapsto X$ does not exist.

In what follows, we fix a universe $\mathscr{U}$, we take $\mathbf{Set} = \mathbf{Set}_{\mathscr{U}}$, we assume that all categories are $\mathscr{U}$-categories and, if we take a limit or colimit in $\mathbf{Set}$, we assume that the indexing category is $\mathscr{U}$-small.

I.5.3 Presheaves and limits

The proof of the following result is straightforward from the definition of a limit and a colimit.

Proposition I.5.3.1. *Let $\mathscr{I}$ and $\mathscr{D}$ be categories.*

(i) Assume that all functors $\mathscr{I} \to \mathscr{D}$ have limits. Then, for every category $\mathscr{C}$, all functors $\mathscr{I} \to \mathrm{Func}(\mathscr{C}, \mathscr{D})$ have limits, and, if $F : \mathscr{I} \to \mathrm{Func}(\mathscr{C}, \mathscr{D})$ is a functor, then $\varprojlim F$ is

the functor from $\mathscr{C}$ to $\mathscr{D}$ sending $X \in \mathrm{Ob}(\mathscr{C})$ to $\varprojlim_{i\in\mathrm{Ob}(\mathscr{I})} F(i)(X)$ and a morphism f of $\mathscr{C}$ to $\varprojlim_{i\in\mathrm{Ob}(\mathscr{I})} F(i)(f)$.

(ii) *Assume that all functors $\mathscr{I} \to \mathscr{D}$ have colimits. Then, for every category $\mathscr{C}$, all functors $\mathscr{I} \to \mathrm{Func}(\mathscr{C},\mathscr{D})$ have colimits, and, if $F : \mathscr{I} \to \mathrm{Func}(\mathscr{C},\mathscr{D})$ is a functor, then $\varinjlim F$ is the functor from $\mathscr{C}$ to $\mathscr{D}$ sending $X \in \mathrm{Ob}(\mathscr{C})$ to $\varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} F(i)(X)$ and a morphism f of $\mathscr{C}$ to $\varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} F(i)(f)$.*

In particular, if $\mathscr{C}$ is a category, then the categories $\mathrm{PSh}(\mathscr{C})$ and $\mathrm{PSh}(\mathscr{C}^{\mathrm{op}})$ have all limits and colimits (index by $\mathscr{U}$-small categories).

We now explain how to see limits and colimits as objects representing functors from $\mathscr{C}$ or $\mathscr{C}^{\mathrm{op}}$ to $\mathbf{Set}$.

Let $\mathscr{I}$ be a ($\mathscr{U}$-small) category. Consider the functor $S\varprojlim : \mathrm{Func}(\mathscr{I},\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$ [5] sending $F : \mathscr{I} \to \mathscr{C}$ to the limit in $\mathrm{PSh}(\mathscr{C})$ of the functor $h_{\mathscr{C}} \circ F$. In other words :

(1) If $F : \mathscr{I} \to \mathscr{C}$ is a functor, then $S\varprojlim F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is the presheaf sending an object X of $\mathscr{C}$ to

$$\begin{aligned} S\varprojlim F(X) &= \{(u_i) \in \prod_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, F(i)) \mid \forall \alpha : i \to j,\ F(\alpha)\circ u_i = u_j\} \\ &= \varprojlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, F(i)) \\ &= \varprojlim \mathrm{Hom}_{\mathscr{C}}(X,\cdot)\circ F, \end{aligned}$$

and a morphism $f : X \to Y$ of $\mathscr{C}$ to the map $(u_i) \longmapsto (u_i \circ f)$.

(2) If $v : F \to G$ is a morphism of functors from $\mathscr{I}$ to $\mathscr{C}$, then, for every $X \in \mathrm{Ob}(\mathscr{C})$, $S\varprojlim v(X)$ is the restriction to $S\varprojlim F(X)$ of the map $(u_i) \longmapsto (v(i)\circ u_i) \in \prod_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, G(i))$. It is easy to check that, if the family (u_i) is in $S\varprojlim F(X)$, then the family $(v(i)\circ u_i)$ is in $S\varprojlim G(X)$.

Proposition I.5.3.2. *Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. Then F has a limit if and only if the presheaf $S\varprojlim F$ is representable, and a limit of F is the same as a pair representing this presheaf.*

Proof. Indeed, if the pair $(X,(u_i))$ represents $S\varprojlim F$, then it is a cone over F with apex X, and, for every cone $(Y,(v_i))$ over F with apex Y, i.e. for every $Y \in \mathrm{Ob}(\mathscr{C})$ and $(v_i) \in S\varprojlim F(Y)$, there is a unique morphism $f : Y \to X$ such that $(v_i) = S\varprojlim F(f)((u_i)) = (u_i \circ f)$, that is, there exists a unique morphism of cones from $(Y,(v_i))$ to $(X,(u_i))$. So $(X,(u_i))$ is a final object in the category of cones over F.

Conversely, suppose that $(X,(u_i))$ is a final object in the category of cones over F. Let $Y \in \mathrm{Ob}(\mathscr{C})$. Then, for every cone $(Y,(v_i))$ with apex Y over F, there exists a unique morphism

[5] "Stupid limit", for lack of a better name.

of cones from $(Y,(v_i))$ to $(X,(u_i))$, that is, there exists a unique morphism $f : Y \to X$ in $\mathscr{C}$ such that $v_i = u_i \circ f$ for every $i \in \mathrm{Ob}(\mathscr{I})$. In other words, the map $\mathrm{Hom}_{\mathscr{C}}(Y,X) \to S\varprojlim F(Y)$, $f \longmapsto (u_i \circ f)$ is bijective. This shows that the couple $(X,(u_i))$ represents the presheaf $S\varprojlim F$.

$\square$

There is another way to state the result of Proposition I.5.3.2 (when $\mathscr{C}$ has all limits indexed by $\mathscr{I}$). Consider the functor $c : \mathscr{C} \to \mathrm{Func}(\mathscr{I},\mathscr{C})$ that sends $X \in \mathrm{Ob}(\mathscr{C})$ to the "constant" functor $c(X) : \mathscr{I} \to \mathscr{C}$ defined by $c(X)(i) = X$ for every $i \in \mathrm{Ob}(\mathscr{I})$ and $c(X)(\alpha) = \mathrm{id}_X$ for every morphism α of $\mathscr{I}$, and that sends a morphism $f : X \to Y$ of $\mathscr{C}$ to the morphism of functors $c(F) : c(X) \to c(Y)$ defined by $c(f)(i) = f$ for every $i \in \mathrm{Ob}(\mathscr{I})$.

If $F : \mathscr{I} \to \mathscr{C}$ is a functor, giving a morphism of $c(X) \to F$ is the same as giving a cone over F with apex X, or, as we have seen above, an element of $S\,\mathrm{proj}\lim F(X)$. So we get :

Corollary I.5.3.3. *If every functor $F : \mathscr{I} \to \mathscr{C}$ has a limit, then the functor $\varprojlim : \mathrm{Func}(\mathscr{C},\mathscr{I}) \to \mathscr{C}$ of Proposition I.5.1.4 is right adjoint to $c : \mathscr{C} \to \mathrm{Func}(\mathscr{C},\mathscr{I})$.*

In other words, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $F \in \mathrm{Func}(\mathscr{I},\mathscr{C})$, we have an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X,\varprojlim F) \simeq \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{C})}(c(X),F) = \varprojlim_{i\in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X,F(i)),$$

and this isomorphism is an isomorphism of functors from $\mathscr{C}^{\mathrm{op}} \times \mathrm{Func}(\mathscr{I},\mathscr{C})$ to $\mathbf{Set}$.

All these results have a dual version for colimits. We will give the statements, the proofs are similar.

We consider the functor $S\varinjlim : \mathrm{Func}(\mathscr{I},\mathscr{C}) \to \mathrm{PSh}(\mathscr{C}^{\mathrm{op}})$ sending $F : \mathscr{I} \to \mathscr{C}$ to the limit of $k_{\mathscr{C}} \circ F$ in $\mathrm{PSh}(\mathscr{C}^{\mathrm{op}})$. [6] In other words :

(1) If $F : \mathscr{I} \to \mathscr{C}$ is a functor, then $S\varinjlim F : \mathscr{C} \to \mathbf{Set}$ is the functor sending an object X of $\mathscr{C}$ to

$$\begin{aligned} S\varinjlim F(X) &= \{(u_i) \in \prod_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(F(i),X) \mid \forall \alpha : i \to j,\ u_i = u_j \circ F(\alpha)\} \\ &= \varprojlim_{i\in\mathrm{Ob}(\mathscr{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathscr{C}}(F(i),X) \\ &= \varprojlim \mathrm{Hom}_{\mathscr{C}}(\cdot,X)\circ F, \end{aligned}$$

and a morphism $f : X \to Y$ of $\mathscr{C}$ to the map $(u_i) \longmapsto (f \circ u_i)$.

(2) If $v : F \to G$ is a morphism of functors from $\mathscr{I}$ to $\mathscr{C}$, then, for every $X \in \mathrm{Ob}(\mathscr{C})$, $S\varinjlim v(X)$ is the restriction to $S\varinjlim F(X)$ of the map $(i,u) \longmapsto (u \circ v(i)) \in \prod_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(G(i),X)$.

[6] We are taking the limit and not the colimit, because the functor $k_{\mathscr{C}}$ is a contravariant functor from $\mathscr{C}$ to $\mathrm{PSh}(\mathscr{C}^{\mathrm{op}})$, so it reverses the direction of arrows.

Proposition I.5.3.4. *Let $F : \mathscr{I} \to \mathscr{C}$ be a functor. Then F has a colimit if and only if the functor $S \varinjlim F$ is representable, and a colimit of F is the same as a pair representing this functor.*

Corollary I.5.3.5. *If every functor $F : \mathscr{I} \to \mathscr{C}$ has a colimit, then the functor $\varinjlim : \mathrm{Func}(\mathscr{C}, \mathscr{I}) \to \mathscr{C}$ of Proposition I.5.1.5 is left adjoint to $c : \mathscr{C} \to \mathrm{Func}(\mathscr{C}, \mathscr{I})$.*

In other words, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $F \in \mathrm{Func}(\mathscr{I}, \mathscr{C})$, we have an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(\varinjlim F, X) \simeq \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{C})}(F, c(X)) = \varprojlim_{i \in \mathrm{Ob}(\mathscr{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathscr{C}}(F(i), X),$$

and this isomorphism of functors from $\mathrm{Func}(\mathscr{I}, \mathscr{C})^{\mathrm{op}} \times \mathscr{C}$ to $\mathbf{Set}$.

I.5.4 General properties of limits and colimits

I.5.4.1 Limits of limits

Let $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{C}$ be categories. We have an equivalence of categories $\mathrm{Func}(\mathscr{I} \times \mathscr{J}, \mathscr{C}) \xrightarrow{\sim} \mathrm{Func}(\mathscr{I}, \mathrm{Func}(\mathscr{J}, \mathscr{C}))$ sending a bifunctor $F : \mathscr{I} \times \mathscr{J} \to \mathscr{C}$ to the functor $F_{\mathscr{I}} : i \longmapsto (F(i, \cdot) : \mathscr{J} \to \mathscr{C})$ (its quasi-inverse sends a functor $G : \mathscr{I} \to \mathrm{Func}(\mathscr{J}, \mathscr{C})$ to the bifunctor $(i, j) \longmapsto G(i)(j)$). Similarly, we have an equivalence of categories $\mathrm{Func}(\mathscr{I} \times \mathscr{J}, \mathscr{C}) \xrightarrow{\sim} \mathrm{Func}(\mathscr{J}, \mathrm{Func}(\mathscr{I}, \mathscr{C}))$, which we denote by $F \longmapsto F_{\mathscr{J}}$.

Let $F : \mathscr{I} \times \mathscr{J} \to \mathscr{C}$. Then, if all the limits appearing in the formulas exist, we have isomorphisms

$$\varprojlim F \simeq \varprojlim \varprojlim F_{\mathscr{I}} \simeq \varprojlim \varprojlim F_{\mathscr{J}},$$

which we also write

$$\varprojlim_{(i,j) \in \mathrm{Ob}(\mathscr{I}) \times \mathrm{Ob}(\mathscr{J})} \simeq \varprojlim_{j \in \mathrm{Ob}(\mathscr{J})} \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} F(i,j) \simeq \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \varprojlim_{j \in \mathrm{Ob}(\mathscr{J})} F(i,j).$$

Similary, if all the colimits appearing in the formulas exist, we have isomorphisms

$$\varinjlim F \simeq \varinjlim \varinjlim F_{\mathscr{I}} \simeq \varinjlim \varinjlim F_{\mathscr{J}},$$

which we also write

$$\varinjlim_{(i,j) \in \mathrm{Ob}(\mathscr{I}) \times \mathrm{Ob}(\mathscr{J})} \simeq \varinjlim_{j \in \mathrm{Ob}(\mathscr{J})} \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} F(i,j) \simeq \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \varinjlim_{j \in \mathrm{Ob}(\mathscr{J})} F(i,j).$$

I.5.4.2 Limits and functors

Let $\mathscr{I}$, $\mathscr{C}$ and $\mathscr{C}'$ be categories, and let $F : \mathscr{C} \to \mathscr{C}'$ and $\alpha : \mathscr{I} \to \mathscr{C}$ be functors.

If the functors α and $F \circ \alpha$ have limits, then, by the definition of limits as terminal cones, we have a morphism of cones over $F \circ \alpha$:

$$F(\varprojlim \alpha) \to \varprojlim(F \circ \alpha).$$

Similarly, if the functors α and $F \circ \alpha$ have colimits, then, by the definition of colimits as initial cones, we have a morphism of cones under $F \circ \alpha$:

$$\varinjlim(F \circ \alpha) \to F(\varinjlim \alpha).$$

Definition I.5.4.1. Let $\mathscr{I}$ be a category and $F : \mathscr{C} \to \mathscr{C}'$ be a functor.

(i). Suppose that $\mathscr{C}$ and $\mathscr{C}'$ admits all limits indexed by $\mathscr{I}$. We say that F *commutes with limits indexed by* $\mathscr{I}$ if, for every $\alpha \in \mathrm{Func}(\mathscr{I}, \mathscr{C})$, the morphism $F(\varprojlim \alpha) \to \varprojlim(F \circ \alpha)$ defined above is an isomorphism.

We say that F commutes with products (resp. kernels, resp. finite limits, resp. finite products) if it commutes with all limits indexed by discrete categories (resp. the category $\mathscr{I}$ of Example I.5.1.3(2), resp. all finite categories, resp. all finite discrete categories).

(ii). Suppose that $\mathscr{C}$ and $\mathscr{C}'$ admits all colimits indexed by $\mathscr{I}$. We say that F *commutes with colimits indexed by* $\mathscr{I}$ if, for every $\alpha \in \mathrm{Func}(\mathscr{I}, \mathscr{C})$, the morphism $\varinjlim(F \circ \alpha) \to F(\varinjlim \alpha)$ defined above is an isomorphism.

We say that F commutes with coproducts (resp. cokernels, resp. finite colimits, resp. finite coproducts) if it commutes with all colimits indexed by discrete categories (resp. the category $\mathscr{I}$ of Example I.5.1.3(2), resp. all finite categories, resp. all finite discrete categories).

Example I.5.4.2. The Yoneda embedding $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ commutes with limits indexed by any ($\mathscr{U}$-small) category. On the other hand, it doesn't always commute with colimits. [7]

The main result about these notions is that right adjoint functors commute with limits and left adjoint functors commute with colimits.

Proposition I.5.4.3. *Let $\mathscr{I}$ be a category and $F : \mathscr{C} \to \mathscr{C}'$ be a functor.*

(i). Suppose that $\mathscr{C}$ and $\mathscr{C}'$ admits all limits indexed by $\mathscr{I}$. If F admits a left adjoint, then it commutes with all limits indexed by $\mathscr{I}$.

(ii). Suppose that $\mathscr{C}$ and $\mathscr{C}'$ admits all colimits indexed by $\mathscr{I}$. If F admits a right adjoint, then it commutes with all colimits indexed by $\mathscr{I}$.

Proof. We prove (i) (the proof of (ii) is similar). Let G be a left adjoint of F, and let $\alpha : \mathscr{I} \to \mathscr{C}$ be a functor. To check that the morphism $F(\varprojlim \alpha) \to \varprojlim(F \circ \alpha)$, it suffices to check that its

[7] See PS 2.

image by the Yoneda embedding $h_{\mathscr{C}'}$ is an isomorphism. So let $Y \in \mathrm{Ob}(\mathscr{C}')$. The morphism $\mathrm{Hom}_{\mathscr{C}'}(Y, F(\varprojlim \alpha)) \to \mathrm{Hom}_{\mathscr{C}'}(Y, \varprojlim(F \circ \alpha))$ is equal to the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}'}(Y, F(\varprojlim \alpha)) &\simeq \mathrm{Hom}_{\mathscr{C}}(G(Y), \varprojlim \alpha) \\
&\simeq \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(G(Y), \alpha(i)) \\
&\simeq \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(F(X), F(\alpha(i))) \\
&\simeq \mathrm{Hom}_{\mathscr{C}'}(Y, \varprojlim(F \circ \alpha)),
\end{aligned}$$

hence it is an isomorphism.

□

I.5.5 Limits and colimits in other categories

We explain some of the constructions of limits and colimits in other categories. As in the category of sets, if we can construct products and kernels (resp. coproducts and cokernels), then we can get all limits (resp. colimits).

I.5.5.1 Some common categories

Topological spaces

Limits If I is a set and $(X_i)_{i \in I}$ is a family of topological spaces, then the product $\prod_{i \in I} X_i$ in the category $\mathbf{Top}$ is the product of the X_i as sets with the product topology. In general, if $F : \mathscr{I} \to \mathbf{Top}$ is a functor, then $\varprojlim F$ is the subset $\{(x_i) \in \prod_{i \in \mathrm{Ob}(\mathscr{I})} F(i) \mid \forall \alpha : i \to j,\ F(\alpha)(x_i) = x_j\}$ of $\prod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$ with the subspace topology.

Colimits If I is a set and $(X_i)_{i \in I}$ is a family of topological spaces, then the coproduct $\coprod_{i \in I} X_i$ in the category $\mathbf{Top}$ is the disjoint union of the X_i (i.e. the coproduct in the category of sets), with the topology such that $U \subset \coprod_{i \in I} X_i$ is open if and only $U \cap X_i$ is open for every $i \in I$. In general, if $F : \mathscr{I} \to \mathbf{Top}$ is a functor, then $\varinjlim F$ is the same quotient of $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$ as in the construction of colimits of sets, with the quotient topology.

In particular, the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ commutes with all limits and all colimits.

Groups

Let $F : \mathscr{I} \to \mathbf{Grp}$ be a functor. Then $\prod_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$ has a natural group structure (multiplying entry by entry), and the subset $\{(x_i) \in \prod_{i\in \mathrm{Ob}(\mathscr{I})} F(i) \mid \forall \alpha : i \to j,\ F(\alpha)(x_i) = x_j\}$ of $\prod_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$ is a subgroup of this group. This is the limit of F.

Colimits also exist in $\mathbf{Grp}$, but they are harder to construct, and the forgetful functor $\mathbf{Grp} \to \mathbf{Top}$ does not commute with colimits. Coproducts in $\mathbf{Grp}$ are called free products. If $f, g : G \to H$ are two morphisms of groups, then $\mathrm{Coker}(f - g)$ is the quotient of H by the normal subgroup of H generated by the subset $\{f(x)g(x)^{-1},\ x \in G\}$.

R-modules

We look at the category ${}_R\mathbf{Mod}$, the case of $\mathbf{Mod}_R$ is similar. (If only because a right R-module is a left module over the opposite ring of R.)

Limits If I is a set and $(M_i)_{i\in I}$ is a family of left R-modules, then the product $\prod_{i\in I} M_i$ has a natural structure of left R-module (if $a \in R$ and $(m_i) \in \prod_{i\in I} M_i$, take $a \cdot (m_i) = (am_i)$), and it is the product in the category ${}_R\mathbf{Mod}$. In general, if $F : \mathscr{I} \to \mathbf{Top}$ is a functor, then the subset $\{(x_i) \in \prod_{i\in \mathrm{Ob}(\mathscr{I})} F(i) \mid \forall \alpha : i \to j,\ F(\alpha)(x_i) = x_j\}$ of $\prod_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$ is a R-submodule, and this is $\varprojlim F$.

Colimits If I is a set and $(M_i)_{i\in I}$ is a family of left R-modules, then the coproduct of $(M_i)_{i\in I}$ in ${}_R\mathbf{Mod}$ is the direct sum $\bigoplus_{i\in I} M_i$. (Indeed, the direct sum has the correct universal property.)

Let $F : \mathscr{I} \to \mathbf{Top}$ be a functor, let $M = \bigoplus_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$ and $M' = \bigoplus_{\alpha\in \mathrm{Mor}(\mathscr{I})} F(\sigma(\alpha))$ (we are using the notation of Remark I.5.2.4), and consider the two maps $f, g : M' \to M$ defined by :

- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, if $x \in F(\sigma(\alpha))$, then $f(x) = x$;
- for every $\alpha \in \mathrm{Mor}(\mathscr{I})$, if $x \in F(\sigma(\alpha))$, then $g(x) = F(\alpha)(x)$.

Then $\varinjlim_F F = \mathrm{Coker}(f - g) = M/\operatorname{Im}(f - g)$.

In particular, the forgetful functor ${}_R\mathbf{Mod} \to \mathbf{Set}$ commutes with limits, but not with colimits in general. This also holds for the inclusion functor $\mathbf{Ab} \to \mathbf{Grp}$; for example, the coproduct of the family $(\mathbb{Z}, \mathbb{Z})$ is the free group on two generators in $\mathbf{Grp}$, but it is $\mathbb{Z} \oplus \mathbb{Z}$ in $\mathbf{Ab}$.

Rings

Just as in the cases of groups and R-modules, if $F : \mathscr{I} \to \mathbf{Ring}$ is a functor, then the limit in $\mathbf{Set}$ of its composition with the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ has a natural structure of ring, and this gives the limit of F.

All colimits exist in $\mathbf{Ring}$, but, as in $\mathbf{Grp}$, they are harder to construct, and the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ does not commute with colimits. For example, if $f, g : R \to S$ are two morphisms of rings, then $\operatorname{Coker}(f, g)$ is the quotient of S by the two-sided ideal of S generated by the subset $\{f(r) - g(r),\ r \in R\}$.

Commutative rings

If $F : \mathscr{I} \to \mathbf{CRing}$ is a functor, then the limit of F as a functor $\mathscr{I} \to \mathbf{Ring}$ is a commutative ring, and this gives the limit of F. So the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ commutes with limits.

We can construct all colimits if we know how to construct cokernels and coproducts. If $f, g : R \to S$ are two morphisms of commutative rings, then $\operatorname{Coker}(f, g)$ is the quotient of S by the ideal of S generated by the subset $\{f(r) - g(r),\ r \in R\}$. The coproduct of a family $(R_i)_{i \in I}$ of commutative rings is the tensor product $\bigotimes_{i \in I} R_i$. Note that, once again, colimits don't commute with the forgetful functors from $\mathbf{CRing}$ to $\mathbf{Ring}$, $\mathbf{Ab}$ or $\mathbf{Set}$.

I.5.5.2 Limits as kernels between products

Let $\mathscr{C}$ be a category, and let $F : \mathscr{I} \to \mathscr{C}$ be a functor. For every morphism $\alpha : i \to j$ in $\mathscr{I}$, we write $i = \sigma(\alpha)$ and $j = \tau(\alpha)$. Let $\operatorname{Mor}(\mathscr{I}) = \coprod_{i,j \in \operatorname{Ob}(\mathscr{I})} \operatorname{Hom}_{\mathscr{I}}(i, j)$.

Suppose that the products $A = \prod_{i \in \operatorname{Ob}(\mathscr{I})} F(i)$ and $B = \prod_{\alpha \in \operatorname{Mor}(\mathscr{I})} F(\tau(\alpha))$ exists. We define two morphisms $f, g : A \to B$ in the following way :

- for every $\alpha \in \operatorname{Mor}(\mathscr{I})$, the composition of f with the projection on the α component of $\prod_{\alpha \in \operatorname{Mor}(\mathscr{I})} F(\tau(\alpha))$ is the projection $A \to F(\tau(\alpha))$;
- for every $\alpha \in \operatorname{Mor}(\mathscr{I})$, the composition of g with the projection on the α component of $\prod_{\alpha \in \operatorname{Mor}(\mathscr{I})} F(\tau(\alpha))$ is the the compositon of $F(\alpha)$ and of the canonical projection $A \to F(\sigma(\alpha))$.

Suppose that $\operatorname{Ker}(f, g)$ exists, and, for every $i \in \operatorname{Ob}(\mathscr{I})$, let $u_i : \operatorname{Ker}(f, g) \to F(i)$ be the composition of the projection $A \to F(i)$ and of the canonical morphism $\operatorname{Ker}(f, g) \to A$.

Then $(\operatorname{Ker}(f, g), (u_i)_{i \in \operatorname{Ob}(\mathscr{I})})$ is a limit of F.

Proof. We first check that $(\operatorname{Ker}(f, g), (u_i)_{i \in \operatorname{Ob}(\mathscr{I})})$ is a cone over F. For every $\alpha \in \operatorname{Mor}(\mathscr{I})$, let $p_\alpha : B \to F(\tau(\alpha))$ be the canonical projection. By definition of the morphisms f and g, for every morphism $\alpha : i \to j$ in $\mathscr{I}$, we have $p_j = p_\alpha \circ f$ and $F(\alpha) \circ p_i = p_\alpha \circ g$, hence $u_j = F(\alpha) \circ u_i$.

Let $(C, (v_i)_{i \in \operatorname{Ob}(\mathscr{I})})$ be a cone over F. The family (v_i) defines a morphism $v : C \to A$. The compatibility condition on the v_i says that, for every morphism $\alpha : i \to j$ in $\mathscr{I}$, the morphisms

$p_\alpha \circ f \circ v = v_j$ and $p_\alpha \circ g \circ v = F(\alpha) \circ v_i$ are equal. So v factors uniquely through $\mathrm{Ker}(f,g)$.

□

There is a similar construction for colimits once we have coproducts and cokernels, see Remark I.5.2.4.

I.5.6 Filtrant colimits

Definition I.5.6.1. A category $\mathscr{I}$ is called *filtrant* if:

(a) $\mathscr{I}$ is nonempty;

(b) for any $i, j \in \mathrm{Ob}(\mathscr{I})$, there exists a $k \in \mathrm{Ob}(\mathscr{I})$ and morphisms $i \to k$ and $j \to k$;

(c) if $f, g : i \to j$ are morphisms in $\mathscr{I}$, there exists a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

This generalizes the definition of a (nonempty) directed poset.

Colimits indexed by filtrant categories are called *filtrant colimits*. They are usually easier to calculate. For example :

Proposition I.5.6.2. *Let $\mathscr{I}$ be a filtrant category, and let $F : \mathscr{I} \to \mathbf{Set}$ be a functor. Consider the following relation $\sim$ on $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$: $(i,x) \sim (j,y)$ if and only if there exists $k \in \mathrm{Ob}(\mathscr{I})$ and morphisms $\alpha : i \to k$, $\beta : j \to k$ such that $F(\alpha)(x) = F(\beta)(y)$. Then $\sim$ is an equivalence relation, $\varinjlim F = \coprod_{i \in \mathrm{Ob}(\mathscr{I})} / \sim$.*

Proof. If $\sim$ is an equivalence relation, then it is clearly the equivalence relation generated by the relation R of Theorem I.5.2.1. So it suffices to show that $\sim$ is an equivalence relation; as it is clearly reflexive and symmetric, it suffices to show that it is transitive. Let $(i_1, x_1), (i_2, x_2), (i_3, x_3) \in \coprod_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$ such that $(i_1, x_1) \sim (i_2, x_2)$ and $(i_2, x_2) \sim (i_3, x_3)$. Then there exists morphisms $\alpha_1 : i_1 \to j$, $\alpha_2 : i_2 \to j$, $\beta_2 : i_2 \to k$ and $\beta_3 : i_3 \to k$ in $\mathscr{I}$ such that $F(\alpha_1)(x_1) = F(\alpha_2)(x_2)$ and $F(\beta_2)(x_2) = F(\beta_3)(x_3)$. By condition (b) of Definition I.5.6.1, there exist morphisms $\gamma : j \to l$ and $\delta : j \to l$. By condition (c) of Definition I.5.6.1, there exists a morphism $\epsilon : l \to l'$ such that $\epsilon \circ \gamma \circ \alpha_2 = \epsilon \circ \delta \circ \beta_2$. Then we get

$$\begin{aligned} F(\epsilon \circ \gamma\alpha_1)(x_1) = F(\epsilon \circ \gamma)(F(\alpha_1)(x_1)) = F(\epsilon \circ \gamma)(F(\alpha_2)(x_2)) &= F(\epsilon \circ \gamma \circ \alpha_2)(x_2) \\ &= F(\epsilon \circ \delta \circ \beta_2)(x_2) \\ &= F(\epsilon \circ \delta)(F(\beta_2)(x_2)) \\ &= F(\epsilon \circ \delta)(F(\beta_3)(x_3)) \\ &= F(\epsilon \circ \delta \circ \beta_3)(x_3), \end{aligned}$$

so $(i_1, x_1) \sim (i_3, x_3)$.

□

Corollary I.5.6.3. *Let R be a ring and let* $\mathrm{For} : {}_R\mathbf{Mod} \to \mathbf{Set}$ *be the forgetful functor. Then* For *commutes with filtrant colimits.*

We have a similar statement for right R-modules. Note that this is false without the hypothesis that $\mathscr{I}$ is filtrant. For example, coproducts in ${}_R\mathbf{Mod}$ are direct sums and coproducts in $\mathbf{Set}$ are disjoint unions.

Proof. See problem A.2.4.

$\square$

Let $F : \mathscr{I} \times \mathscr{J} \to \mathscr{C}$ be a functor. By the universal properties of limits and colimits, there is always a canonical morphism

(*) $$\varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \varprojlim_{j\in\mathrm{Ob}(\mathscr{J})} F(i,j) \to \varprojlim_{j\in\mathrm{Ob}(\mathscr{J})} \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} F(i,j).$$

(If all limits and colimits in this formula exist.) This is not an isomorphism in general. However, we have the following result :

Proposition I.5.6.4. *Let $F : \mathscr{I} \times \mathscr{J} \to \mathbf{Set}$ be a functor. Suppose that the category $\mathscr{I}$ is filtrant and that the category $\mathscr{J}$ is finite. Then the morphism (*) is an isomorphism.*

In other words, for every filtrant category $\mathscr{I}$, the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, \mathbf{Set}) \to \mathbf{Set}$ commutes with finite limits.

Proof. It suffices to show that $\varinjlim : \mathrm{Func}(\mathscr{I}, \mathbf{Set}) \to \mathbf{Set}$ commutes with finite products and with kernels.

Kernels: Let $F, G : \mathscr{I} \to \mathbf{Set}$ be two functors, and let $u, v : F \to G$ be two morphisms of functors. Let $H : \mathscr{I} \to \mathbf{Set}$ be the functor $i \in \mathrm{Ob}(\mathscr{I})$ to $\mathrm{Ker}(u(i), v(i) : F(i) \to G(i))$, and a morphism $\alpha : i \to j$ to the map induced by $F(\alpha)$. Let $Y = \mathrm{Ker}(\varinjlim u, \varinjlim v : \varinjlim F \to \varinjlim G)$. We have a natural map $\phi : \varinjlim H \to Y$, and we want to show that it is a bijection.

We first prove that ϕ is surjective. Let $y \in Y$. Then there exists $i \in Ob(\mathscr{I})$ and $x \in F(i)$ such that y is the image of (i,x) by the canonical map $\coprod_{i'\in\mathrm{Ob}(\mathscr{I})} F(i') \to Z$. As $\varinjlim u(y) = \varinjlim v(y)$, the images of $u_i(x)$ and $v_i(x)$ in $\varinjlim G$ are equal, so there exists $\alpha : i \to j$ such that $G(\alpha)(u_i(x)) = G(\alpha)(v_i(x))$. Let $x' = F(\alpha)(x) \in F(j)$. Then $u_j(x') = G(\alpha)(u_i(x)) = G(\alpha)(v_i(x)) = v_j(x')$, so $x' \in H(j)$. Also, (i,x) and (j,x') have the same image in $\varinjlim H$, so ϕ the image of (j,x') in $\varinjlim H$ to y.

We now show that ϕ is injective. Let $i, j \in \mathrm{Ob}(\mathscr{I})$, $x \in H(i)$ and $x' \in H(j)$ such that the images of (i,x) and (j,x') in $\varinjlim H$ have the same image by ϕ. This means that (i,x) and (j,x') have the same image in $\varinjlim F$, so there exists $\alpha : i \to k$ and $\beta : j \to k$ such that $F(\alpha)(x) = F(\beta)(x')$. But then (i,x) and (j,x') also have the same image in $\varinjlim H$, as desired.

Finite products: Let $F_1, \ldots, F_n : \mathscr{I} \to \mathbf{Set}$ be functors, and let $G = F_1 \times \ldots \times F_n$. We want to show that the canonical map $\lambda : \varinjlim G \to (\varinjlim F_1) \times \ldots \times (\varinjlim F_n)$ is bijective.

We show that λ is injective. Let $x, y \in\in \varinjlim G$ such that $\lambda(x) = \lambda(y)$. We represent x (resp. y) by $(i, x_1, \ldots, x_n)$ (resp. $(j, y_1, \ldots, y_n)$ with $i \in \mathrm{Ob}(\mathscr{I})$ and $x_m \in F_m(i)$ (resp. $j \in \mathrm{Ob}(\mathscr{I})$ and $y_m \in F_m(j)$). The assumption that $\lambda(x) = \lambda(y)$ says that, for every $m \in \{1, \ldots, m\}$, (i, x_m) and (j, y_m) have the same image in $\varinjlim F_m$, so there exist morphisms $\alpha_m : i \to k_m$ and $\beta_m : j \to k_m$ in $\mathscr{I}$ such that $F_m(\alpha_m)(x_m) = F_m(\beta_m)(y_m)$. By condition (b) of Definition I.5.6.1, we can find $l \in \mathrm{Ob}(\mathscr{I})$ and morphisms $\gamma_1 : k_1 \to l, \ldots, \gamma_n : k_n \to l$. By condition (c) of the same definition, we can find a morphism $\delta : l \to l'$ such that $\delta \circ \gamma_1 \circ \alpha_1 = \ldots = \delta \circ \gamma_n \circ \alpha_n$ and $\delta \circ \gamma_1 \circ \beta_1 = \ldots = \delta \circ \gamma_n \circ \beta_n$. Then $G(\delta \circ \gamma_1 \circ \alpha_1)(x_1, \ldots, x_n) = G(\delta \circ \gamma_1 \circ \beta_1)(y_1, \ldots, y_n)$, so $x = y$.

We show that λ is surjective. Let $(z_1, \ldots, z_n) \in (\varinjlim F_1) \times \ldots \times (\varinjlim F_n)$. For every $m \in \{1, \ldots, n\}$, we choose $i_m \in \mathrm{Ob}(\mathscr{I})$ and $x_m \in F_m(i_m)$ such that (i_m, x_m) represents z_m. By condition (b) of Definition I.5.6.1, we can find $j \in \mathrm{Ob}(\mathscr{I})$ and morphisms $\alpha_1 : i_1 \to j$, $\ldots$, $\alpha_n : i_n \to j$. Let z be the image in $\varinjlim G$ of $(j, F_1(\alpha_1)(x_1), \ldots, F_n(\alpha_n)(x_n))$. Then $\lambda(z) = (z_1, \ldots, z_n)$.

$\square$

Corollary I.5.6.5. *Let R be a ring and $\mathscr{I}$ be a filtrant category. Then the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ commutes with finite limits.*

II Additive and abelian categories

II.1 Additive categories

II.1.1 Preadditive categories

Definition II.1.1.1. (i). A *preadditive category* is a category $\mathscr{C}$ with the additional data of an abelian group structure on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, such that the composition maps are $\mathbb{Z}$-bilinear. More generally, if k is a commutative ring, a *k-linear category* is a category $\mathscr{C}$ with the additional data of a k-module structure on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, such that the composition maps are k-bilinear.

(ii). Let $\mathscr{C}$ and $\mathscr{D}$ be preadditive (resp. k-linear) categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is called *additive* (resp. *k-linear*) if, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, the map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is a morphism of groups (resp. of k-modules).

Notation II.1.1.2. If $\mathscr{C}$ and $\mathscr{D}$ are two preadditive categories, the full subcategory of $\mathrm{Func}(\mathscr{C}, \mathscr{D})$ whose objects are additive functors is denoted by $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}, \mathscr{D})$.

Note that a preadditive category is just a $\mathbb{Z}$-linear category, and an additive functor is a $\mathbb{Z}$-linear functor. If $\mathscr{C}$ is an additive category and $X, Y \in \mathrm{Ob}(\mathscr{C})$, we denote by $0 : X \to Y$ the unit element of the abelian group structure on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and call it the *zero morphism* from X to Y. We will see in Proposition II.1.2.4 that, if $\mathscr{C}$ has a zero object, then this notation agrees with the one introduced in Definition I.2.1.9.

Remark II.1.1.3. If $\mathscr{C}$ is a preadditive category, then, for every $X \in \mathrm{Ob}(\mathscr{C})$, the set $\mathrm{End}_{\mathscr{C}}(X)$ has a natural ring structure, whose addition is given by the abelian group structure of $\mathrm{End}_{\mathscr{C}}(X)$ and whose multiplication is given by composition. Similarly, if $\mathscr{C}$ is a k-linear category, then $\mathrm{End}_{\mathscr{C}}(X)$ is a k-algebra for every $X \in \mathrm{Ob}(\mathscr{C})$.

Example II.1.1.4. (1) If R is a k-algebra, then the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ are k-linear, as are the full subcategories of finitely generated (resp. initely presented, torsion, torsion-free) objects.

(2) Consider the category $\mathbb{Z}[\mathbf{Set}]$ defined by :

   (a) $\mathrm{Ob}(\mathbb{Z}[\mathbf{Set}]) = \mathrm{Ob}(\mathbf{Set})$;

   (b) for all sets X and Y, $\mathrm{Hom}_{\mathbb{Z}[\mathbf{Set}]}(X, Y)$ is the free abelian group on $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$;

(c) the composition law is the unique $\mathbb{Z}$-bilinear extension of the composition law of $\mathbf{Set}$.

Then $\mathbb{Z}[\mathbf{Set}]$ is a preadditive category.

(3) If $\mathscr{D}$ is a preadditive category, then, for every category $\mathscr{C}$, the category $\mathrm{Func}(\mathscr{C}, \mathscr{D})$ is preadditive, with the following addition on its Hom sets : if $F, G : \mathscr{C} \to \mathscr{D}$ are functors, if $u, v : F \to G$ are morphisms of functors, we define $u + v : F \to G$ by $(u+v)(X) = u(X) + v(X) \in \mathrm{Hom}_{\mathscr{D}}(F(X), G(X))$, for every $X \in \mathrm{Ob}(\mathscr{C})$.

If $\mathscr{C}$ is preadditive, this also makes the subcategory $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}, \mathscr{D})$ into a preadditive category.

(4) If R is a ring, then the category $\mathscr{C}$ with one object $*$ and such that $\mathrm{End}_{\mathscr{C}}(*) = R$ is a preadditive category.

(5) If $\mathscr{C}$ is a preadditive (resp. k-linear) category, then $\mathscr{C}^{\mathrm{op}}$ has a natural structure of preadditive (resp. k-linear) category.

(6) If $\mathscr{C}$ and $\mathscr{C}'$ are preadditive categories, then $\mathscr{C} \times \mathscr{C}'$ has a natural structure of preadditive category.

Example II.1.1.5. If $\mathscr{C}$ is a preadditive category, then we can see the functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$ as a functor from $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to $\mathbf{Ab}$, and it is an additive functor.

Proposition II.1.1.6. *Let $\mathscr{C}$ be a preadditive category, and let $X_1, \dots, X_n$ be objects of $\mathscr{C}$. Then the following assertions are equivalent :*

(i) the family $(X_1, \dots, X_n)$ has a product in $\mathscr{C}$;

(ii) the family $(X_1, \dots, X_n)$ has a coproduct in $\mathscr{C}$;

(iii) there exists an object Z of $\mathscr{C}$ and morphisms $i_r : X_r \to Z$, $p_r : Z \to X_r$, for $1 \leq r \leq n$, such that :

(a) $p_r \circ i_r = \mathrm{id}_{X_r}$ for every $r \in \{1, \dots, n\}$;

(b) if $r, s \in \{1, \dots, n\}$ and $r \neq s$, then $p_s \circ i_r = 0$;

(c) $i_1 \circ p_1 + \ldots + i_2 \circ p_2 = \mathrm{id}_Z$.

If these conditions holds, then $(Z, (i_r)_{1 \leq r \leq n})$ is a coproduct of $(X_1, \dots, X_n)$, and $(Z, (p_r)_{1 \leq r \leq n})$ is a product of $(X_1, \dots, X_n)$. In particular, we get a canonical isomorphism between $\prod_{r=1}^n X_r$ and $\coprod_{r=1}^n X_r$, and we usually denote both the product and coproduct by $\bigoplus_{r=1}^n X_r$ and call it the biproduct *of $(X_1, \dots, X_n)$.*

Proof. We prove that (i) implies (iii). Let $Z = \prod_{r=1}^n X_r$, and let $p_r : Z \to X_r$ be the canonical morphisms, for $1 \leq r \leq n$. For every $r \in \{1, \dots, n\}$, the pair $(X_r, (u_s)_{1 \leq s \leq n})$ with $u_r = \mathrm{id}_{X_r}$ and $u_s = 0$ if $s \neq r$ is a cone over $(X_1, \dots, X_n)$, hence there is a morphism $i_r : X_r \to Z$ such that $p_s \circ i_r = u_s$ for every $s \in \{1, \dots, n\}$. This gives Z and morphisms i_r and p_r satusfying conditions (a) and (b) of (iii). We need to check condition (c) of (iii). Let $u = i_1 \circ p_1 + \ldots + i_n \circ p_n : Z \to Z$. For every $r \in \{1, \ldots, n\}$, we have $p_r \circ u = \sum_{s=1}^n (p_r \circ i_s) \circ p_s = \mathrm{id}_{X_r} \circ p_r = p_r$. So the morphism u is a morphism of cones over $(X_1, \ldots, X_n)$ from $(Z, (p_r)_{1\leq r\leq n})$ to itself. By the universal property of the product, u must be equal to id_Z.

We prove that (iii) implies (i). In fact, we will prove that $(Z, (p_r)_{1\leq r\leq n})$ is a product of the family $(X_1, \ldots, X_n)$. This pair is clearly a cone over $(X_1, \ldots, X_n)$; we show that it is a terminal cone. Let $(Y, (u_r)_{1\leq r\leq n})$ be a cone over $(X_1, \ldots, X_n)$. Let $f = i_1 \circ u_1 + \ldots + i_n \circ u_n : Y \to Z$. For every $r \in \{1, \ldots, n\}$, we have $p_r \circ f = \sum_{s=1}^n (p_r \circ i_s) \circ u_s = u_r$, so f is a morphism of cones. Suppose that $g : Y \to Z$ is another morphism of cones. Then

$$g = \mathrm{id}_Z \circ g = \sum_{r=1}^n i_r \circ p_r \circ g = \sum_{r=1}^n i_r \circ u_r = f.$$

This shows that Z is indeed a terminal object in the category of cones over $(X_1, \ldots, X_n)$, i.e. a product of $(X_1, \ldots, X_n)$.

To prove that (ii) and (iii) are equivalent, and the fact that a pair $(Z, (i_r))$ as in (iii) is a coproduct of $(X_1, \ldots, X_n)$, it suffices to apply what we already proved to $\mathscr{C}^{\mathrm{op}}$.

□

Matrix notation for morphisms : Let $\mathscr{C}$ be a preadditive category, let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be objects of $\mathscr{C}$, and let $f_{rs} : X_r \to Y_s$ be morphisms of $\mathscr{C}$ for every $r \in \{1, \ldots, n\}$ and $s \in \{1, \ldots, m\}$. We denote by $i_r : X_r \to X_1 \oplus \ldots \oplus X_n$ and $p_s : Y_1 \oplus \ldots \oplus Y_m \to Y_s$ the canonical morphisms, for $1 \leq r \leq n$ and $1 \leq s \leq m$. Then there exists a unique morphism $F : X_1 \oplus \ldots \oplus X_n \to Y_1 \oplus \ldots \oplus Y_m$ such that $p_s \circ F \circ i_r = f_{rs}$ for every $r \in \{1, \ldots, n\}$ and $s \in \{1, \ldots, m\}$.

We write this morphism F as the $m \times n$ matrix $(f_{rs})_{1\leq sm, 1\leq r\leq n}$, so we never have to write the previous paragraph again. The convention is chosen so that matrix multiplication corresponds to composition of morphisms in $\mathscr{C}$.

Notation II.1.1.7. More generally, if $(X_i)_{i\in I}$ is a family of objects in a preadditive category, and if the coproduct of this family exists, then we denote this coproduct by $\bigoplus_{i\in I} X_i$ and we call it the *direct sum* of the family $(X_i)_{i\in I}$.

II.1.2 Additive categories

Definition II.1.2.1. An *additive category* (resp. k-linear additive category) is a preadditive (resp. k-linear) category that has all finite products.

By Proposition II.1.1.6, it is equivalent to require that $\mathscr{C}$ has all finite coproducts, and the product and coproduct of a finite family of objects in an additive category are canonically equivalent.

Remark II.1.2.2. In particular, an additive category has a zero object (see Definition I.2.1.9), that is, an object 0 that is both initial and terminal. Indeed, such an object is the biproduct of the empty family of objects.

Example II.1.2.3. (1) In Example II.1.1.4, the categories in (1) are additive, and the category in (3) (resp. (5), resp. (6)) is additive if $\mathscr{D}$ (resp. $\mathscr{C}$, resp. $\mathscr{C}$ and $\mathscr{C}'$) is; but the categories in (2) and (4) are not additive.

(2) The category of Banach spaces over $\mathbb{R}$ (resp. $\mathbb{C}$) is $\mathbb{R}$-linear additive (resp. $\mathbb{C}$-linear additive).

Proposition II.1.2.4. *Let $\mathscr{C}$ be an additive category, and let $X, Y \in \mathrm{Ob}(\mathscr{C})$. Then :*

(i). The zero morphism from X to Y is the composition $X \to 0 \to Y$, where the morphisms $X \to 0$ and $0 \to Y$ are the unique morphisms.

(ii). Let $f, g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$. Consider the morphisms $F = (\mathrm{id}_X, f) \in \mathrm{Hom}_{\mathscr{C}}(X, X \oplus Y) = \mathrm{Hom}_{\mathscr{C}}(X,X) \times \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $G = (g, \mathrm{id}_Y) \in \mathrm{Hom}_{\mathscr{C}}(X \oplus Y, Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Y)$. Then $f + g = G \circ F$.

In particular, this proposition says that the group law on $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is uniquely determined by the category $\mathscr{C}$. (It is not actually an extra structure.) In fact, we can show that, if $\mathscr{C}$ is a category in which all finite biproducts exist (i.e. objects that are both products and coproducts), then the formulas of the proposition define a structure of commutative monoid on $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, such that the composition law is biadditive. [1]

Proof. Note that (i) follows from (ii) and from the fact that $X \oplus 0$ (resp. $Y \oplus 0$) is canonically isomorphic to X (resp. Y), by Subsection I.5.4.1.

We prove (ii). Let $i_1 : X \to X \oplus Y$, $i_2 : X \to X \oplus Y$, $p_1 : X \oplus Y \to X$ and $p_2 : X \oplus Y \to Y$ be the morphisms of Proposition II.1.1.6(iii). By definition of F and G, we have $p_1 \circ F = \mathrm{id}_X$, $p_2 \circ F = f$, $G \circ i_1 = g$ and $G \circ i_2 = \mathrm{id}_Y$. So

$$G \circ F = G \circ \mathrm{id}_{X \oplus Y} \circ F = G \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ F = g \circ \mathrm{id}_X + \mathrm{id}_Y \circ f = g + f.$$

□

Corollary II.1.2.5. *Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then the following assertions are equivalent :*

(i) F is an additive functor;

(ii) F commutes with finite products.

Proof. The fact that (ii) implies (i) follows immediately from Proposition II.1.2.4. The fact that (i) implies (ii) follows from the characterization of finite products in Proposition II.1.1.6.

□

[1] See problem set 3 maybe ?

II.1.3 Kernels and cokernels

Definition II.1.3.1. Let $\mathscr{C}$ be a category.

(i). Let $f, g : X \to Y$ be two morphisms. If (f, g) has a kernel, the morphism $\mathrm{Ker}(f, g) \to X$ is called a *kernel morphism*; if (f, g) has a cokernel, the morphism $Y \to \mathrm{Coker}(f, g)$ is called a *cokernel morphism*.

(ii). Suppose that $\mathscr{C}$ is a preadditive category. If f is a morphism of $\mathscr{C}$, we write $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ for $\mathrm{Ker}(f, 0)$ and $\mathrm{Coker}(f, 0)$ and call them the *kernel* and *cokernel* of f.

Definition II.1.3.2. Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ be a morphism of $\mathscr{C}$.

(i). Suppose that f has a kernel. A *coimage of f* is a cokernel of the kernel morphism $\mathrm{Ker}(f) \to X$; we denote it by $X \to \mathrm{Coim}(f)$ (if it exists).

(ii). Suppose that f has a cokernel. An *image of f* is a kernel of the cokernel morphism $Y \to \mathrm{Coker}(f)$; we denote it by $\mathrm{Im}(f) \to Y$ (if it exists).

Lemma II.1.3.3. *Let $\mathscr{C}$ be a category.*

(i). Every kernel morphism in $\mathscr{C}$ is a monomorphism, and every cokernel morphism in $\mathscr{C}$ is an epimorphism.

(ii). Suppose that $\mathscr{C}$ is an additive category, and let $f : X \to Y$ be a morphism of $\mathscr{C}$ that admits a kernel (resp. a cokernel). Then the following assertions are equivalent :

a) f is a monomorphism (resp. an epimorphism);

b) $\mathrm{Ker}\, f = 0$ *(resp.* $\mathrm{Coker}\, f = 0$*);*

c) the kernel morphism $\mathrm{Ker}\, f \to X$ *(resp. the cokernel morphism* $Y \to \mathrm{Coker}\, f$*) is* 0*;*

d) the coimage (resp. image) of f exists, and the canonical morphism $X \to \mathrm{Coim}(f))$ *(resp.* $\mathrm{Im}(f) \to Y$*) is an isomorphism;*

e) the coimage (resp. image) of f exists, and the canonical morphism $X \to \mathrm{Coim}(f))$ *(resp.* $\mathrm{Im}(f) \to Y$*) is a monomorphism (resp. an epimorphism).*

Proof. We only prove the assertions about kernels (then we apply them to $\mathscr{C}^{\mathrm{op}}$ to get the assertions about cokernels).

(i). Let k be a kernel morphism in $\mathscr{C}$; we write $k : \mathrm{Ker}(f, g) \to X$, where $f, g : X \to Y$ are two morphisms of $\mathscr{C}$. Let $h_1, h_2 : Z \to \mathrm{Ker}(f, g)$ be two morphisms such that $k \circ h_1 = k \circ h_2$, and write $k' = k \circ h_1$. We have $f \circ k' = g \circ k'$, so, by the universal property of the kernel, there exists a unique $h : Z \to \mathrm{Ker}(f, g)$ such that $k' = k \circ h$. As both h_1 and h_2 satisfy that property, we have $h_1 = h_2$.

(ii). Let $K = \mathrm{Ker}(f)$ and $k : K \to X$ be the kernel morphism.

We have $f \circ k = f \circ 0 = 0$, so, if f is a monormophism, then $k = 0$. This shows that (a) implies (c). Suppose that $K = 0$, and let $h_1, h_2 : Z \to X$ be two morphisms such that $f \circ h_1 = f \circ h_2$. Then $f \circ (h_1 - h_2) = f \circ 0 = 0$, so there exists a unique morphism $u : Z \to K$ such that $h_1 - h_2 = k \circ u$; as $K = 0$, this means that $h_1 - h_2 = 0$, i.e. that $h_1 = h_2$. This shows that (b) implies (a).

We show that (c) implies (b). As $k = 0$, we have $k \circ \mathrm{id}_K = k \circ 0$, so $\mathrm{id}_K = 0$. This implies that the composition $K \to 0 \to K$ (where the morphisms are the unique morphisms) is the identity of K, so the unique morphism $K \to 0$ is an isomorphism, i.e. $K = 0$.

Suppose that (b) holds. Then the coimage of f (if it exists) is the cokernel of the unique morphism $z : 0 \to X$. We claim that $X \overset{\mathrm{id}_X}{\to} X$ is a cokernel of this morphism. Indeed, we have $\mathrm{id}_X \circ z = 0$. Let $u : X \to Y$ be any morphism such that $u \circ z = 0$; then there exists a unique $v : X \to Y$ such that $v \circ \mathrm{id}_X = u$, which is $v = u$. This proves (d).

It is clear that (d) implies (e), so it remains to show that (e) implies (c). Let $v : X \to \mathrm{Coim}(f)$ and $u : \mathrm{Ker}(f) \to X$ be the canonical morphisms. Then $v \circ u = 0 = v \circ 0$; as v is a monomorphism by assumption, this implies that $u = 0$.

□

Lemma II.1.3.4. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$. Suppose that f has a kernel, a cokernel, an image and a coimage. Then there exists a unique morphism $u : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ making the following diagram commute (where the unmarked arrows are canonical morphisms) :*

$$\begin{tikzcd}
\mathrm{Ker}\, f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \mathrm{Coker}(f) \\
& \mathrm{Coim}(f) \arrow[r, dashed, "u"'] & \mathrm{Im}(f) \arrow[u] &
\end{tikzcd}$$

Proof. The uniqueness of u follows from the fact that the kernel morphism $k : \mathrm{Im}(f) \to Y$ is a monomorphism and the cokernel morphism $c : X \to \mathrm{Coim}(f)$ is an epimorphism (by Lemma II.1.3.3); indeed, if we have $u, u' : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ satisfying the conditions of the lemma, then $k \circ u \circ c = k \circ u' \circ c$, so $u \circ c = u' \circ c$, so $u = u'$.

For the existence, note that f composed with the kernel morphism $g : \mathrm{Ker}(f) \to X$ is 0; so, by definition of the cokernel of g, there exists a morphism $h : \mathrm{Coker}(g) = \mathrm{Coim}(f) \to Y$ such that $h \circ c = f$. Let $l : Y \to \mathrm{Coker}(f)$ be the canonical morphism. Then $l \circ h \circ c = l \circ f = 0 = 0 \circ c$; as c is an epimorphism, this implies that $l \circ h = 0$. By the universal property of the kernel of l, this implies that there exists a morphism $u : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that $k \circ u = h$. Then $k \circ u \circ c = h \circ c = f$, as desired.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, "g"] & X \arrow[r, "f"] \arrow[d, "c"'] & Y \arrow[r, "l"] & \operatorname{Coker}(f) \\
 & \operatorname{Coim}(f) \arrow[r, dashed, "u"'] \arrow[ur, dashed, "h"] & \operatorname{Im}(f) \arrow[u, "k"'] &
\end{tikzcd}$$

□

Proposition II.1.3.5. *Let $\mathscr{C}$ be an additive category in which every morphism has a kernel and a cokernel.*[2] *Let $f : X \to Y$ be a morphism in $\mathscr{C}$.*

(i). Let $i = \begin{pmatrix} \mathrm{id}_Y \\ \mathrm{id}_Y \end{pmatrix} : Y \to Y \oplus Y$, $F = \begin{pmatrix} f \\ f \end{pmatrix} : X \to Y \oplus Y$ *and* $c : Y \oplus Y \to Y \oplus_X Y := \operatorname{Coker}(F)$ *be the canonical morphism. Then there is a unique morphism* $u : \operatorname{Im}(f) \to \operatorname{Ker}(c \circ i)$ *such that the triangle*
$$\begin{tikzcd}
\operatorname{Ker}(c \circ i) \arrow[r] & Y \\
\operatorname{Im}(f) \arrow[u, "u"] \arrow[ur] &
\end{tikzcd}$$
commutes (where the unmarked arrows are the canonical maps), and it is an isomorphism.

(ii). Let $p = \begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \end{pmatrix} : X \times X = X \oplus X \to X$ *be the sum of the two projections from* $X \times X$ *to* X, $f_1, f_2 : X \times X \to Y$ *be the composition of f with these two projections, and* $k : X \times_Y X = \operatorname{Ker}(f_1 - f_2) \to X \times X$ *be the canonical morphism. Then there exists a unique morphism* $v : \operatorname{Coker}(k) \to \operatorname{Coim}(f)$ *such that the triangle*
$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \operatorname{Coker}(k) \arrow[d, "v"] \\
 & \operatorname{Coim}(f)
\end{tikzcd}$$
commutes (where the unmarked arrows are the canonical maps), and it is an isomorphism.

Remark II.1.3.6. This proposition gives an alternative definition of the image and the coimage of f that makes sense in any category admitting finite limits and finite colimits (after a small modification). See Definition 5.1.1 of [8].

Proof of Proposition II.1.3.5. Note that (ii) is just (i) in the opposite category. So we just prove (i). The uniqueness of u follows from the fact that $\operatorname{Ker}(c \circ i) \to Y$ is a monomorphism (by Lemma II.1.3.3). Now consider the commutative diagram :

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr, "f"] & & Y \oplus Y \arrow[d, "c"] \\
\operatorname{Ker}(c \circ i) \arrow[r, "j"] & Y \arrow[ur, "i"] \arrow[r, "c \circ i"] \arrow[dr, "q"'] & Y \oplus_X Y \\
\operatorname{Im}(f) = \operatorname{Ker}(q) \arrow[u, dashed, "u"] \arrow[ur, "k"'] & & \operatorname{Coker}(f) \arrow[u, dashed, "h"']
\end{tikzcd}$$

[2] In fact, we only need the kernels and the cokernels that actually appear in the statement to exist.

As $(c \circ i) \circ f = c \circ F = 0$, there exists a unique morphism $h : \mathrm{Coker}(f) \to Y \oplus_X Y$ such that $h \circ q = c \circ i$. This implies that $(c \circ i) \circ k = h \circ q \circ k = 0$, so there exists a unique morphism $u : \mathrm{Im}(f) \to \mathrm{Ker}(c \circ i)$ such that $j \circ u = k$.

To show that u is an isomorphism, it suffices to construct a morphism $v : \mathrm{Ker}(c \circ i) \to \mathrm{Im}(f)$ such that $k \circ v = j$. Indeed, we will then have $j \circ (u \circ v) = j = j \circ \mathrm{id}_{\mathrm{Ker}(c \circ i)}$, so $u \circ v = \mathrm{id}_{\mathrm{Ker}(c \circ i)}$ because j is a monomorphism, and similarly $k \circ (v \circ u) = k$, so $v \circ u = \mathrm{id}_{\mathrm{Im}(f)}$ because k is a monomorphism. To show the existence of v, it suffices to prove that $q \circ j = 0$. Let $p_1 : Y \oplus Y = Y \times Y \to Y$ be the first projection.

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "f"'] & Y \oplus Y \arrow[r, "c"] \arrow[d, "p_1"] & Y \oplus_X Y \arrow[d, dashed, "h'"] \\
& Y \arrow[r, "q"'] & \mathrm{Coker}(f)
\end{tikzcd}$$

Then $p_1 \circ i = \mathrm{id}_Y$, so $p_1 \circ F = f$, so $q \circ p_1 \circ F = 0$, so there exists a unique morphism $h' : Y \oplus_X Y \to \mathrm{Coker}(f)$ such that $h' \circ c = q \circ p_1$. So $q \circ j = q \circ \mathrm{id}_Y \circ j = (q \circ p_1) \circ i \circ j = h' \circ (c \circ i) \circ j = h' \circ 0 = 0$.

□

II.2 Abelian categories

II.2.1 Definition

Definition II.2.1.1. Let $\mathscr{C}$ be an additive category. We say that $\mathscr{C}$ is an *abelian category* if :

(a) Every morphism of $\mathscr{C}$ has a kernel and a cokernel.

(b) For every morphism f of $\mathscr{C}$, the morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma II.1.3.4 is an isomorphism.

Remark II.2.1.2. Let $\mathscr{C}$ be an additive category. Then $\mathscr{C}$ is an abelian category if and only if $\mathscr{C}^{\mathrm{op}}$ is an abelian category.

Example II.2.1.3. (1) If R is a ring, the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ are abelian. If R is commutative and Noetherian, then the category of finitely generated R-modules is also abelian, but this is not true if R is not Noetherian. More generally, we say that a commutative ring R is *coherent* if the category of finitely presented R-modules is abelian. Noetherian rings are coherent, but the converse is not true; for example, if R is Noetherian, then any polynomial ring over RF is coherent (even a polynomial ring in an infinite number of indeterminates).

(2) If $\mathscr{A}$ is an abelian category and $\mathscr{C}$ is any category, then the additive category $\mathrm{Func}(\mathscr{C}, \mathscr{A})$ is abelian. If $F, G : \mathscr{C} \to \mathscr{A}$ are two functors and $u : F \to G$ is a morphism

of functors, then $\operatorname{Ker} u$ is the functor $X \longmapsto \operatorname{Ker}(u(X))$ and $\operatorname{Coker} u$ is the functor $X \longmapsto \operatorname{Coker}(u(X))$.

In particular, for a category $\mathscr{C}$, the category $\operatorname{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ of presheaves in abelian groups over $\mathscr{C}$ is an abelian category.

(3) The additive category $\mathbf{Ban}$ of Banach spaces over $\mathbb{R}$ (or over $\mathbb{C}$) has all kernels and cokernels, but it is not abelian. Note that kernels in this category are kernels in the category of vector spaces, but the cokernel in $\mathbf{Ban}$ of a morphism $f : E \to F$ is $E/\overline{f(F)}$.

For example, the inclusion $L^2([0,1]) \subset L^1([0,1])$ (where we use Lebesgue measure to define these spaces) is $\mathbb{C}$-linear continuous with dense image, so it has kernel and cokernel equal to 0, but it is not an isomorphism of Banach.

Similarly, the category of topological abelian groups is an additive category that has all kernels and cokernels but is not abelian.

(4) Let R be a ring. A *filtered left R-module* is a left R-module M together with a family $\operatorname{Fil}_* M = (\operatorname{Fil}_n M)_{n\in\mathbb{Z}}$ of R-submodules of M such that $\operatorname{Fil}_n M \subset \operatorname{Fil}_{n+1} M$ for every $n \in \mathbb{Z}$. [3] A morphism of filtered R-modules from $(M, \operatorname{Fil}_* M)$ to $(N, \operatorname{Fil}_* N)$ is a R-linear map $f : M \to N$ such that $f(\operatorname{Fil}_n M) \subset \operatorname{Fil}_n N$ for every $n \in \mathbb{Z}$. The category of filtered R-modules is additive and has all kernels and cokernels, but it is not abelian.

For example, take $R = \mathbb{Z}$, take $M = N = \mathbb{Z}^2$ with $\operatorname{Fil}_n M = \begin{cases} 0 & \text{if } n \leq 0 \\ \mathbb{Z} \oplus 0 & \text{if } n = 1 \\ \mathbb{Z}^2 & \text{if } n \geq 2 \end{cases}$ and $\operatorname{Fil}_n N = \begin{cases} 0 & \text{if } n \leq 0 \\ \mathbb{Z}^2 & \text{if } n \geq 1 \end{cases}$. Then the identity from M to N is a morphism of filtered $\mathbb{Z}$-modules, it has kernel and cokernel 0, but it is not an isomorphism.

(5) If R is a commutative ring, the category of torsionfree R-modules is additive and has all kernels and cokernels, but it is not abelian in general. For example, take $R = \mathbb{Z}$, and take $f : \mathbb{Z} \to \mathbb{Z}$, $a \longmapsto 2a$. Then $\operatorname{Ker}(f) = 0$ and $\operatorname{Coker}(f) = 0$, but f is not an isomorphism.

Remark II.2.1.4. Examples (3), (4) and (5) are examples of what are called quasi-abelian categories. It is still possible to define derived functors and derievd categories in the setting of quasi-abelian categories, but it takes more work.

Proposition II.2.1.5. *An abelian category has all finite limits and colimits.*

Proof. This follows from Subsection I.5.5.2.

□

The following proposition is easy but very useful.

[3] Technically, this is a $\mathbb{Z}$-filtered R-module. We can use any poset instead of $\mathbb{Z}$, or even any category.

Proposition II.2.1.6. *Let $\mathscr{C}$ be an abelian category, and let f be a morphism of $\mathscr{C}$. Then f is an isomorphism if and only if* $\mathrm{Ker}(f) = 0$ *and* $\mathrm{Coker}(f) = 0$.

Proof. If f is an isomorphism, then it is a monomorphism and an epimorphism, so $\mathrm{Ker}(f) = 0$ and $\mathrm{Coker}(f) = 0$ by Lemma II.1.3.3(i).

Conversely, suppose that $\mathrm{Ker}(f) = 0$ and $\mathrm{Coker}(f) = 0$. By Lemma II.1.3.3(ii), the canonical morphisms $X \to \mathrm{Coim}(f)$ and $\mathrm{Im}(f) \to Y$ are isomorphisms. By Lemma II.1.3.4 and the definition of an abelian category, the morphism $X \to Y$ is an isomorphism.

□

Until the end of this subsection, we fix an abelian category $\mathscr{A}$.

Remark II.2.1.7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms of $\mathscr{A}$ such that $g \circ f = 0$. If $i : \mathrm{Ker}\, g \to Y$ is the canonical morphism, the fact that $g \circ f = 0$ implies that there exists a unique $u : X \to \mathrm{Ker}\, g$ such that $i \circ u = f$. Let $j : \mathrm{Ker}\, f \to X$ be the canonical morphism. Then $i \circ (h \circ j) = f \circ f = 0$, so, as i is a monomorphism, we get $h \circ j = 0$, so there exists a unique morphism $v : \mathrm{Coker}(j) = \mathrm{Coim}(f) \to \mathrm{Ker}(g)$ such that $i \circ v \circ q = f$, where $q : X \to \mathrm{Coim}(f)$ is the canonical morphism. Composing this with the inverse of the canonical morphism $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$, we get a morphism $w : \mathrm{Im}(f) \to \mathrm{Ker}(g)$.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "j"] & X \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, dashed, "h"] & Y \arrow[r, "g"] & Z \\
& \mathrm{Coim}(f) \arrow[r, dashed, "v"'] & \mathrm{Ker}(g) \arrow[u, "i"'] &
\end{tikzcd}$$

Definition II.2.1.8. Let $X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} X_2 \to \ldots \to X_n \xrightarrow{d_n} X_{n+1}$ be a sequence of composable morphisms of $\mathscr{A}$. We say that this sequence is a *complex* if $d_i \circ d_{i-1} = 0$ for $1 \leq i \leq n$, and that it is an *exact sequence* if it is a complex and if the canonical morphism $\mathrm{Im}(d_{i-1}) \to \mathrm{Ker}(d_i)$ of Remark II.2.1.7 is an isomorphism for $1 \leq i \leq n$.

A *short exact sequence* is an exact sequence of the form $0 \to X \to X' \to X'' \to 0$.

Remark II.2.1.9. (1) Let $f : X \to Y$ be a morphism in $\mathscr{A}$. Then the sequence $0 \to X \xrightarrow{f} Y$ is exact if and only if f is a monomorphism, and the sequence $X \xrightarrow{f} Y \to 0$ is exact if and only if f is a epimorphism. Indeed, saying that the first (resp. second sequence) is exact is equivalent to saying that $\mathrm{Ker}(f) = 0$ (resp. that the canonical morphism $\mathrm{Im}(f) \to Y$ is an isomorphism), and we can apply Lemma II.1.3.3.

(2) Let $0 \to X \xrightarrow{f} X' \xrightarrow{g} X''$ be a complex in $\mathscr{A}$. Then we have a unique morphism $X \xrightarrow{u} \mathrm{Ker}(g)$ such that the composition of $\mathrm{Ker}(g) \to X'$ and of u is equal to f, and the complex is exact if and only if u is an isomorphism. (In other words, the complex is exact if and only if the morphism $X \to \mathrm{Ker}(g)$ induced by f is an isomorphism.) Indeed,

u decomposes as $X \to \mathrm{Im}(f) \to \mathrm{Ker}(g)$ as in Remark II.2.1.7; if the complex is exact, then $X \to \mathrm{Im}(f)$ is an isomorphism by Lemma II.1.3.3(ii) and $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is an isomorphism by assumption; if u is an isomorphism, then $X \to \mathrm{Im}(f)$ is a monomorphism so $\mathrm{Ker}\, f = 0$ and $X \to \mathrm{Im}(f)$ is an isomorphism by Lemma II.1.3.3(ii), and then we deduce that $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is also an isomorphism.

Similarly, a complex $X \xrightarrow{f} X' \xrightarrow{g} X'' \to 0$ is exact if and only if the morphism $\mathrm{Coker}\, f \to X''$ induced by g is an isomorphism.

Lemma II.2.1.10. *Let $f : X \to Y$ be a morphism in $\mathscr{A}$. Then the following two sequences are exact :*

$$0 \to \mathrm{Ker}\, f \to X \to \mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f) \to 0$$

and

$$0 \to \mathrm{Im}(f) \to Y \to \mathrm{Coker}(f) \to 0.$$

Proof. It suffices to prove that the first sequence is exact. (The second is its analogue in $\mathscr{A}^{\mathrm{op}}$.) As $\mathrm{Coim}(f)$ is the cokernel of the morphism $\mathrm{Ker}(f) \to X$, it suffices to prove that, if $g : A \to B$ is a monomorphism in $\mathscr{A}$, then the sequence $0 \to A \xrightarrow{g} B \xrightarrow{h} \mathrm{Coker}(g) \to 0$ is exact. By the remark above, the exactness on the left follows from the fact that g is a monomorphism, and the exactness on the left follows from the fact that h is an epimorphism (because it is a cokernel morphism). Also, the fact that $h \circ g = 0$ follows from the definition of the cokernel. So we just have to check that the canonical morphism $\mathrm{Im}(g) \to \mathrm{Ker}(h)$ is an isomorphism; but this is true by definition of $\mathrm{Im}(g)$.

□

Proposition II.2.1.11. *Let $0 \to X \xrightarrow{f} X' \xrightarrow{g} X'' \to 0$ be a short exact sequence in $\mathscr{A}$. Then the following assertions are equivalent :*

(a) there exists $h : X'' \to X'$ such that $g \circ h = \mathrm{id}_{X''}$;

(b) there exists $k : X' \to X$ such that $k \circ f = \mathrm{id}_X$;

(c) there exists $\varphi = \begin{pmatrix} k \\ g \end{pmatrix} : X' \to X \oplus X''$ and $\psi = \begin{pmatrix} f & h \end{pmatrix} : X \oplus X'' \to X'$ that are mutually inverse isomorphisms;

(d) there exist morphisms $k : X' \to X$ and $h : X'' \to X'$ such that $\mathrm{id}_X = k \circ f$, $\mathrm{id}_X = f \circ k + h \circ g$ and $\mathrm{id}_{X''} = g \circ h$; [4]

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & X' \arrow[r, "g"] \arrow[d, "\mathrm{id}_{X'}"] \arrow[dl, "k"'] & X'' \arrow[r] \arrow[d, "\mathrm{id}_{X''}"] \arrow[dl, "h"'] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"'] & X' \arrow[r, "g"'] & X'' \arrow[r] & 0 \end{tikzcd}$$

[4]In other words, the complex $0 \to X \to X' \to X'' \to 0$ is homotopic to 0.

(e) there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & X' \arrow[r, "g"] \arrow[d, "\varphi"'] & X'' \arrow[r] \arrow[d, "\mathrm{id}_{X''}"'] & 0 \\
0 \arrow[r] & X \arrow[r, "i"'] & X \oplus X'' \arrow[r, "p"'] & X'' \arrow[r] & 0
\end{tikzcd}$$

where $i = \begin{pmatrix} \mathrm{id}_X \\ 0 \end{pmatrix}$ *and* $0 = \begin{pmatrix} 0 & \mathrm{id}_{X''} \end{pmatrix}$.

Proof. We show that (a) implies (c). Take $\psi = \begin{pmatrix} f & h \end{pmatrix} : X \oplus X'' \to X'$. As $g = g \circ h \circ g$, we have $g \circ (\mathrm{id}_{X'} - h \circ g) = 0$, so there exists $k' : X' \to \mathrm{Ker}(g)$ such that $\mathrm{id}_{X'} - h \circ g$ is the composition of the canonical morphism $\mathrm{Ker}(g) \to X'$ and of k'. Composing this with the inverse of $X \xrightarrow{\sim} \mathrm{Im}(f) \xrightarrow{\sim} \mathrm{Ker}(g)$, we get $k : X' \to X$ such that $f \circ k = \mathrm{id}_{X'} - h \circ g$. We have $f \circ k \circ h = h - h \circ g \circ h = h - h = 0$, hence $k \circ h = 0$ because f is a monomorphism. Similary, $f \circ k \circ f = f - h \circ g \circ f = f$, so $k \circ f = \mathrm{id}_X$.

Let $\varphi = \begin{pmatrix} k \\ g \end{pmatrix} : X' \to X \oplus X''$. Then $\psi \circ \varphi = f \circ k + h \circ g = \mathrm{id}_{X'}$, and $\varphi \circ \psi = \begin{pmatrix} k \circ f & k \circ h \\ g \circ f & g \circ h \end{pmatrix} = \mathrm{id}_{X \oplus X''}$.

The proof that (b) implies (c) is similar (it is the proof that (a) implies (c) in $\mathscr{A}^{\mathrm{op}}$).

The fact that (c) implies (a), (b) and (e) is clear, and (d) is just another way to write (c). Finally, (e) implies (b), by taking for $k : X' \to X$ the first component of φ.

□

Definition II.2.1.12. If a short exact sequence satisfies the equivalent conditions of Proposition II.2.1.11, we say that it is *split*.

We say that the abelian category $\mathscr{A}$ is *semisimple* if every short exact sequence in $\mathscr{A}$ is split.

Example II.2.1.13. (1) The category $\mathbf{Ab}$ is not semisimple. More generally, if R is a ring, the catgeory ${}_R\mathbf{Mod}$ is not semisimple in general. In fact, ${}_R\mathbf{Mod}$ is semisimple if and only the ring R is semisimple, which, by the Artin-Wedderburn structure theorem, is equivalent to the fact that R is a finite direct product of matrix rings over division algebras.

(2) If k is a field, the category of k-vector spaces is semisimple.

(3) Let G be a finite group and k be a field. Then the category of representations of G on a finite-dimensional k-vector spaces is semisimple if and only if $\mathrm{char}(k)$ does not divide the order of G.

Remark II.2.1.14. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor, with $\mathscr{A}$ and $\mathscr{B}$ abelian categories. if $0 \to X \to X' \to X'' \to 0$ is a split exact sequence in $\mathscr{A}$, then the sequence $0 \to F(X) \to F(X') \to F(X'') \to 0$ is exact. Indeed, the split exact sequence

$0 \to X \to X' \to X'' \to 0$ is isomorphism to the sequence $0 \to X \to X \oplus X'' \to X'' \to 0$, where the morphism are the obvious ones, so the sequence $0 \to F(X) \to F(X') \to F(X'') \to 0$ is isomorphic to $0 \to F(X) \to F(X) \oplus F(X'') \to F(X'') \to 0$, which is exact.

The following result is proved in problem set 4.

Proposition II.2.1.15. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathscr{A}$. Then the following statements are equivalent :

(i) The canonical morphism $A \to B \times_D C$ is an epimorphism.

(ii) The canonical morphism $B \sqcup_A C \to D$ is a monomorphism.

In particular, if the morphism $A \to B \oplus C$ is a monomorphism and the morphism $B \oplus C \to D$ is an epimorphism, then $A \overset{\sim}{\to} B \times_D C$ if and only if $B \sqcup_A C \overset{\sim}{\to} D$.

Consider a commutative square as in the proposition. If $A \overset{\sim}{\to} B \times_D C$, we say that the square is *cartesian* or a *pullback square*. If $B \sqcup_A C \overset{\sim}{\to} D$, we say that the square is *cocartesian* or a *pushout square*.

Corollary II.2.1.16. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

in $\mathscr{A}$.

(i). Suppose that the square is cocartesian and that f is injective. Then g is injective.

(ii). Suppose that the square is cartesian and that g is surjective. Then f is surjective.

Proof. It suffices to prove (i). If f is injective, then the morphism $A \to B \times C$ is also injective, so the square is cartesian by Proposition II.2.1.15. Let $h : Z \to B$ be a morphism such that $g \circ h = 0$. Then $g \circ h = v \circ 0$, so the morphism $\begin{pmatrix} h \\ 0 \end{pmatrix} : Z \to B \oplus C$ factors through $k : Z \to A$, that is, we have $f \circ k = 0$ and $u \circ k = h$. As f is injective, this implies that $k = 0$, and so $h = 0$. This shows that g is injective.

□

II.2.2 Subobjects and quotients

We start with some useful conventions. Let $f : X \to Y$ be a morphism. If $\mathrm{Ker}(f) = 0$, we say that f is *injective* (this is also equivalent to f being a monomorphism). If $\mathrm{Coker}(f) = 0$, we say that f is surjective (this is also equivalent to f being an epimorphism).

Definition II.2.2.1. Let X be an object of $\mathscr{A}$.

(i). The set of *objects* of X is the quotient set $\{(Y, f) \text{ with } Y \to X \text{ injective}\}/\sim$, where $(Y_1, f_1) \sim (Y_2, f_2)$ if there exists an isomorphism $\varphi : Y_1 \to Y_2$ such that $f_1 = f_1 \circ \varphi$. (This is clearly an equivalence relation.)

(ii). The set of *quotients* of X is the quotient set $\{(Y, f) \text{ with } X \to Y \text{ surjective}\}/\sim$, where $(Y_1, f_1) \sim (Y_2, f_2)$ if there exists an isomorphism $\varphi : Y_1 \to Y_2$ such that $f_2 = \varphi \circ f_1$. (This is clearly an equivalence relation.)

(iii). If $0 \to X \to X' \to X'' \to 0$ is a short exact sequence, we say that X' is an *extension* of X by X''.

We often write a subobject (resp. quotient) of X as $Y \subset X$(resp. $X \twoheadrightarrow Y$); the fact that we took the quotient by the equivalence relation $\sim$ are implicit.

Lemma II.2.2.2. *The map sending an injective morphism $f : Y \to X$ to the canonical morphism $X \to \mathrm{Coker}(f)$ induces a bijection from the set of subobjects of X to the set of quotients of X, whose inverse comes from the map sending a surjective morphism $f' : X \to Y'$ to the canonical morphism $\mathrm{Ker}(f') \to X$.*

If $Y \subset X$ is a subobject of X, we denote its image by this bijection by $X \to X/Y$.

Lemma II.2.2.3. *(i). If $f : Y_1 \to X$ and $f_2 : Y_2 \to X$ are two injective morphisms, and if $p_1 : Y_1 \times_X Y_2 \to Y_1$ and $p_2 : Y_1 \times_X Y_2 \to Y_2$ are the two canonical projections, then $g = f_1 \circ p_1 = f_2 \circ p_2$ is also an injective morphism.*

(ii). If $f : X \to Y_1$ and $f : X \to Y_2$ are two surjective morphisms, and if $q_1 : Y_1 \to Y_1 \oplus_X Y_2$ and $q_2 : Y_2 \to Y_1 \oplus_X Y_2$ are the canonical morphisms, then $g = q_1 \circ p_1 = q_2 \circ f_2$ is also surjective.

Proof. We prove (i). For every object Z of $\mathscr{A}$, the morphism $\mathrm{Hom}_{\mathscr{A}}(Z, g) : \mathrm{Hom}_{\mathscr{A}}(Z, Y_1 \times_X Y_2) \to \mathrm{Hom}_{\mathscr{A}}(Z, X)$ is the composition

$$\mathrm{Hom}_{\mathscr{A}}(Z, Y_1 \times_X Y_2) \simeq \mathrm{Hom}_{\mathscr{A}}(Z, Y_1) \times_{\mathrm{Hom}_{\mathscr{A}}(Z,X)} \mathrm{Hom}_{\mathscr{A}}(Z, Y_2) \to \mathrm{Hom}_{\mathscr{A}}(Z, X),$$

where the second map is $u \longmapsto f_1 \circ u = f_2 \circ u$, so it is injective; this shows that g is a monomorphism.

□

We have just shown that, if $Y_1 \subset X$ and $Y_2 \subset X$ are subobjects of X, then $Y_1 \times_X Y_2$ is also a subobject of X; we denote this subobject by $Y_1 \cap Y_2$. [5] We also denote the image of the map $\begin{pmatrix} f_1 & f_2 \end{pmatrix} : Y_1 \oplus Y_2 \to X$ by $Y_1 + Y_2 \subset X$.

Notation II.2.2.4. If $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are subobjects of X, we write $Y_1 \subset Y_2$ if there exists a morphism from Y_1 to Y_2 in the category $\mathscr{A}/X$ (if $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are the injective morphisms, this means that there exists $g : Y_1 \to Y_2$ such that $f_1 = f_2 \circ g$).

Note that there is at most one morphism $g : Y_1 \to Y_2$ such that $f_1 = f_2 \circ g$, and that such a morphism $g : Y_1 \to Y_2$ is automatically injective. (Both statements follow from the injectivity of f_2 and f_1.)

Lemma II.2.2.5. *The relation "$Y_1 \subset Y_2$" is a partial order on the set of subobjects of X. This partially ordered set is a lattice, with* $\max(Y_1, Y_2) = Y_1 + Y_2$ *and* $\min(Y_1, Y_2) = Y_1 \cap Y_2$.

Proof. The relation of the lemma is clearly transitive and reflexive. We show that it is antisymmetric. Suppose that $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ are subobjects of X such that $Y_1 \subset Y_2$ and $Y_2 \subset Y_1$, and let $g : Y_1 \to Y_2$ and $h : Y_2 \to Y_1$ be morphisms such that $f_1 = f_2 \circ g$ and $f_2 = f_1 \circ h$. Then $f_2 = f_2 \circ (g \circ h)$ and f_2 is a monomorphism, so $g \circ h = \mathrm{id}_{Y_2}$; similarly, we get that $h \circ g = \mathrm{id}_{Y_1}$. So (Y_1, f_1) and (Y_2, f_2) define the same subobject of X.

We show the second statement. Let Y_1 and Y_2 be as before. We clearly have $Y_1 \cap Y_2 \subset Y_1$ and $Y_1 \cap Y_2 \subset Y_2$; indeed, the two projections p_1 and p_2 from $Y_1 \cap Y_2$ to Y_1 and Y_2 are morphisms in $\mathscr{A}/X$. If $f : Z \to X$ is a subobject of X such that $Z \subset Y_1$ and $Z \subset Y_2$, then we have morphisms $g_1 : Z \to Y_1$ and $g_2 : Z \to Y_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$; by the universal property of the fiber product, there is a unique morphism $h : Z \to Y_1 \cap Y_2$ such that $g_1 = p_1 \circ h$ and $g_2 = p_2 \circ h$; in particular, h is a morphism in $\mathscr{A}/X$, so $Z \subset Y_1 \cap Y_2$.

For the upper bound, let $i_1 : Y_1 \to Y_1 \oplus Y_2$ and $i_2 : Y_2 \to Y_1 \oplus Y_2$ be the canonical morphisms, and let $p : Y_1 \oplus Y_2 \to Y_1 + Y_2$ be the canonical surjection. Then $p \circ i_1$ and $p \circ i_2$ are morphisms in $\mathscr{A}/X$, so $Y_1 \subset Y_1 + Y_2$ and $Y_2 \subset Y_1 + Y_2$. Let $f : Z \to X$ be a subobject of X such that $Y_1 \subset Z$ and $Y_2 \subset Z$. This means that we have two morphisms $Y_1 \to Z$ and $Y_2 \to Z$ in $\mathscr{A}/X$, which induce a morphism $h : Y_1 \oplus Y_2 \to Z$ in $\mathscr{A}/X$. Let $i : \operatorname{Ker} p \to Y_1 \oplus Y_2$ and $j : Y_1 + Y_2 \to X$ be the canonical injections. We have $f \circ h = j \circ p$, so $f \circ h \circ i = 0$; as f is injective, this implies that $h \circ i = 0$, so there exists a unique morphism $k : Y_1 + Y_2 \to Z$ such that $h = k \circ p$. This morphism is a morphism of $\mathscr{A}/X$, and so we have $Y_1 + Y_2 \subset Z$.

$\square$

Lemma II.2.2.6. *If Y_1 and Y_2 are subobjects of X, then the square*

$$\begin{tikzcd} Y_1 \cap Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_1 + Y_2 \end{tikzcd}$$

[5] There is no special notation for the pushout of two quotients of X.

is cartesian and cocartesian.

Proof. This follows immediately from the definition of $Y_1 \cap Y_2$ and from Proposition II.2.1.15. □

In particular, if Y_1 and Y_2 are two subobjects of X and $u : Y_1 \to Z$ and $v : Y_2 \to Z$ are two morphisms, then there exists a morphism $w : Y_1 + Y_2 \to Z$ such that $w_{|Y_1} = u$ and $w_{|Y_2} = v$ if and only if $w_{|Y_1 \cap Y_2} = w_{|Y_1 \cap Y_2}$.

Let I be a family of subobjects of X; in particular, I is a poset, so we can think of it as a category. We can lift this family to a functor $I \to \mathscr{A}/X$ in the following way : For each $i \in I$, choose a representative $f_i : Y_i \to X$ of i. If i and j are elements of I such that $i \leq j$, then there exists a unique morphism $u_{ij} : Y_i \to Y_j$ such that $f_i = f_j \circ u_{ij}$; thanks to this uniqueness, if k is another element of I such that $j \leq k$, then $u_{ik} = u_{jk} \circ u_{ij}$. This defines a functor Φ from I to $\mathscr{A}/X$; also, if we choose different representatives for the elements of I and construct another $\Phi' : I \to \mathscr{A}/X$ as before, there is a unique isomorphism from Φ to Φ'. In particular, the colimit of Φ is well-defined up to unique isomorphism if it exists. Now suppose that I is filtrant (for example totally ordered) and that $\varinjlim \Phi$ exists. If we assume that filtrant colimits are exact in $\mathscr{A}$ (which we often will), then the canonical morphism $\varinjlim \Phi \to X$ is injective; in other words, $\varinjlim \Phi$ is also a subobject of X, and we will denote it by $\bigcup_{i \in I} Y_i$.

Here is some more useful notation : Let $g : X \to X'$ be a morphism. If $f : Y \to X$ is a subobject of X, we often write $g_{|Y} : Y \to X'$ instead of $g \circ f$ and $g(Y)$ instead of $\mathrm{Im}(g_{|Y})$ (this is a subobject of X'). Also, if $Y' \subset X'$ is a subobject of X', we write $g^{-1}(Y') \to X$ for the second projection $Y' \times_{X'} X \to X$; this is a monomorphism (the proof is the same as in Lemma II.2.2.3(i)), so $g^{-1}(Y')$ is a subobject of X. Moreover, we have $\mathrm{Ker}\, g = g^{-1}(0) \subset g^{-1}(Y')$; indeed, $g^{-1}(0) = 0 \times_{Y'} Y = \mathrm{Ker}(g, 0) = \mathrm{Ker}(g)$.

II.2.3 Exact functors

In this subsection, $\mathscr{A}$ and $\mathscr{B}$ are abelian categories.

Definition II.2.3.1. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. We say that F is :

(i). *left exact* if it commutes with finite limits;

(ii). *right exact* if it commutes with finite colimits;

(iii). *exact* if it is both left and right exact.

Lemma II.2.3.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor.*

(i). The following assertions are equivalent :

a) F is left exact;

b) F commutes with kernels, that is, for every morphism $f : X \to Y$ is $\mathscr{A}$, the canonical morphism $F(\mathrm{Ker}(f)) \to \mathrm{Ker}(F(f))$ is an isomorphism;

c) for any exact sequence $0 \to X \to X' \to X''$ in $\mathscr{A}$, the sequence $0 \to F(X) \to F(X') \to F(X'')$ is exact in $\mathscr{B}$;

d) for any exact sequence $0 \to X \to X' \to X'' \to 0$ in $\mathscr{A}$, the sequence $0 \to F(X) \to F(X') \to F(X'')$ is exact in $\mathscr{B}$.

(ii). The following assertions are equivalent :

a) F is right exact;

b) F commutes with cokernels, that is, for every morphism $f : X \to Y$ is $\mathscr{A}$, the canonical morphism $\mathrm{Coker}(F(f)) \to F(\mathrm{Coker}(f))$ is an isomorphism;

c) for any exact sequence $X \to X' \to X'' \to 0$ in $\mathscr{A}$, the sequence $F(X) \to F(X') \to F(X'') \to 0$ is exact in $\mathscr{B}$;

d) for any exact sequence $0 \to X \to X' \to X'' \to 0$ in $\mathscr{A}$, the sequence $F(X) \to F(X') \to F(X'') \to 0$ is exact in $\mathscr{B}$.

(iii). The following assertions are equivalent :

a) F is exact;

b) for any exact sequence $X \to X' \to X''$ in $\mathscr{A}$, the sequence $F(X) \to F(X') \to F(X'')$ is exact in $\mathscr{B}$;

c) for any exact sequence $0 \to X \to X' \to X'' \to 0$ in $\mathscr{A}$, the sequence $0 \to F(X) \to F(X') \to F(X'') \to 0$ is exact in $\mathscr{B}$.

Proof. Point (ii) is point (i) for the opposite categories. We prove (i). As additive functors commute with finite products by Corollary II.1.2.5, the equivalence of (a) and (b) follows from Subsection I.5.5.2. It is clear that (c) implies (d), and the fact that (b) implies (c) follows from Remark II.2.1.9(2). It remains to show that (d) implies (b). Suppose that (d) holds, and let $f : X \to Y$ be a morphism. If f is a monomorphism, then the sequence $0 \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0$ is exact, so the sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \to F(\mathrm{Coker}\, f)$ is also exact, which means that $F(f)$ is a monomorphism. In general, we have a short exact sequence $0 \to \mathrm{Ker}(f) \to X \to \mathrm{Im}(f) \to 0$, so we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(\mathrm{Ker}(f)) \arrow[r] \arrow[d, shift left, "u"] & F(X) \arrow[r] \arrow[d, equal] & F(\mathrm{Im}(f)) \arrow[d, "i"] \\
0 \arrow[r] & \mathrm{Ker}(F(f)) \arrow[r] \arrow[u, dashed, shift left, "v"] & F(X) \arrow[r, "F(f)"'] & F(Y)
\end{tikzcd}$$

where both rows are exact. As $\mathrm{Im}(f) \to Y$ is a monomorphism, so is i; so the composition $\mathrm{Ker}(F(f)) \to F(X) \to F(\mathrm{Im}(f))$ is 0, and so we get a unique morphism

$v : \mathrm{Ker}(F(f)) \to F(\mathrm{Ker}(f))$ making the diagram commute. As $F(\mathrm{Ker}\, f) \to F(X)$ and $\mathrm{Ker}(F(f)) \to F(X)$ are monomorphisms, this forces $u \circ v$ and $v \circ u$ to be the identity morphisms, so we see that $u : F(\ker f) \to \mathrm{Ker}(F(f))$ is indeed an isomorphism.

Now we prove (iii). The equivalence of (iii)(a) and (iii)(c) follows from the equivalence of (a) and (d) in points (i) and (ii), and (iii)(b) obviously implies (iii)(c). Also, by (i) and (ii), if F is exact, then, for every morphism $f : X \to Y$, the canonical morphisms $F(\mathrm{Ker}\, f) \to \mathrm{Ker}(F(f))$and $\mathrm{Coker}(F(f)) \to F(\mathrm{Coker}\, f)$ are isomorphisms, and so we get morphisms $F(\mathrm{Im}\, f) \to \mathrm{Im}\, F(f)$ and $\mathrm{Coim}(F(f)) \to F(\mathrm{Coim}\, f)$, which are also isomorphisms. So (a) implies (b).

□

Proposition II.2.3.3. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be additive functors, and suppose that (F, G) is a pair of adjoint functors. Then F is right exact and G is left exact.*

Proof. This follows from the definition of left and right exact functors and from Proposition I.5.4.3.

□

Corollary II.2.3.4. *(i). For every $X \in \mathrm{Ob}(\mathscr{A})$, the functors $\mathrm{Hom}_{\mathscr{A}}(X, \cdot) : \mathscr{A} \to \mathbf{Ab}$ and $\mathrm{Hom}_{\mathscr{A}}(\cdot, X) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ are left exact.*

(ii). Let $\mathscr{I}$ be a category, and suppose that $\mathscr{A}$ has all limits (resp. colimits) indexed by $\mathscr{I}$. Then the functor $\varprojlim : \mathrm{Func}(\mathscr{I}, \mathscr{A}) \to \mathscr{A}$ (resp. $\varinjlim : \mathrm{Func}(\mathscr{I}, \mathscr{A}) \to \mathscr{A}$) is left exact (resp. right exact).

(iii). Let R be a ring and I be a set. Then the functor $\prod_I : \mathrm{Func}(I, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ and $\bigoplus_I : \mathrm{Func}(I, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ are exact.

(iv). Let R be a ring and $\mathscr{I}$ be a filtrant category. Then the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is exact.

(v). Let R be a ring, M be a left R-module and N be a right R-module. Then the functors $M \otimes_R (\cdot) : {}_R\mathbf{Mod} \to \mathbf{Ab}$ and $(\cdot) \otimes_R N : \mathbf{Mod}_R \to \mathbf{Ab}$ are right exact. If R is commutative, the functor $M \otimes_R (\cdot) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ is right exact.

Proof. Points (i), (ii) and (v) follow from Proposition II.2.3.3. In fact, point (i) also follows from the definitions and Propositions I.5.3.2 and I.5.3.4.

Point (iv) follows from (ii) and Proposition I.5.6.5.

We prove (iii). We already know that $\prod_I$ is left exact and $\bigoplus_I$ is right exact by (ii). To show that $\prod_I$ is exact, it suffices to show that it preserves kernels, which is clear on the explicit description of kernels in the category ${}_R\mathbf{Mod}$. The proof that $\bigoplus_I$ is right exact is similar.

□

Remark II.2.3.5. Points (iii) and (iv) don't hold in a general abelian category, even if we suppose that (co)products indexed by I or colimits indexed by $\mathscr{I}$ exist.

Example II.2.3.6. Here are examples showing that Hom and $\otimes$ are not always exact. We take $\mathscr{A} = \mathbf{Ab}$.

The canonical projection $f : Z \to \mathbb{Z}/2\mathbb{Z}$ is surjective, but there is no morphism $g : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ such that $f \circ g = \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}$; so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, f)$ is not surjective.

Consider the morphism $g : \mathbb{Z} \to \mathbb{Z}$, $x \longmapsto 2x$. Then g is injective, but the morphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ induced by g is the zero morphism; so $g \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is not injective. Also, there is no morphism $h : \mathbb{Z} \to \mathbb{Z}$ such that $h \circ g = \mathrm{id}_{\mathbb{Z}}$; so $\mathrm{Hom}_{\mathbb{Z}}(g, \mathbb{Z})$ is not surjective.

Example II.2.3.7. Let $\mathscr{C}$ be a category, and let A be an object of $\mathscr{C}$. The functor $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \to \mathbf{Ab}$, $F \longmapsto F(A)$ is exact. In other words, the functor from the category of presheaves of abelian groups on $\mathscr{C}$ to $\mathbf{Ab}$ sending a presheaf to its group of sections on A is exact.

In particular, if $\mathscr{C}$ is the category of open subsets of a topological space X, then the global sections functor is exact on the category $\mathrm{PSh}(X, \mathbf{Ab})$ of presheaves of abelian groups on X. On the other, here is a non-exact functor. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open cover of X. If F is a presheaf on X, we write

$$\check{\mathrm{H}}^0(\mathscr{U}, F) = \{(s_i)_{i \in I} \in \prod_{i \in I} F(U_i) \mid \forall i, j \in I,\ s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}\}.$$

Then $F \longmapsto \check{\mathrm{H}}^0(\mathscr{U}, F)$ defines a left exact functor from $\mathrm{PSh}(X, \mathbf{Ab})$ to $\mathbf{Ab}$; this functor is not exact in general.

Example II.2.3.8. Let $\mathscr{I}$ be a category, and suppose that all limits indexed by $\mathscr{I}$ exist in $\mathscr{A}$. Then the left exact functor $\varprojlim : \mathrm{Func}(\mathscr{I}, \mathscr{A}) \to \mathscr{A}$ is not exact in general.

For example, take $\mathscr{I} = \mathbb{N}^{\mathrm{op}}$ and $\mathscr{A} = \mathbf{Mod}_k$, where k is a field. Let J be the ideal (x) in $k[x]$; for every $n \in \mathbb{N}$, we have $J^n = (x^n)$. Consider the functor $F : \mathscr{I} \to \mathscr{A}$ sending $n \in \mathbb{N}$ to the quotient $k[x]/J^n$, and sending a morphism $n \to m$ in $\mathbb{N}$ to the canonical quotient morphism $k[x]/J^m \to k[x]/J^n$. Then $\varprojlim F = k[[x]]$. On the other hand, consider the functor $G : \mathscr{I} \to \mathbf{Mod}_k$ sending $n \in \mathbb{N}$ to J^n, and sending a morphism $n \to m$ in $\mathbb{N}$ to the inclusion $J^m \subset J^n$. Then $\varprojlim G = \bigcap_{n \geq 0} J^n = \{0\}$. Finally, let $H : \mathscr{I} \to \mathbf{Mod}_k$ be the constant functor $n \longmapsto k[x]$. For every $n \in \mathbb{N}$, we have an exact sequence $0 \to J^n \to k[x] \to k[x]/J^n \to 0$. The morphisms in this exact sequence define morphisms of functors $G \to H$ and $H \to F$, so we get an exact sequence $0 \to G \to H \to F \to 0$ in $\mathrm{Func}(\mathscr{I}, \mathbf{Mod}_k)$. But the sequence $0 \to \varprojlim G \to \varprojlim H \to \varprojlim F \to 0$ is $0 \to 0 \to k[x] \to k[[x]] \to 0$, which is not exact.

II.2.4 Injective and projective objects

In this subsection, $\mathscr{A}$ is an abelian category.

Definition II.2.4.1. (i). An object I of $\mathscr{A}$ is called *injective* if the left exact functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, I) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

(ii). An object P of $\mathscr{A}$ is called *projective* if the left exact functor $\mathrm{Hom}_{\mathscr{C}}(P, \cdot) : \mathscr{A} \to \mathbf{Ab}$ is exact.

(iii). We say that $\mathscr{A}$ has *enough injectives* (resp. *enough projectives*) if, for every $X \in \mathrm{Ob}(\mathscr{A})$, there exists a monomorphism $A \to I$ with I injective (resp. there exists an epimorphism $P \to A$ with P projective).

Note that an object is projective in $\mathscr{A}$ if and only if it is injective in $\mathscr{A}^{\mathrm{op}}$.

Proposition II.2.4.2. *(i). Let I be an object of $\mathscr{A}$. Then I is injective if and only if, for any $X, Y \in \mathrm{Ob}(\mathscr{A})$, any monomorphism $f : X \to Y$ and any morphism $u : X \to I$, there exists $v : Y \to I$ such that $v \circ f = u$.*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[dl, dashed, "v"] \\ & I & \end{tikzcd}$$

(ii). Let P be an object of $\mathscr{A}$. Then P is projective if and only if, for any $X, Y \in \mathrm{Ob}(\mathscr{A})$, any epimorphism $f : X \to Y$ and any morphism $u : P \to Y$, there exists $v : P \to X$ such that $u = f \circ v$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "v"'] \arrow[d, "u"] & \\ X \arrow[r, "f"'] & Y \arrow[r] & 0 \end{tikzcd}$$

Proof. We prove (i). (Point (ii) is (i) in the opposite category.)

Assume that I is injective. Applying $\mathrm{Hom}_{\mathscr{A}}(\cdot, I)$ to the exact sequence $0 \to X \to Y \xrightarrow{f} \mathrm{Coker}\, f \to 0$, we get an exact sequence $0 \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Coker}\, f, I) \to \mathrm{Hom}_{\mathscr{A}}(Y, I) \to \mathrm{Hom}_{\mathscr{A}}(X, I) \to 0$. In particular, the map $\mathrm{Hom}_{\mathscr{A}}(Y, I) \to \mathrm{Hom}_{\mathscr{A}}(X, I)$, $v \longmapsto v \circ f$ is surjective, which is what we wanted.

Conversely, assume that the map $\mathrm{Hom}_{\mathscr{A}}(Y, I) \to \mathrm{Hom}_{\mathscr{A}}(X, I)$, $v \longmapsto v \circ f$ is surjective for any monomorphism $f : X \to Y$. Let $0 \to X \xrightarrow{f} X' \xrightarrow{g} X'' \to 0$ be a short exact sequence. Then we know that the sequence $0 \to \mathrm{Hom}_{\mathscr{A}}(X'', I) \to \mathrm{Hom}_{\mathscr{A}}(X', I) \to \mathrm{Hom}_{\mathscr{A}}(X, I)$ is exact because $\mathrm{Hom}_{\mathscr{A}}(\cdot, I)$ is left exact, and that the map $\mathrm{Hom}_{\mathscr{A}}(X', I) \to \mathrm{Hom}_{\mathscr{A}}(X, I)$ is surjective by assumption. So the sequence $0 \to \mathrm{Hom}_{\mathscr{A}}(X'', I) \to \mathrm{Hom}_{\mathscr{A}}(X', I) \to \mathrm{Hom}_{\mathscr{A}}(X, I) \to 0$ is exact. By Lemma II.2.3.2, the functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, I)$ is exact, which means that I is injective.

□

Lemma II.2.4.3. *Let $(X_i)_{i\in I}$ be a family of objects of $\mathscr{A}$, and suppose that its product (resp. coproduct) exists. Then $\prod_{i\in I} X_i$ is injective (resp. $\bigoplus_{i\in I} X_i$ is projective) if and only if all the X_i are.*

Proof. We have isomorphisms of functors $\mathrm{Hom}_{\mathscr{A}}(\prod_{i\in I} X_i, \cdot) \simeq \prod_{i\in I} \mathrm{Hom}_{\mathscr{A}}(X_i, \cdot)$ and $\mathrm{Hom}_{\mathscr{A}}(\cdot, \bigoplus_{i\in I} X_i) \simeq \prod_{i\in I} \mathrm{Hom}_{\mathscr{A}}(\cdot, X_i)$, and, in $\mathbf{Ab}$, a product of sequences is exact if and only if each of the sequences is exact.

□

Here is a related result.

Lemma II.2.4.4. *Let $(F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A})$ be a pair of additive adjoint functors between abelian categories.*

(i). If G is exact, then F sends projective objects to projective objects.

(ii). If F is exact, then G sends injective objects to injective objects.

Proof. As usual, it suffices to prove (i). Let P be a projective object of $\mathscr{A}$. Then the functor $\mathrm{Hom}_{\mathscr{B}}(F(P), \cdot) : \mathscr{B} \to \mathbf{Ab}$ is isomorphic to the exact functor $\mathrm{Hom}_{\mathscr{A}}(P, G(\cdot))$, so it is exact.

□

Corollary II.2.4.5. *Let $0 \to X \xrightarrow{f} X' \xrightarrow{g} X'' \to 0$ be a short exact sequence in $\mathscr{A}$. If X is injective (resp. if X'' is projective), then this sequence splits.*

Proof. Suppose that X is injective. By Proposition II.2.4.2, there exists $v : X' \to X$ such that $v \circ f = \mathrm{id}_X$, so the sequence splits. To get the second assertion, apply this to $\mathscr{A}^{\mathrm{op}}$.

□

Applying Lemma II.2.4.3, we immediately get the following corollary.

Corollary II.2.4.6. *Let $0 \to X \to X' \to X'' \to 0$ be a short exact sequence in $\mathscr{A}$. If X and X' are injective, then X'' is also injective. If X' and X'' are projective, then X is also projective.*

Example II.2.4.7. (1) Let R be a ring. We claim that projective objects in ${}_R\mathbf{Mod}$ (resp. $\mathbf{Mod}_R$) are exactly direct summands of free R-modules, and finitely generated projective R-modules are direct summands of finitely generated free R-modules. In particular, the catgeories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ have enough projectives.

Indeed, notice first that R itself (with its obvious structure of left R-module) is projective, because we have an isomorphism of functors from $\mathrm{Hom}_R(R, \cdot)$ to $\mathrm{id}_{{}_R\mathbf{Mod}}$ given by $\mathrm{Hom}_R(R, M) \to M$, $u \longmapsto u(1)$. By Lemma II.2.4.3, every free R-module is projective. By the same lemma, every direct summand of a free R-module is projective.

Now let P be a projective R-module. Then there exists a free R-module F and a surjective R-linear map $f : F \to P$. (Take X a set of generators of P, for example P itself, take F equal to the free R-module on the set X, and f sending the generator corresponding to $x \in X$ to x.) By Corollary II.2.4.5, the short exact sequence $0 \to \mathrm{Ker}(f) \to F \overset{f}{\twoheadrightarrow} P \to 0$ split, so P is a direct factor of F. Note that, if P is finitely generated, we can choose F to be finitely generated.

(2) If R is a principal ideal domain, projective R-modules are the same as free R-modules. For finitely generated projective R-modules, this is a consequence of the structure theorem for finitely generated R-modules.

Example II.2.4.8. If $\mathscr{A} = {}_R\mathbf{Mod}$, then an object of $\mathscr{A}$ is injective if and only if it satisfies the condition of Proposition II.2.4.2(i) for f the injection of a left ideal of R into R. This is an immediate corollary of Proposition II.3.2.3, as ${}_R\mathbf{Mod}$ admits colimits and R is a generator of ${}_R\mathbf{Mod}$.

In particular, if R is a principal ideal domain, then a R-module I is injective if and only if it is divisible (we say that I is divisible if, for every $a \in R - \{0\}$, the map $I \to I$, $x \longmapsto ax$ is surjective.)

II.3 Generators and cogenerators

II.3.1 Morita's theorem

Definition II.3.1.1. Let $\mathscr{C}$ be a category. We say that an object X of $\mathscr{C}$ is a *generator* (resp. a *cogenerator*) if the functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \to \mathbf{Set}$ (resp. $\mathrm{Hom}_{\mathscr{C}}(\cdot, X) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$) is conservative.

Example II.3.1.2. (1) A singleton is a generator in $\mathbf{Set}$.

(2) The left R-module R is a projective generator in ${}_R\mathbf{Mod}$.

(3) The abelian group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbf{Ab}$.

(4) Let $\mathscr{U}$ be a universe, $\mathscr{C}$ be a $\mathscr{U}$-small category and $\mathbf{Ab} = \mathbf{Ab}_{\mathscr{U}}$. For every $X \in \mathrm{Ob}(\mathscr{C})$, we denote by $\mathbb{Z}^{(X)}$ the presheaf in $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ sending $Y \in \mathrm{Ob}(\mathscr{C})$ to the free abelian group on $\mathrm{Hom}_{\mathscr{C}}(Y, X)$; we call $\mathbb{Z}^{(X)}$ the free presheaf of abelian groups on the presheaf $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$. Then $\bigoplus_{X \in \mathrm{Ob}(\mathscr{C})} \mathbb{Z}^{(X)} = \bigoplus_{X \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a projective generator in $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$. [6] Indeed, for every object F of $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$, the Yoneda lemma and the universal property of free abelian groups and of the direct sum gives an

[6] The restriction on the size of $\mathscr{C}$ is just there to make sure this direct sum exists.

isomorphism

$$\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}^{\mathrm{op}},\mathbf{Ab})}(\bigoplus_{X\in\mathrm{Ob}(\mathscr{C})}\mathbb{Z}^{(X)},F)\overset{\sim}{\to}\prod_{X\in\mathrm{Ob}(\mathscr{C})}\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}^{\mathrm{op}},\mathbf{Set})}(h_X,F)\overset{\sim}{\to}\prod_{X\in\mathrm{Ob}(\mathscr{C})}F(X),$$

which is easily seen to be an isomorphism of abelian groups. So the functor $\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}^{\mathrm{op}},\mathbf{Ab})}(\bigoplus_{X\in\mathrm{Ob}(\mathscr{C})}\mathbb{Z}^{(X)},\cdot)$ is faithful, and it is exact because products preserve exact sequences in $\mathbf{Ab}$.

We fix a universe $\mathscr{U}$ and an abelian $\mathscr{U}$-category $\mathscr{A}$.

Proposition II.3.1.3. *(i). Let Q be a generator of $\mathscr{A}$. Then :*

a) The functor $\mathrm{Hom}_{\mathscr{A}}(Q,\cdot):\mathscr{A}\to\mathbf{Ab}$ is faithful.

b) If $X\in\mathrm{Ob}(\mathscr{A})$, then $X\simeq 0$ if and only if $\mathrm{Hom}_{\mathscr{A}}(Q,X)=\{0\}$.

c) A morphism $f:X\to Y$ is of $\mathscr{A}$ injective if and only if $\mathrm{Hom}_{\mathscr{A}}(Q,f)$ is.

d) Let $f:X\to Y$ be a morphism of $\mathscr{A}$. If $\mathrm{Hom}_{\mathscr{A}}(Q,f)$ is surjective, then f is surjective.

e) Suppose that $\mathscr{A}$ admits all direct sums indexed by sets $I\in\mathscr{U}$. For any $X\in\mathrm{Ob}(\mathscr{A})$, consider the morphism $\bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,X)}Q\to X$ whose composition with the injection corresponding to $f\in\mathrm{Hom}_{\mathscr{A}}(Q,X)$ is f; then this morphism is surjective.

f) For every object X of $\mathscr{A}$, the set of subobjects of X and the set of quotients of X are isomorphic to elements of $\mathscr{U}$.

(ii). Let J be a cogenerator of $\mathscr{A}$. Then :

a) The functor $\mathrm{Hom}_{\mathscr{A}}(\cdot,J):\mathscr{A}^{\mathrm{op}}\to\mathbf{Ab}$ is faithful.

b) If $X\in\mathrm{Ob}(\mathscr{A})$, then $X\simeq 0$ if and only if $\mathrm{Hom}_{\mathscr{A}}(X,J)=\{0\}$.

c) A morphism $f:X\to Y$ is of $\mathscr{A}$ surjective if and only if $\mathrm{Hom}_{\mathscr{A}}(f,J)$ is injective.

d) Let $f:X\to Y$ be a morphism of $\mathscr{A}$. If $\mathrm{Hom}_{\mathscr{A}}(f,J)$ is surjective, then f is injective.

e) Suppose that $\mathscr{A}$ admits all direct project indexed by sets $I\in\mathscr{U}$. For any $X\in\mathrm{Ob}(\mathscr{A})$, consider the morphism $X\to\prod_{\mathrm{Hom}_{\mathscr{A}}(X,J)}J$ whose composition with the projection on the factor corresponding to $f\in\mathrm{Hom}_{\mathscr{A}}(X,J)$ is f; then this morphism is injective.

f) For every object X of $\mathscr{A}$, the set of subobjects of X and the set of quotients of X are elements of $\mathscr{U}$.

Proof. We only prove (i) (point (ii) follows by applying (i) to $\mathscr{A}^{\mathrm{op}}$).

Let $f:X\to Y$ be a morphism of $\mathscr{A}$ such that $\mathrm{Hom}_{\mathscr{A}}(Q,f)=0$. Let $u:\mathrm{Ker}(f)\to X$ be the kernel of f. As the functor $\mathrm{Hom}_{\mathscr{A}}(Q,\cdot)$ is left exact, the canonical morphism $\mathrm{Hom}_{\mathscr{A}}(Q, \operatorname{Ker} f) \to \operatorname{Ker}(\mathrm{Hom}_{\mathscr{A}}(Q, f))$ is an isomorphism, so $\mathrm{Hom}_{\mathscr{A}}(Q, u) : \mathrm{Hom}_{\mathscr{A}}(Q, \operatorname{Ker} f) \to \mathrm{Hom}_{\mathscr{A}}(Q, X)$ is an isomorphism. As Q is a generator, this implies that u is an isomorphism, hence that $f = 0$. So $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ is conservative. This proves (a)

Let $X \in \mathrm{Ob}(\mathscr{A})$. If $X \simeq 0$, then obviously $\mathrm{Hom}_{\mathscr{A}}(Q, X) = \{0\}$. Conversely, if $\mathrm{Hom}_{\mathscr{A}}(Q, X) = \{0\}$, then the zero map in $\mathrm{End}_{\mathscr{A}}(X)$ induces an isomorphism of $\mathrm{Hom}_{\mathscr{A}}(Q, X)$, so it is an isomorphism, so $X \simeq 0$. This proves (b).

Let $f : X \to Y$ be a morphism. As $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ is left exact, we have $\mathrm{Hom}_{\mathscr{A}}(Q, \operatorname{Ker} f) \simeq \operatorname{Ker}(\mathrm{Hom}_{\mathscr{A}}(Q, f))$, so, by (b), we get that $\operatorname{Ker}(\mathrm{Hom}_{\mathscr{A}}(Q, f)) \simeq 0$ if and only if $\operatorname{Ker} f \simeq 0$; this shows that f is injective if and only if $\mathrm{Hom}_{\mathscr{A}}(Q, f)$ is injective. This proves (c). Now assume that $\mathrm{Hom}_{\mathscr{A}}(Q, f)$ is surjective, and let $p : Y \to \operatorname{Coker} f$ be the canonical morphism. For every $g : Q \to Y$, the hypothesis says that there exists $h : Q \to X$ such that $g = f \circ h$, so $p \circ g = p \circ f \circ h = 0$. In other words, we have $\mathrm{Hom}_{\mathscr{A}}(Q, p) = 0$. By (a), this implies that $p = 0$, hence that f is surjective. This proves (d).

Let $X \in \mathrm{Ob}(\mathscr{A})$, and let $g : \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,X)} Q \to X$ be the morphism of (e). The map $\mathrm{Hom}_{\mathscr{A}}(Q, g) : \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,X)} \mathrm{Hom}_{\mathscr{A}}(Q, Q) \to \mathrm{Hom}_{\mathscr{A}}(Q, X)$ is the map whose composition with the injection $\mathrm{Hom}_{\mathscr{A}}(Q, Q) \to \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,X)} \mathrm{Hom}_{\mathscr{A}}(Q, Q)$ corresponding to $f \in \mathrm{Hom}_{\mathscr{A}}(Q, X)$ sends $u \in \mathrm{Hom}_{\mathscr{A}}(Q, Q)$ to $f \circ u$. Any $f \in \mathrm{Hom}_{\mathscr{A}}(Q, X)$ is clearly in the image of this map (because $f = f \circ \mathrm{id}_Q$), so g is surjective by (d). This proves (e).

We finally prove (f). Let $X \in \mathrm{Ob}(\mathscr{A})$. As the set of subobjects and the set of quotients of X are in bijection, it suffices to prove the statement about the set of subobjects. Let $\Omega = \mathrm{Hom}_{\mathscr{A}}(Q, X)$, and consider the map spending a couple $(Y, f : Y \to X)$ to the subset $f \circ \mathrm{Hom}_{\mathscr{A}}(Q, Y)$ of Ω. This induces a map from the set of subobjects of X to $\mathfrak{P}(\Omega)$, and it suffices to show that this map is injective. Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two injective morphisms, and suppose that $f_1 \circ \mathrm{Hom}_{\mathscr{A}}(Q, Y_1) = f_2 \circ \mathrm{Hom}_{\mathscr{A}}(Q, Y_2)$. Let $p_1 : Y_1 \cap Y_2 = Y_1 \times_X Y_2 \to Y_1$ and $p_2 : Y_1 \cap Y_2 \to Y_2$ be the canonical projections. As $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ is left exact, we have $\mathrm{Hom}_{\mathscr{A}}(Q, Y_1 \times_X Y_2) = \mathrm{Hom}_{\mathscr{A}}(Q, Y_1) \times_{\mathrm{Hom}_{\mathscr{A}}(Q,X)} \mathrm{Hom}_{\mathscr{A}}(Q, Y_2)$, so the hypothesis implies that $\mathrm{Hom}_{\mathscr{A}}(Q, p_1)$ and $\mathrm{Hom}_{\mathscr{A}}(Q, p_2)$ are bijections. As $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ is conservative, this shows that p_1 and p_2 are isomorphisms, so (Y_1, f_1) and (Y_2, f_2) represent the same subobject of X.

□

Proposition II.3.1.4. *(i). Suppose that $\mathscr{A}$ admits all direct sums indexed by sets $I \in \mathscr{U}$, and let Q be an object of $\mathscr{A}$. Then the following are equivalent :*

a) Q is a generator;

b) the functor $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot) : \mathscr{A} \to \mathbf{Ab}$ is faithful;

c) for any object X of $\mathscr{A}$, there exist a set $I \in \mathscr{U}$ and a surjective morphism $\bigoplus_I Q \to X$.

(ii). Suppose that $\mathscr{A}$ admits all direct products indexed by sets $I \in \mathscr{U}$, and let J be an object of $\mathscr{A}$. Then the following are equivalent :

a) *J is a cogenerator;*

b) *the functor* $\mathrm{Hom}_{\mathscr{A}}(\cdot, J) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ *is faithful;*

c) *for any object X of $\mathscr{A}$, there exist a set $I \in \mathscr{U}$ and an injective morphism $X \to \prod_I J$.*

Proof. We only prove (i) (point (ii) follows by applying (i) to $\mathscr{A}^{\mathrm{op}}$). The fact (a) implies (b) and (c) is part of Proposition II.3.1.3.

We prove that (b) implies (a). Suppose that (b) holds. Let $f : X \to Y$ be a morphism in $\mathscr{A}$, and assume that $\mathrm{Hom}_{\mathscr{A}}(Q, f)$ is a bijection. Let $i : \mathrm{Ker}\, f \to X$ and $p : Y \to \mathrm{Coker}\, f$ be the canonical morphisms. We have $\mathrm{Hom}_{\mathscr{A}}(Q, f) \circ \mathrm{Hom}_{\mathscr{A}}(Q, i) = 0$, so $\mathrm{Hom}_{\mathscr{A}}(Q, i) = 0$, so $i = 0$, so $\mathrm{Ker}\, f \simeq 0$. Similarly, we have $\mathrm{Hom}_{\mathscr{A}}(Q, p) \circ \mathrm{Hom}_{\mathscr{A}}(Q, f) = 0$, so $\mathrm{Hom}_{\mathscr{A}}(Q, p) = 0$, so $p = 0$, so $\mathrm{Coker}\, f \simeq 0$. By Proposition II.2.1.6, this implies that f is an isomorphism.

Finally, we prove that (c) implies (b). Suppose that (c) holds. Let $f : X \to Y$ be a morphism such that $\mathrm{Hom}_{\mathscr{A}}(Q, f) = 0$. By (c), there exists a set $I \in \mathscr{U}$ and a surjective morphism $u : \bigoplus_I Q \to X$; as $\mathrm{Hom}_{\mathscr{A}}(\bigoplus_I Q, X) \simeq \mathrm{Hom}_{\mathscr{A}}(Q, X)^I$, we can write u as a family $(u_i)_{i \in I}$ of elements of $\mathrm{Hom}_{\mathscr{A}}(Q, X)$. By the hypothesis on f, we then have $f \circ u = 0$, which implies that $f = 0$ because u is surjective.

□

Corollary II.3.1.5. *(i). Suppose that $\mathscr{A}$ admits all direct sums indexed by sets in $\mathscr{U}$, and let P be an object of $\mathscr{A}$. Then the following are equivalent :*

a) *P is a projective generator;*

b) *the functor* $\mathrm{Hom}_{\mathscr{A}}(P, \cdot) : \mathscr{A} \to \mathbf{Ab}$ *is exact and faithful;*

c) *P is projective, and for every nonzero $X \in \mathrm{Ob}(\mathscr{C})$, there exists a nonzero morphism from P to X.*

Moreover, if $\mathscr{A}$ has a projective generator, then it has enough projective objects.

(ii). Suppose that $\mathscr{A}$ admits all direct products indexed by sets in $\mathscr{U}$, and let I be an object of $\mathscr{A}$. Then the following are equivalent :

a) *P is an injective cogenerator;*

b) *the functor* $\mathrm{Hom}_{\mathscr{A}}(\cdot, I) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ *is exact and faithful;*

c) *I is injective, and for every nonzero $X \in \mathrm{Ob}(\mathscr{C})$, there exists a nonzero morphism from X to I.*

Moreover, if $\mathscr{A}$ has a injective cogenerator, then it has enough injective objects.

Proof. We only prove (i) (point (ii) follows by applying (i) to $\mathscr{A}^{\mathrm{op}}$).

We prove that (a) implies (b). Suppose that P is a projective generator. As P is projective, the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ is exact. The fact that it is faithful follows from Proposition II.3.1.3.

We show that (b) implies (c) and (a). Suppose that the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ is exact and faithful. Then P is projective (by definition of "projective"). Let X be a nonzero object of $\mathscr{A}$. As $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ is faithful, we have $\mathrm{Hom}_{\mathscr{A}}(P,\mathrm{id}_X)\neq 0$, which means that there exists $g : P \to X$ such that $g = \mathrm{id}_X \circ g \neq 0$. Also, the fact that P is a generator follows from Proposition II.3.1.4.

Finally, we show that (c) implies (b). Suppose that (c) holds. As P is projective, the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ is exact. Let $f : X \to Y$ be a morphism, and suppose that $f \neq 0$. We want to show that $\mathrm{Hom}_{\mathscr{A}}(P,f)\neq 0$. Write $f = i\circ p$, where $p : X \to \mathrm{Coim}(f) \overset{\sim}{\to} \mathrm{Im}(f)$ is the canonical surjection and $i : \mathrm{Im}(f) \to Y$ is the canonical injection. As $f\neq 0$, the object $\mathrm{Im}(f)$ is not zero, so there exists a nonzero morphism $u : P \to \mathrm{Im}(f)$. As P is projective and p is surjective, there exists $v : P \to X$ such that $u = p\circ v$. Then $f\circ v = i\circ p\circ v = i\circ u \neq 0$, because $u\neq 0$ and i is injective. So $\mathrm{Hom}_{\mathscr{A}}(P,f)$ is not the zero map.

□

We come to our main reason for introducing generators. Let P be an object of $\mathscr{A}$, and let R be the ring $\mathrm{End}_{\mathscr{A}}(P)$. If X is any object of $\mathscr{A}$, we can make R act on $\mathrm{Hom}_{\mathscr{A}}(P,X)$ by : if $r\in R$ and $f\in \mathrm{Hom}_{\mathscr{A}}(P,X)$, then $rf = f\circ r$. This makes $\mathrm{Hom}_{\mathscr{A}}(P,X)$ into a right R-module. It is easy to see that the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ factors through the forgetful functor $\mathbf{Mod}_R \to \mathbf{Ab}$ (this just follows from the associativity of composition); we will still denote the resulting functor $\mathscr{A}\to\mathbf{Mod}_R$ by $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$.

Theorem II.3.1.6 (Morita's theorem). *Suppose that $\mathscr{A}$ has all direct sums indexed by sets in $\mathscr{U}$ and that P is a projective generator of $\mathscr{A}$ such that $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ commutes with direct sums (indexed by sets in $\mathscr{U}$). (Such a P is sometimes called a* progenerator *of $\mathscr{A}$.) Let $R = \mathrm{End}_{\mathscr{A}}(P)$. Then the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot) : \mathscr{A}\to\mathbf{Mod}_R$ is an equivalence of categories.*

Moreover, if S is a ring and $G : \mathscr{A}\to\mathbf{Mod}_S$ is an equivalence of categories, then there exists a projective generator P' of $\mathscr{A}$ such that $F\simeq \mathrm{Hom}_{\mathscr{A}}(P',\cdot)$.

Proof. Let $G = \mathrm{Hom}_{\mathscr{A}}(P,\cdot) : \mathscr{A}\to\mathbf{Mod}_R$. We already know that G is exact and faithful. For every set $X\in\mathscr{U}$, we write $P^{(X)}$ for the colimit of the constant functor $X\to\mathscr{A}$ sending every element of X to P. As $G = \mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ commutes with direct sums, the canonical morphism $R^{(X)} = G(P)^{(X)} \to G(P^{(X)})$ is an isomorphism for every $X\in\mathscr{U}$. (Remember that $R^{(X)}$ is the free R-module with basis X.)

We claim that, if X and Y are sets, then the map $G : \mathrm{Hom}_{\mathscr{A}}(P^{(X)},P^{(Y)}) \to \mathrm{Hom}_R(G(P^{(X)}),G(P^{(Y)})) \simeq \mathrm{Hom}_R(R^{(X)},R^{(Y)})$ is an isomorphism. Indeed, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(P^{(X)}, P^{(Y)}) \arrow[r] \arrow[d, "(1)"'] & \mathrm{Hom}_R(R^{(X)}, R^{(Y)}) \arrow[d, "(1')"] \\
\prod_X \mathrm{Hom}_{\mathscr{A}}(P, P^{(Y)}) \arrow[r] & \prod_X \mathrm{Hom}_R(R, R^{(Y)}) \\
\prod_X \bigoplus_Y \mathrm{Hom}_{\mathscr{A}}(P, P) \arrow[r, "(3)"'] \arrow[u, "(2)"] & \prod_X \bigoplus_Y \mathrm{Hom}_R(R, R) \arrow[u, "(2')"']
\end{tikzcd}$$

where all the horizontal maps are induced by G, maps (1) and (1') are the isomorphisms given by the universal property of the direct product, and maps (2) and (2') are the canonical maps of Subsection I.5.4.2. We know that (2) is an isomorphism by the assumption on P, and (2') is an isomorphism because the functor $\mathrm{Hom}_R(R, \cdot) : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is isomorphic to $\mathrm{id}_{\mathbf{Mod}_R}$, hence commutes with direct sums. Also, the map (3) is an isomorphism because the map $R = \mathrm{Hom}_{\mathscr{A}}(P, P) \xrightarrow{G} \mathrm{Hom}_R(G(P), G(P)) = \mathrm{Hom}_R(R, R)$ is an isomorphism. (Indeed, if $f \in R = \mathrm{Hom}_{\mathscr{A}}(P, P)$, then the map $G(f) = \mathrm{Hom}_{\mathscr{A}}(P, f) : \mathrm{Hom}_{\mathscr{A}}(P, P) \to \mathrm{Hom}_{\mathscr{A}}(P, P)$ sends $g \in \mathrm{Hom}_{\mathscr{A}}(P, P)$ to $f \circ g$; in other words, $G(f) : R \to R$ is left multiplication by f. But we know that every right R-module endomorphism of R is of that type, because R is a free R-module on $1 \in R$.)

We now prove prove that G admits a left adjoint functor $F : \mathbf{Mod}_R \to \mathscr{A}$, and that the unit of this adjunction is an isomorphism. By Proposition I.4.7, it suffices to show that, for every right R-module M, the functor $\Phi_M : \mathscr{A} \to \mathbf{Set}$, $A \longmapsto \mathrm{Hom}_R(M, G(A))$ is representable; then a couple representing this functor is $(F(M), \eta(M))$, where $\eta(M) \in \Phi_M(F(M)) = \mathrm{Hom}_R(M, G(F(M)))$ is the value at M of the unit of the adjunction. If $M = R^{(X)}$ with X a set, then Φ_M is canonically isomorphic to the functor $\prod_X \mathrm{Hom}_R(R, G(\cdot)) \simeq \prod_X G(\cdot) \simeq \prod_X \mathrm{Hom}_{\mathscr{A}}(P, \cdot) \simeq \mathrm{Hom}_{\mathscr{A}}(P^{(X)}, \cdot)$, so it is representable by $P^{(X)}$. Note also that $\eta_M : R^{(X)} \to G(P^{(X)})$ is the inverse of the isomorphism $R^{(X)} \to G(P^{(X)})$ of the first paragraph of the proof. In general, we have an exact sequence $R^{(X)} \xrightarrow{u} R^{(Y)} \to M \to 0$, with X and Y sets. By the previous paragraph, there exists a unique morphism $f : P^{(X)} \to P^{(Y)}$ in $\mathscr{A}$ such that $G(f) = u$. Let $B = \mathrm{Coker}\, f$; by Subsection I.5.4.2, there is a canonical morphism $M = \mathrm{Coker}\, G(f) \to G(\mathrm{Coker}\, f) = G(B)$, which is an isomorphism because G is exact. This isomorphism induces a morphism of functors $\mathrm{Hom}_{\mathscr{A}}(B, \cdot) \xrightarrow{G} \mathrm{Hom}_R(G(B), G(\cdot)) \xrightarrow{\sim} \mathrm{Hom}_R(M, G(\cdot)) = \Phi_M$. We show that this morphism is an isomorphism. For every $A \in \mathrm{Ob}(\mathscr{A})$, we have a commutative diagram with exact columns (in the category of abelian groups) :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathrm{Hom}_R(M, G(A)) \arrow[r, "(1)"] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(B, A) \arrow[d] \\
 & \mathrm{Hom}_R(R^{(Y)}, G(A)) \arrow[r, "(2)"] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(P^{(Y)}, A) \arrow[d] \\
 & \mathrm{Hom}_R(R^{(X)}, G(A)) \arrow[r, "(3)"] & \mathrm{Hom}_{\mathscr{A}}(P^{(X)}, A)
\end{tikzcd}$$

We have already shown that maps (2) and (3) are isomorphisms. So we can deduce that (1) is an isomorphism. So Φ_M is representable by B, and we have also shown that the morphism $\eta_M : M \to G(B)$ is an isomorphism.

Let $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{A}}$ be the counit of the adjunction. We claim that ε is also an isomorphism; this implies that G is an equivalence of categories, with quasi-inverse F.

Let $A \in \mathrm{Ob}(\mathscr{A})$. By Proposition I.4.6, the composition

$$G(A) \xrightarrow{\eta(G(A))} G(F(G(A))) \xrightarrow{G(\varepsilon(A))} G(A)$$

is equal to $\mathrm{id}_{G(A)}$. We have already shown that $\eta(G(A))$ is an isomorphism, so this implies that $G(\varepsilon(A))$ is an isomorphism. As G is conservative, we deduce that $\varepsilon(A)$ is an isomorphism.

We finally prove the last statement. Let $G : \mathscr{A} \to \mathbf{Mod}_S$ be an equivalence of categories, and let $F' : \mathbf{Mod}_S \to \mathscr{A}$ be a quasi-inverse of G. In particular, for every right S-module M, the functor $\mathscr{A} \to \mathbf{Set}$, $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(M, G(A))$ is representable by $F'(M)$, so G admits a left adjoint that is isomorphic to F'. In other words, we may choose a quasi-inverse F of G such that (F, G) is a pair of adjoint functors. Also, the functors F and G preserve finite limits and finite colimits (because they are equivalences of categories), so they are additive and exact functors (by Corollary II.1.2.5). Let $Q = F(S)$. As S is a progenerator of $\mathbf{Mod}_S$, the object Q is a progenerator of $\mathscr{A}$. Also, we have an isomorphism of functors $G(\cdot) \simeq \mathrm{Hom}_S(S, G(\cdot)) \simeq \mathrm{Hom}_{\mathscr{A}}(F(S), \cdot) = \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$. This finishes the proof.

□

II.3.2 Grothendieck abelian categories

Definition II.3.2.1. Let $\mathscr{U}$ be a universe. A *Grothendieck abelian category* is an abelian $\mathscr{U}$-category $\mathscr{A}$ such that :

- colimits indexed by $\mathscr{U}$-small categories exist in $\mathscr{A}$;

- if $\mathscr{I}$ is a $\mathscr{U}$-small filtrant category, the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, \mathscr{A}) \to \mathscr{A}$ is exact;
- $\mathscr{A}$ has a generator.

Example II.3.2.2. (1) For any ring R, the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ are Grothendieck abelian categories.

(2) If $\mathscr{A}$ is a Grothendieck abelian category and $\mathscr{C}$ is a $\mathscr{U}$-small category, then $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathscr{A})$ is a Grothendieck abelian category.

One advantage of Grothendieck abelian categories is that there is a simpler way to characterize injective objects in them. This generalizes the criterion for R-modules, and the proof is almost the same.

Proposition II.3.2.3. *Suppose that $\mathscr{A}$ is a Grothendieck abelian category, and let Q be a generator of $\mathscr{A}$. Then an object I of $\mathscr{A}$ is injective if and only if, for every monomorphism $f : A \to Q$ and any morphism $u : A \to I$, there exists $v : Q \to I$ such that $v \circ f = u$.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"'] & Q \arrow[dl, dashed, "v"] \\ & I & \end{tikzcd}$$

Proof. It is obvious that an injective object satisfies the condition of the proposition. We prove the converse. Let I be an object of $\mathscr{A}$, and suppose that any morphism from a subobject of Q to I extends to Q.

Let $f : A \to B$ be an injective morphism and $u : A \to I$ be a morphism. We consider the set X of pairs (A', u'), where A' is a subobject of B such that $A \subset A'$ and $u' : A' \to I$ is a morphism such that $u'_{|A} = u$. If $x_1 = (A'_1, u'_1)$ and $x_2 = (A'_2, u'_2)$ are two elements of X, we write $x_1 \leq x_2$ if $A'_1 \subset A'_2$ and $u'_1 = u'_{2|A'_1}$; it is easy to see that this is an order relation on X.

We want to apply Zorn's lemma to show that X has s maximal element. We already know that X is not empty (because $(A, u) \in X$), so it suffices to check that every totally ordered subset of X has an upper bound. Let $Y \subset X$ be a totally ordered subset. For every $y \in Y$, we denote the corresponding pair by (A'_y, u_y). Then $\{A'_y,\ y \in Y\}$ is a totally ordered set of subobjects of B; by Proposition II.3.1.3(i)(f), it is in bijection with an element of $\mathscr{U}$, so its colimit A' exists in $\mathscr{A}$, and, as $\mathscr{A}$ is a Grothendieck abelian category, this colimit is also a subobject of B. Also, the colimit of the family of morphisms $(u'_y)_{y \in Y}$ is a morphism $u' : A' \to I$ such that $u'_{|A'_y} = u'_y$ for every $y \in Y$, and in particular $u'_{|A} = u$. So $(A', u') \in X$, and it is clearly an upper bound of Y.

We now apply Zorn's lemma to get a maximal element x of X. Let (A', u') be the corresponding pair. We claim that the canonical morphism $A' \to B$ is an isomorphism, which finishes the proof (take v equal to u' times the inverse of this isomorphism). To prove the claim, we assume that $A' \to B$ is not an isomorphism. As Q is a generator of $\mathscr{A}$, there exists a morphism $\psi : Q \to B$ that does not factor through A'. Let $N = A' \cap \operatorname{Im} \psi$ (a subobject of B), and let

$M = \psi^{-1}(N)$ (a subobject of Q); we still denote by $\psi : M \to N$ the morphism induced by ψ. Let $g : M \to I$ be the composition $M \stackrel{\psi}{\to} N \to A' \stackrel{u'}{\to} I$, where $N \to A'$ is the canonical injection. By the hypothesis on I, there exists a morphism $h : Q \to I$ extending g. As $\operatorname{Ker}\psi \subset M$, we have $\operatorname{Ker}\psi \subset \operatorname{Ker} g \subset \operatorname{Ker} h$, so h induces a morphism $k : \operatorname{Im}\psi \to I$. By definition of g, the composition of u' and of k with the canonical injections of N in A' and $\operatorname{Im}\psi$ are equal, so u' and k define a morphism u'' from $A'' = A' + \operatorname{Im}\psi$ to I extending u'. As $A' \subsetneq A''$, this contradicts the maximality of x.

□

The main result of this subsection is the following :

Theorem II.3.2.4. *Let $\mathscr{A}$ be a Grothendieck abelian category. Then there exists an additive endofunctor I of $\mathscr{A}$ and a morphism of functors $\iota : \mathrm{id}_{\mathscr{A}} \to I$ such that, for every $A \in \operatorname{Ob}(\mathscr{A})$, the object $I(A)$ of $\mathscr{A}$ is injective and the morphism $\iota(A) : A \to I(A)$ is a monomorphism.*

In particular, the abelian category $\mathscr{A}$ has enough injective objects.

Proof. Let Q be a generator of $\mathscr{A}$. For every $A \in \operatorname{Ob}(\mathscr{A})$, we denote by $\Phi(A)$ a pushout of the following diagram :

$$\begin{array}{ccc} \bigoplus_{M\subset Q} \bigoplus_{f \in \operatorname{Hom}_{\mathscr{A}}(M,A)} M & \xrightarrow{F} & A \\ {\scriptstyle i}\downarrow & & \\ \bigoplus_{M\subset Q} \bigoplus_{f \in \operatorname{Hom}_{\mathscr{A}}(M,A)} Q & & \end{array}$$

where i is the direct of the injections $M \subset Q$ and F is the unique morphism whose composition with the injection $M_0 \subset \bigoplus_{M\subset Q} \bigoplus_{f \in \operatorname{Hom}_{\mathscr{A}}(M,A)} M$ corresponding to $f_0 : M_0 \to A$ is f_0. Note that the direct sums exist by Proposition II.3.1.3(i)(f), and that we have a canonical morphism $A \to \Phi(A)$, which is injective by Corollary II.2.1.16(i).

The diagram defining $\Phi(A)$ is functorial in A, so we can extend Φ to an endofunctor of $\mathscr{A}$ such that the morphisms $A \to \Phi(A)$ define a morphism of functors $\iota_{01} : \mathrm{id}_{\mathscr{A}} \to \Phi$.

Using transfinite induction, we now define a family Φ_α of endofunctors of $\mathscr{A}$ indexed by the ordinals $\alpha \in \mathscr{U}$, together with morphisms of functors $\iota_{\alpha\beta} : \Phi_\alpha \to \Phi_\beta$ for $\alpha \leq \beta$, such that $\iota_{\alpha\beta}(A)$ is injective for every $A \in \operatorname{Ob}(\mathscr{A})$, that $\iota_{\alpha\alpha} = \mathrm{id}_{\Phi_\alpha}$ and that $\iota_{\alpha\gamma} = \iota_{\beta\gamma} \circ \iota_{\alpha\beta}$ if $\alpha \leq \beta \leq \gamma$. (In other words, we define a functor from the ordered set of ordinals $\alpha \in \mathscr{U}$ to the category of endofunctors of $\mathscr{A}$.)

(1) We take $\Phi_0 = \mathrm{id}_{\mathscr{A}}$, $\Phi_1 = \Phi$ and ι_{01} the morphism defined above.

(2) Suppose that α is a successor ordinal, so that $\alpha = \alpha' + 1$ for some ordinal α' (uniquely determined by α). We take $\Phi_\alpha = \Phi \circ \Phi_{\alpha'}$ and $\iota_{\alpha'\alpha}(A) = \iota_{01}(\Phi_{\alpha'}(A))$. If $\beta \leq \alpha'$, we take $\iota_{\beta\alpha}(A) = \iota_{\alpha'\beta}(A) \circ \iota_{\beta\alpha'}(A)$.

(3) If α is a limit ordinal, we take $\Phi_\alpha = \varinjlim_{\beta<\alpha} \Phi_\beta$ (with the transition maps $\iota_{\beta\beta'} : \Phi_\beta \to \Phi_{\beta'}$, for $\beta \leq \beta' < \alpha$. For every ordinal $\beta < \alpha$, we have a canonical morphism $\Phi_\beta \to \Phi_\alpha$, and we take this to be $\iota_{\beta\alpha}$; the morphism $\iota_{\beta\alpha}(A)$ is injective for every $A \in \mathrm{Ob}(\mathscr{A})$ because it is equal to $\varinjlim_{\beta\leq\beta'<\alpha} \iota_{\beta\beta'}(A)$ and because filtrant inductive limits are exact in $\mathscr{A}$.

Let $\mathfrak{c}$ be the cardinality of the set of subobjects of Q. By Proposition II.3.1.3(i)(f), we have $\mathfrak{c} \in \mathscr{U}$, so we can choose an ordinal $\alpha \in \mathscr{U}$ whose cofinality is striclty greater than $\mathfrak{c}$. (See Definition II.3.2.5 and Lemma II.3.2.6.) We take $I = \Phi_\alpha$ and $\iota = \iota_{0\alpha}$. Let $A \in \mathrm{Ob}(\mathscr{A})$. We want to show that $I(A)$ is an injective object of $\mathscr{A}$, which will finish the proof. Let $M \subset Q$ be a subobject and $u : N \to I(A)$ be a morphism. Thanks to Proposition II.3.2.3, it suffices to show that u extends to a morphism $Q \to I(A)$. By Lemma II.3.2.7, the morphism $u : M \to I(A)$ factors as $M \overset{u'}{\to} \Phi_\beta(A) \overset{\iota_{\beta\alpha}(A)}{\to} I(A)$, for some $\beta < \alpha$. By construction of Φ, the morphism $u' : M \to \Phi_\beta(A)$ extends to a morphism $v' : Q \to \Phi(\Phi_\beta(A)) = \Phi_{\beta+1}(A)$. Composing with $\iota_{\beta+1,\alpha}(A) : \Phi_{\beta+1}(A) \to I(A)$, we get a morphism $v : Q \to I(A)$ such that $v_{|M} = u$.

□

Definition II.3.2.5. Let I be an ordered set. A subset J of I is called *cofinal* if, for every $i \in I$, there exists $j \in J$ such that $i \leq j$. The *cofinality* of I the least of the cardinalities of the cofinal subsets of I.

Lemma II.3.2.6. *Let $\mathscr{U}$ be a universe and $\mathfrak{c} \in \mathscr{U}$ be a cardinal. There exists an ordinal $\alpha \in \mathscr{U}$ whose cofinality is strictly greater than $\mathfrak{c}$.*

Proof. If $\mathfrak{c}$ is finite, we can take $\alpha = \omega$ (whose cofinality is ω). Suppose that $\mathfrak{c}$ is finite. Let α be the smalles ordinal in $\mathscr{U}$ whose cardinality is striclty greater than $\mathfrak{c}$. (Such ordinals exist, for example $\mathrm{card}(2^{\mathfrak{c}})$.) If there were another ordinal β such that $\alpha = \beta + 1$, we would have $\mathrm{card}(\beta) = \mathrm{card}(\alpha) > \mathfrak{c}$, contradicting the minimality of α; so α is a limit ordinal. Suppose that there exists a cofinal subset $J \subset \alpha$ such that $\mathrm{card}(J) \leq \mathfrak{c}$. For every $\beta \in J$, we have $\mathrm{card}(\beta) < \mathrm{card}(\alpha)$ (because α is a limit ordinal), so $\mathrm{card}(\beta) \leq \mathfrak{c}$ by minimality of α. On the other hand, as J is cofinal, we have $\alpha = \bigcup_{\beta\in J} \beta$, so $\mathrm{card}(\alpha) \leq \mathfrak{c}^2 = \mathfrak{c}$. This is a contradiction, so the cofinality of α is $> \mathfrak{c}$.

□

Lemma II.3.2.7. *Let $\mathscr{A}$ be an abelian $\mathscr{U}$-category in which colimits indexed by filtrant $\mathscr{U}$-small categories exist and are exact, let A be an object of $\mathscr{A}$, let $\mathfrak{c}$ be the cardinality of the set of subobjects of A and $F : \alpha \to \mathscr{A}$ be a functor, where $\alpha \in \mathscr{U}$ is an ordinal whose cofinality is $> \mathfrak{c}$. We also suppose that all the morphisms $u_{\beta\gamma} : F(\beta) \to F(\gamma)$, for $\beta \leq \gamma$ in α, are injective.*

Then any morphism $A \to \varinjlim F$ factors through one of the $F(\beta)$.

Proof. Let $B = \varinjlim F$ and let $f : A \to B$ be a morphism. For every $\beta \in \alpha$, the canonical morphism $F(\beta) \to B$ is equal to $\varinjlim_{\gamma\geq\beta} u_{\beta\gamma}$, so it is injective; so we can see $F(\beta)$ as a subobject of B, and define a subobject A_β of A by $A_\beta = f^{-1}(F(\beta))$. For every $\beta \in \alpha$, the morphism

$f : A \to B$ factors through $F(\beta)$ if and only if $A_\beta = A$. So it suffices to find $\beta \in \alpha$ such that $A = A_\beta$.

As filtrant colimits are exact in $\mathscr{A}$, we have

$$\bigcup_{\beta\in\alpha} A_\beta = \bigcup_{\beta\in\alpha} f^{-1}(F(\beta)) = f^{-1}(\bigcup_{\beta\in\alpha} F(\beta)) = f^{-1}(B) = A.$$

On the other hand, as the cardinality of the set of subobjects of A is $\mathfrak{c}$, there exists a subset S of α of cardinality $\leq \mathfrak{c}$ such that, for every $\beta \in \alpha$, there exists $\beta' \in S$ such that $A_\beta = A_{\beta'}$. As α has cofinality $> \mathfrak{c}$, the subset S is not cofinal in α, so there exists $\beta \in \alpha$ such that $\gamma < \beta$ for every $\gamma \in S$. By the property of S, we have $A_\gamma \subset A_\beta$ for every $\gamma \in \alpha$, so $A_\beta \supset \bigcup_{\gamma\in\alpha} A_\gamma = A$, that is, $A_\beta = A$.

□

Corollary II.3.2.8. *Let $\mathscr{A}$ be a Grothendeick abelian category. Then $\mathscr{A}$ has an injective cogenerator.*

Proof. By corollary II.3.1.5, it suffices to construct an injective object I of $\mathscr{A}$ such that every nonzero object has a nonzero morphism to I.

Let Q be a generator of $\mathscr{A}$. By Proposition II.3.1.3, the direct sum $N := \bigoplus_{M\subset Q} Q/M$ exists in $\mathscr{A}$. Choose an injection $N \to I$, with I an injective object. Let A be a nonzero object of $\mathscr{A}$. As Q is a generator, there exists (by Proposition II.3.1.3) a nonzero morphism $u : Q \to A$. If $M = \operatorname{Ker} u$, the morphism u induces an injective morphism $v : Q/M \to A$. We have a canonical injective morphism $Q/M \to N$, hence an injective morphism $Q/M \to I$. As I is injective, this extends to a morphism $A \to I$, which is nonzero because $Q/M \neq 0$.

□

Corollary II.3.2.9. *Let R be a ring. Then the categories ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ have enough injective and projective objects.*

Proof. As $\mathbf{Mod}_R = {}_{R^{\mathrm{op}}}\mathbf{Mod}$, it suffices to treat the case of ${}_R\mathbf{Mod}$. We have seen in Example II.2.4.7(1) that any free R-module is projective; as every R-module is a quotient of a free R-module, the category ${}_R\mathbf{Mod}$ has enough projective objects. On the other hand, the category ${}_R\mathbf{Mod}$ has all limits and colimits (see Subsection I.5.5.1), and filtrant colimits in ${}_R\mathbf{Mod}$ are exact by Corollary I.5.6.5. Also, the R-module R is a projective generator of ${}_R\mathbf{Mod}$. So ${}_R\mathbf{Mod}$ is a Grothendieck abelian category, hence it has enough injective objects by Theorem II.3.2.4.

□

III Sheaves

In this chapter, we will define sheaves and use them to illustrate some of the methods and constructions that we have defined so far. Sheaves also give important examples of Grothendieck abelian categories.

III.1 Sheaves on a topological space

We quickly review the classical theory of sheaves on topological spaces.

Let X be a topological space. We denote by $\mathrm{PSh}(X)$ (resp. $\mathrm{PSh}(X,R)$ if R is a ring) the category of presheaves of sets (resp. left R-modules) on X. If $\mathrm{Open}(X)$ is the category of open subsets of X, we can think of $\mathrm{PSh}(X)$ (resp. $\mathrm{PSh}(X,R)$) as the functor categories $\mathrm{Func}(\mathrm{Open}(X)^{\mathrm{op}},\mathbf{Set})$ (resp. $\mathrm{Func}(\mathrm{Open}(X)^{\mathrm{op}},{}_R\mathbf{Mod})$.) If F is a presheaf on X and $V\subset U$ are open subsets of X, we often denote the map $F(U)\to F(V)$ by $s\longmapsto s_{|V}$.

Definition III.1.1. A sheaf F on X is called a *sheaf* if, for every open subset U of X and every open cover $(U_i)_{i\in I}$ of U, the two following conditions hold :

(a) the map $F(U)\to\prod_{i\in I}F(U_i)$, $s\longmapsto(s_{|U_i})_{i\in I}$ is injective;

(b) the map of (a) identifies $F(U)$ to the kernel of the two maps $f,g:\prod_{i\in I}F(U_i)\to\prod_{i,j\in I}F(U_i\cap U_j)$ defined by $f((s_i))=(s_{i|U_i\cap U_j})_{i,j\in I}$ and $f((s_i))=(s_{j|U_i\cap U_j})_{i,j\in I}$ (in other words, if $(s_i)\in\prod_{i\in I}F(U_i)$, there exists $s\in F(U)$ such that (s_i) is the family $(s_{|U_i})$ if and only, for all $i,j\in I$, we have $s_{i|U_i\cap U_j}=s_{j|U_i\cap U_j}$).

If the presheaf F satisfies condition (a) for every open cover of an open subset of X, we say that it is a *separated presheaf*.

Definition III.1.2. We denote by $\mathrm{Sh}(X)$ (resp. $\mathrm{Sh}(X,R)$) the full subcategory of $\mathrm{PSh}(X)$ (resp. $\mathrm{PSh}(X,R)$) whose objects are sheaves, and call it the category of sheaves of sets (resp. of left R-modules) on X.

Definition III.1.3. Let U be an open subset of X. The *category of open covers of* U is the category $\mathscr{I}_U$ whose objects are open covers $(U_i)_{i\in I}$, and in which a morphism from $(U_i)_{i\in I}$ to $(V_j)_{j\in J}$ is a map $\alpha:I\to J$ such that, for every $i\in I$, we have $U_i\subset V_{\alpha(i)}$.

In particular, if $\mathscr{U},\mathscr{V}\in\mathrm{Ob}(\mathscr{I}_U)$ are open covers of U, there exists a morphism $\mathscr{U}\to\mathscr{V}$ if and only if $\mathscr{U}$ refines $\mathscr{V}$. Note that there can be more than one morphism from $\mathscr{U}$ to $\mathscr{V}$.

Let F be a presheaf on X, U be an open subset of X and $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of U. We set

$$\check{\mathrm{H}}^0(\mathscr{U}, F) = \{(s_i) \in \prod_{i\in I} F(U_i) \mod \forall i, j \in I,\ s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}\}.$$

Remark III.1.4. By definition of sheaves, the presheaf F is a sheaf (resp. a separated presheaf) if and only if, for every open subset U of X and every open cover $\mathscr{U}$ of U, the canonical morphism $F(U) \to \check{\mathrm{H}}^0(\mathscr{U}, F)$ induced by restriction is an isomorphism (resp. an injection).

Now suppose that we are given two open covers $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ of an open subset U of X and a morphism $\alpha : \mathscr{U} \to \mathscr{V}$ in $\mathscr{I}_U$. We define a map $\alpha^* : \check{\mathrm{H}}^0(\mathscr{V}, F) \to \check{\mathrm{H}}^0(\mathscr{U}, F)$ in the following way : if $(s_j) \in \prod_{j\in J} F(V_j)$ is an element of $\check{\mathrm{H}}^0(\mathscr{V}, F)$, we set $\alpha^*((s_j)_{j\in J}) = (s_{\alpha(i)|U_i})_{i\in I}$. It is clear that this makes $\mathscr{U} \longmapsto \check{\mathrm{H}}^0(\mathscr{U}, F)$ into a functor from $\mathscr{I}_U^{\mathrm{op}}$ to $\mathbf{Set}$.

Lemma III.1.5. *Let F be a presheaf on X, U an open subset of X, $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ two open covers of U and $\alpha, \beta : \mathscr{U} \to \mathscr{V}$ two morphisms in $\mathscr{I}_U$. Then we have $\alpha^* = \beta^* : \check{\mathrm{H}}^0(\mathscr{V}, F) \to \check{\mathrm{H}}^0(\mathscr{U}, F)$.*

Proof. Let $s = (s_j)_{j\in J} \in \check{\mathrm{H}}^0(\mathscr{V}, F)$, and let $(t_i)_{i\in I} = \alpha^*(s)$ and $(t'_i)_{i\in I} = \beta^*(s)$. Let $i \in I$. We have $t_i = s_{\alpha(i)|U_i}$ and $t'_{i|\beta(i)|U_i}$. We want to show that $t_i = t'_i$. As $U_i \subset V_{\alpha(i)} \cap V_{\beta(i)}$, it suffices to show that $s_{\alpha(i)|V_{\alpha(i)}\cap V_{\beta(i)}} = s_{\beta(i)|V_{\alpha(i)}\cap V_{\beta(i)}}$; but this follows from the fact that $s \in \check{\mathrm{H}}^0(\mathscr{U}, F)$.

$\square$

Corollary III.1.6. *Let $\mathscr{I}_U^0$ be the category such that $\mathrm{Ob}(\mathscr{I}_U^0) = \mathrm{Ob}(\mathscr{I}_U)$ and such that $\mathrm{Hom}_{\mathscr{I}_U^0}(\mathscr{U}, \mathscr{V})$ is empty if $\mathrm{Hom}_{\mathscr{I}_U}(\mathscr{U}, \mathscr{V})$ is empty, and a singleton if $\mathrm{Hom}_{\mathscr{I}_U}(\mathscr{U}, \mathscr{V})$ is nonempty. There is a unique way to define the composition, and we have an obvious functor $\mathscr{I}_U \to \mathscr{I}_U^0$. Then the functor $F \longmapsto \check{\mathrm{H}}^0(\mathscr{U}, F)$ factors through $(\mathscr{I}_U)^{\mathrm{op}} \to (\mathscr{I}_U^0)^{\mathrm{op}}$, and induces a functor $(\mathscr{I}_U^0)^{\mathrm{op}} \to \mathbf{Set}$.*

Remark III.1.7. Note that $\mathscr{I}_U^0$ is just the category associated to the preordered set of open covers of U, preordered by refinement; this is not an ordered set, because we have can two open covers of the same U who are refinements of each other. Also, as two open covers of U always have a common refinement (taking intersections of the sets in these two covers, for example), the category $(\mathscr{I}_U^0)^{\mathrm{op}}$ is filtrant.

Definition III.1.8. We construct an endofunctor $F \longmapsto F^+$ of the category $\mathrm{PSh}(X)$ in the following way : For every presheaf F on X, for every open subset U of X, we set

$$F^+(U) = \varinjlim_{\mathscr{U}\in\mathrm{Ob}((\mathscr{I}_U^0)^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{U}, F).$$

The set $F^+(U)$ is also denoted by $\check{\mathrm{H}}^0(U, F)$, and called the 0*th Čech cohomology of F on U*.

If $U \subset V$ are open subsets of X, then we have a functor $\mathscr{I}_V^0 \to \mathscr{I}_U^0$ sending an open cover $\mathscr{V} = (V_i)_{i \in I}$ of V to the open cover $U \cap \mathscr{V} = (U \cap V_i)_{i \in I}$. The restriction maps give, for every open cover $\mathscr{V}$ of V, a map $\check{\mathrm{H}}^0(\mathscr{V}, F) \to \check{\mathrm{H}}^0(U \cap \mathscr{V}, F)$, so we get a map

$$F^+(V) = \varinjlim_{\mathscr{V} \in \mathrm{Ob}((\mathscr{I}_V^0)^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{V}, F) \to \varinjlim_{\mathscr{V} \in \mathrm{Ob}((\mathscr{I}_V^0)^{\mathrm{op}})} \check{\mathrm{H}}^0(U \cap \mathscr{V}, F) \to \varinjlim_{\mathscr{U} \in \mathrm{Ob}((\mathscr{I}_U^0)^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{U}, F) = F^+(U).$$

This makes F^+ into a presheaf on X. Also, the canonical maps $F(U) \to \check{\mathrm{H}}^0(\mathscr{U}, F)$ (for $\mathscr{U}$ an open cover of U) induce a morphism of presheaves $\iota_0(F) : F \to F^+$; it is easy to see that this is actually a morphism of functors on $\mathrm{PSh}(X)$.

Remark III.1.9. As $(\mathscr{I}_U^0)^{\mathrm{op}}$ is a filtrant category, the set $F^+(U)$ is the set of families $(s_i)_{i \in I} \in \check{\mathrm{H}}^0((U_i)_{i \in I}, F)$, for $(U_i)_{i \in I}$ an open cover of U, modulo the following equivalence relation : if $\mathscr{U}_1$ and $\mathscr{U}_2$ are open covers of U and $s_1 \in \check{\mathrm{H}}^0(\mathscr{U}_1, F)$, $s_2 \in \check{\mathrm{H}}^0(\mathscr{U}_2, F)$, then s_1 and s_2 are equivalent if and only if there exists an open cover $\mathscr{V}$ of U refining both $\mathscr{U}_1$ and $\mathscr{U}_2$ and refinements $\alpha_1 : \mathscr{V} \to \mathscr{U}_1$ and $\alpha_2 : \mathscr{V} \to \mathscr{U}_2$ such that $\alpha_1^*(s_1) = \alpha_2^*(s_2)$ in $\check{\mathrm{H}}^0(\mathscr{V}, F)$.

Proposition III.1.10. *Let F be a presheaf on X.*

(i). The presheaf F^+ is separated.

(ii). If F is a separated presheaf, then $\iota_0(F) : F \to F^+$ is injective.

(iii). If F is a separated presheaf, then F^+ is a sheaf.

(iv). The presheaf F^{++} is a sheaf.

(v). If F is a sheaf, then the morphism $\iota_0(F) : F \to F^+$ is an isomorphism.

(vi). If G is a sheaf and $u : F \to G$ is a morphism of presheaves, then there exists a unique morphism of presheaves $u' : F^{++} \to G$ such that $u = u' \circ \iota_0(F^+) \circ \iota_0(F)$:

$$\begin{tikzcd} F \arrow[r, "u"] \arrow[d] & G \\ F^{++} \arrow[ur, "u'"'] & \end{tikzcd}$$

In particular, the functor $F \longmapsto F^{++}$ can be seen as a functor $\mathrm{PSh}(X) \to \mathrm{Sh}(X)$; this is called the *sheafification functor* and denote by $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$. We have a morphism of functors $\iota : \mathrm{id}_{\mathrm{PSh}(\mathscr{C})} \to (\cdot)^{\mathrm{sh}}$ given by $\iota(F) = \iota_0(F^+) \circ \iota_0(F)$.

Proof. (i). Let U be an open subset of X and $\mathscr{U} = (U_i)_{i \in I}$ be an open cover of U. Let $s, t \in F^+(U)$ such that $s_{|U_i} = t_{|U_i}$ for every $i \in I$. We choose an open cover $\mathscr{V} = (V_j)_{j \in J}$ of U and elements $(s_j)_{j \in J}$, $(t_j)_{j \in J}$ of $\check{\mathrm{H}}^0(\mathscr{V}, F)$ representing s and t. After replacing $\mathscr{V}$ and $\mathscr{U}$ by a common refinement, we may assume that $\mathscr{V}$ refines $\mathscr{U}$. Let $j \in J$. Then V_j is contained in some U_i, and so the fact that $s_{|U_i} = t_{|U_i}$ in $F^+(U_i)$ implies that s_j and t_j define the same element of $F^+(V_j)$; by Remark III.1.9, this means that there exists an open cover $(W_{jk})_{k\in K_j}$ of V_j such that $s_{j|W_{jk}} = t_{j|W_{jk}}$ in $F(W_{jk})$ for every $k \in J_j$. Consider the open cover $\mathscr{W} = (W_{jk})_{j\in J, k\in J_k}$ of U. Then $\mathscr{W}$ refines $\mathscr{V}$, and the images of (s_j) and (t_j) in $\check{\mathrm{H}}^0(\mathscr{W}, F)$ are equal, so (s_j) and (t_j) define the same element of $F^+(U)$.

(ii). This follows immediately from Remarks III.1.4 and III.1.9.

(iii). Let U be an open subset of X, $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of U and $(s_i) \in \prod_{i\in I} F^+(U_i)$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all $i, j \in I$. For every $i \in I$, choose an open cover $\mathscr{U}_i = (U_{ij})_{j\in J_i}$ of U_i and an element $(s_{ij})_{j\in I_j}$ of $\check{\mathrm{H}}^0(\mathscr{U}_i, F)$ representing s_i. Let $\mathscr{V} = (U_{ij})_{i\in I, j\in J_i}$; this is an open cover of U. We claim that the family $(s_{ij})_{i\in I, j\in J_i}$ is an element of $\check{\mathrm{H}}^0(\mathscr{V}, F)$. Let $i, i' \in I$, $j \in J_i$ and $j' \in J_{i'}$; we need to show that $s_{ij|U_{ij}\cap U_{i'j'}} = s_{i'j'|U_{ij}\cap U_{i'j'}}$ in $F(U_{ij} \cap U_{i'j'})$. We know that $s_{i|U_i\cap U_{i'}} = s_{i'|U_i\cap U_{i'}}$ in $F^+(U_i \cap U_{i'})$; as $U_{ij} \cap U_{i'j'} \subset U_i \cap U_{i'}$, this implies that $s_{ij|U_{ij}\cap U_{i'j'}}$ and $s_{i'j'|U_{ij}\cap U_{i'j'}}$ define the same element of $F^+(U_{ij} \cap U_{i'j'})$, and, by (ii), we get that they are equal in $F(U_{ij} \cap U_{i'j'})$.

Now let s' be the element of $F^+(U)$ defined by the family $(s_{ij})_{i\in I, j\in J_i} \in \check{\mathrm{H}}^0(\mathscr{V}, F)$. It remains to show that $s'_{|U_i} = s_i$ in $F^+(U_i)$ for every $i \in I$. Let $i \in I$. Then $s'_{|U_i} \in F^+(U_i)$ is represented by the family $(s_{ij})_{j\in J_i} \in \check{\mathrm{H}}^0(\mathscr{U}_i, F)$, so it is equal to s_i.

(iv). This follows immediately from (i) and (iii).

(v). This follows immediately from Remark III.1.4.

(vi). We have a commutative square

$$\begin{tikzcd} F \arrow[r] \arrow[d, "u"'] & F^{++} \arrow[d, "u^{++}"] \\ G \arrow[r] & G^{++} \end{tikzcd}$$

where the horizontal morphisms are given by two applications of ι_0. By (v), the morphism $G \to G^{++}$ is an isomorphism, so we get the desired $u' : F^{++} \to G$ by composing the inverse of this isomorphism and u^{++}.

Suppose that we have another morphism $u'' : F^{++} \to G$ such that $u = u'' \circ \iota_0(F^+) \circ \iota_0(F)$. Let U be an open subset of X, let $s \in F^{++}(U)$. We want to show that $u'(s) = u''(s)$. By Remark III.1.9 (applied twice), we can find an open cover $(U_i)_{i\in I}$ of U and sections $s_i \in F(U_i)$ such that $s_{|U_i}$ is equal to the image of s_i in $F^{++}(U_i)$ for every $i \in I$. By the condition on u' and u'', we have, for every $i \in I$, $u'(s)_{|U_i} = u(s_i) = u''(s)_{|U_i}$. As G is a sheaf, this implies that $u'(s) = u''(s)$.

□

III.2 Grothendieck pretopologies

In this section, we study a generalization of topological spaces called Grothendieck pretopologies, and the associated category of sheaves.

We fix a category $\mathscr{C}$, and we always assume that fibered products exist in $\mathscr{C}$.

III.2.1 Pretopologies and sheaves

Definition III.2.1.1. A *Grothendieck pretopology* $\mathscr{T}$ on $\mathscr{C}$ is the data, for every object X of $\mathscr{C}$, of a set of families of morphisms $(u_i : X_i \to X)_{i \in I}$, called *covering families*, and satisfying the following axioms :

(CF1) If $Y \to X$ is a morphism and $(X_i \to X)_{i \in I}$ is a covering family, then the family $(X_i \times_X Y \to Y)_{i \in I}$ is a covering family.

(CF2) If $(u_i : X_i \to X)_{i \in I}$ is a covering, and if, for every $i \in I$, $(v_{ij} : X_{ij} \to X_i)_{j \in J_i}$ is a covering family, then the family $(u_i \circ v_{ij} : X_{ij} \to X)_{i \in I, j \in J_i}$ is a covering family.

(CF3) If $u : X' \to X$ is an isomorphism, then $(u : X' \to X)$ is a covering family.

A category $\mathscr{C}$ (having fiber products) with a Grothendieck pretopology is called a *site*. If $\mathscr{T}$ and $\mathscr{T}'$ are two pretopologies on $\mathscr{C}$, we say that $\mathscr{T}$ is *coarser* than $\mathscr{T}'$ (or that $\mathscr{T}'$ is *finer* than $\mathscr{T}$) if every covering family for $\mathscr{T}$ is also covering for $\mathscr{T}'$.

Remark III.2.1.2. Obligatory set-theoretical remark : if $\mathscr{U}$ is the ambient universe (that, we use the category $\mathbf{Set}_{\mathscr{U}}$ as coefficients for the presheaves), then the indexing set I of covering families is also assumed to be an element of $\mathscr{U}$.

Remark III.2.1.3. There is a more general notion of Grothendieck topology that makes sense for categories $\mathscr{C}$ that don't necessarily have fiber products. It is formulated using *sieves* on $X \in \mathrm{Ob}(\mathscr{C})$, which are by definition subpresheaves of the representable presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$. Instead of covering families, we have to give for each object X of $\mathscr{C}$ the data of a collection of covering sieves, satisfying conditions similar to (CF1)-(CF3). The connection with covering families is the following : Any family of morphisms $(u_i : X_i \to X)$ defines a sieve F on X, by taking $F(Y)$ to be the set of morphisms $Y \to X$ that factor through one of the u_i; we call this sieve the sieve generated by the family. A sieve is called covering if it is generated by a covering family.

For a category that has fiber products, the two notions turn out to be equivalent, so we chose to only consider Grothendieck pretopologies in these notes, because they are closer to the geometric intuition. See for example Chapter 16 of [8] for more about Grothendieck topologies.

Example III.2.1.4. (0) The trivial topology on $\mathscr{C}$ is the topology for which the covering families are exactly the isomorphisms.

(1) Let X be a topological space, and $\mathscr{C} = \mathrm{Open}(X)$ be the category of its open subsets. Then $\mathscr{C}$ has fiber products : if $U \to V$ and $W \to V$ are two morphisms of $\mathscr{C}$, this just means that $U \subset V$ and $W \subset V$, and it si easy to see that $U \times_V W = U \cap W$. We say that a family of morphisms $(U_i \to U)_{i \in I}$ is covering if $U = \bigcup_{i \in I} U_i$. This defines a pretopology on $\mathscr{C}$, which we will call the usual pretopology.

(2) Let $\mathscr{C} = \mathrm{Open}(\mathbb{Q})$. We say that a family of morphisms $(U_i \to U)_{i\in I}$ is covering if $(U_i)_{i\in I}$ is an admissible open cover of U in the sense of Problem A.3.5. This defines a pretopology on $\mathscr{C}$, which is weaker than the usual one.

(3) Let $\mathscr{A}$ be an abelian category. We define a pretopology on $\mathscr{A}$ by taking the admissible coverings of X to be the epimorphisms $Y \to X$. The topology that this defines is called the *canonical topology* on $\mathscr{A}$; see Remark III.2.1.8 for an explanation of the name.

(4) Let G be a topological group. A *G-set* is a set X with an action of G such that the action map $G \times X \to X$ is continuous if we put the discrete topology on X. A morphism of G-sets is a G-equivariant map. We get a category $G - \mathbf{Set}$, called the category of G-sets. This category has fiber products, given by the usual fiber product of sets with the product action of G, and a final object, which is a singleton with the trivial action of G. We put a Grothendieck pretopology on $G - \mathbf{Set}$ by taking covering families to be the families $(u_i : X_i \to X)_{i\in I}$ such that $X = \bigcup_{i\in I} u_i(X_i)$.

(5) A topological space X is called *profinite* if it is homeomorphic to $\varprojlim F$, where $F : \mathscr{I} \to \mathbf{Top}$ is a functor from a $\mathscr{U}$-small category $\mathscr{I}$ such that $F(i)$ is a finite discrete space for every $i \in \mathrm{Ob}(\mathscr{I})$. It is equivalent to ask that X be a compact Hausdorff totally disconnected topological space. We denote by $*_{\text{proét}}$ the full subcategory of $\mathbf{Top}$ whose objects are profinite sets, equipped with the Grothendieck pretopology for which covering families are finite families $(u_i : X_i \to X)_{1\leq i\leq n}$ such that $X = \bigcup_{i=1}^{n} u_i(X_i)$.

(6) Let $\mathbf{CRing}$ be the category of commutative rings, and let $\mathscr{C} = \mathbf{CRing}^{\mathrm{op}}$. (This is the category of affine schemes.) Note that $\mathscr{C}$ has fiber product : if $A \to B$ and $A \to C$ are morphisms of commutative rings, their fiber product in $\mathscr{C}$ is $B \otimes_A C$. There are many useful pretopologies on $\mathscr{C}$. We will define two here :

- the *fpqc topology* [1] is the pretopology for which covering families are morphisms of commutative rings that are faithfully flat;
- the *fppf topology* [2] is the pretopology for which covering families are morphisms of commutative rings that are faithfully flat and of finite presentation.

Definition III.2.1.5. Let $\mathscr{T}$ be a Grothendieck pretopology on $\mathscr{C}$. A presheaf $F \in \mathrm{PSh}(\mathscr{C})$ (resp. $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$, resp. $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, {}_R\mathbf{Mod})$, resp. $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Mod}_R)$) is called a *sheaf* (resp. *sheaf of abelian groups*, *sheaf of left R-modules*, *sheaf of right R-modules*) if, for every covering family $(u_i : X_i \to X)_{i\in I}$, the two following conditions hold :

(a) the map $F(X) \to \prod_{i\in I} F(X_i)$, $s \longmapsto (u_i^*(s))_{i\in I}$ is injective;

(b) the map of (a) identifies $F(X)$ to the kernel of the two maps $f, g : \prod_{i\in I} F(X_i) \to \prod_{i,j\in I} F(X_i \times_X X_j)$ defined by $f((s_i)) = (p_{ij,i}^* s_i)_{i,j\in I}$ and $f((s_i)) = (p_{ij,j}^* s_j)_{i,j\in I}$, where $p_{ij,i} : X_i \times_X X_j \to X_i$ and $p_{ij,j} : X_i \times_X X_j \to X_j$ are the two projections.

[1] Fpqc means “fidèlement plat quasi-compact”, which is French for “faithfully flat quasi-compact”.
[2] Fppf means “fidèlement plat de présentation finie”, which is French for “faithfully flat of finite presentation”.

If the presheaf F satisfies condition (a) for every covering family of an object X of $\mathscr{C}$, we say that it is a *separated presheaf*.

Remark III.2.1.6. If X is an object of $\mathscr{C}$ such that the empty family is a covering family of X and if $\mathscr{F}$ is a sheaf (resp. a sheaf of left or right R-modules) on $\mathscr{C}$, then $\mathscr{F}(X)$ is a singleton (resp. $\mathscr{F}(X) = 0$). Indeed, the empty product in the category $\mathbf{Set}$ (resp. ${}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$) is a terminal object, that is, a singleton (resp. 0).

Definition III.2.1.7. We denote by $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ (resp. $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, R)$) the full subcategory of $\mathrm{PSh}(\mathscr{C})$ (resp. $\mathrm{PSh}(\mathscr{C}, R) := \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, {}_R\mathbf{Mod})$) whose objects are sheaves, and call it the category of sheaves of sets (resp. of left R-modules) on $\mathscr{C}$. If $\mathbb{Z} = R$, we also call objects of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ (resp. $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z})$ *abelian presheaves* (resp. *abelian sheaves*).

The category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ is also called the *topos* associated to the site $(\mathscr{C}, \mathscr{T})$.

Remark III.2.1.8. We say that a Grothendieck (pre)topology is *subcanonical* if every representable presheaf is a sheaf. There always existsa finest subcanonical topology on a category; it is called the *canonical topology*. [3]

Example III.2.1.9. (0) The sheaves for the trivial topology on $\mathscr{C}$ are the presheaves.

(1) The sheaves for the topology of Example III.2.1.4(1) on the category of open subsets of a topological space X are the sheaves on X in the usual sense.

(3) The sheaves of abelian groups on an abelian category $\mathscr{A}$ with its canonical topology are the left exact functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$. (See problem A.3.6.)

(5) The sheaves on $*_{\text{proét}}$ (see Example III.2.1.4(5)) are called *condensed sets*.

III.2.2 Sheafification

Let $\mathscr{C}$ be a category admitting fibered products, and let $\mathscr{T}$ be a Grothendieck pretopology on $\mathscr{C}$.

Definition III.2.2.1. Let $f : X \to Y$ be a morphism of $\mathscr{C}$, let $\mathscr{X} = (u_i : X_i \to X)_{i \in I}$ be a covering family of X and $\mathscr{Y} = (v_j : Y_j \to Y)_{j \in J}$ be a covering family of Y. A *morphism of covering families* from $\mathscr{X}$ to $\mathscr{Y}$ over the morphism f is the data of a map $\alpha : I \to J$ and, for every $i \in I$, of a morphism $f_i : X_i \to Y_{v(i)}$ such that the following diagram commutes :

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & Y_{\alpha(i)} \arrow[d, "v_{\alpha(i)}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $\mathscr{C}$, $\mathscr{X} = (X_i)_{i \in I}$, $\mathscr{Y} = (Y_j)_{j \in J}$ and $\mathscr{Z} = (Z_k)_{k \in K}$ are covering families of X, Y and Z and $u = (\alpha, (f_i)) : \mathscr{X} \to \mathscr{Y}$ and

[3] See for example Exercise 17.6 of [8].

$v = (\beta, (v_j)) : \mathscr{Y} \to \mathscr{Z}$ are morphisms of coverings over f and g, then we get a morphism of coverings $v \circ u : \mathscr{X} \to \mathscr{Z}$ over $g \circ f$ by taking the map $\beta \circ \alpha : I \to K$ and, for every $i \in I$, the morphism $g_{\alpha(i)} \circ f_i : X_i \to Z_{\beta(\alpha(i))}$.

In this way, we get a category $\mathbf{Cov}(\mathscr{C}, \mathscr{T})$ of covering families of $\mathscr{C}$, with a functor $\mathbf{Cov}(\mathscr{C}, \mathscr{T}) \to \mathscr{C}$ sending a covering family to the object it covers.

Definition III.2.2.2. The *category of coverings* of an object X of $\mathscr{C}$ is the category $\mathscr{I}_X$ whose objects are covering families of X and whose morphisms are morphisms of covering families over id_X. [4] We also denote by $\mathscr{I}_X^0$ the category such that $\mathrm{Ob}(\mathscr{I}_X^0) = \mathrm{Ob}(\mathscr{I}_X)$ and $\mathrm{Hom}_{\mathscr{I}_X^0}(\mathscr{X}, \mathscr{X}')$ is empty if $\mathrm{Hom}_{\mathscr{I}_X}(\mathscr{X}, \mathscr{X}')$ is and a singleton if $\mathrm{Hom}_{\mathscr{I}_X}(\mathscr{X}, \mathscr{X}')$ is nonempty (with the only possible composition law). We have an obvious functor $\mathscr{I}_X \to \mathscr{I}_X^0$.

Remark III.2.2.3. The category $\mathscr{I}_X^0$ is the category associated to a preordered set.

Definition III.2.2.4. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$. If X is an object of $\mathscr{C}$ and $\mathscr{X} = (X_i)_{i \in I}$ is a covering family of X, we set

$$\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \{(s_i) \in \prod_{i \in I} \mathscr{F}(X_i) \mid \forall i, j \in I,\ p_{ij,i}^* s_i = p_{ij,j}^* s_j\},$$

where $p_{ij,i} : X_i \times_X X_j \to X_i$ and $p_{ij,j} : X_i \times_X X_j \to X_j$ are the two projections.

Note that the map sending $s \in \mathscr{F}(X)$ to the family $(u_i^*(s))_{i \in I} \in \prod_{i \in I} \mathscr{F}(X_i)$ induces a map $\mathscr{F}(X) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$.

Suppose that $u = (\alpha, (f_i)) : \mathscr{X} \to \mathscr{Y} = (Y_j)_{j \in J}$ is a morphism of coverings over a morphism $f : X \to Y$. We define a map $u^* : \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ in the following way : Let $(t_j)_{j \in J} \in \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F})$. Then $u^*((t_j)_{j \in J}) = (f_i^*(t_{\alpha(i)}))_{i \in I}$. This element of $\prod_{i \in I} \mathscr{F}(U_i)$ is in $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ because the following two diagrams commute :

$$\begin{tikzcd} Y_i \times_Y Y_j \arrow[r] \arrow[d] & Y_i \arrow[d, "f_i"] \\ X_i \times_X X_j \arrow[r] & X_i \end{tikzcd} \qquad \begin{tikzcd} Y_i \times_Y Y_j \arrow[r] \arrow[d] & Y_j \arrow[d, "f_j"] \\ X_i \times_X X_j \arrow[r] & X_j \end{tikzcd}$$

By construction of u^*, the following diagram commutes:

$$\begin{tikzcd} \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) \arrow[r, "u^*"] & \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) \\ \mathscr{F}(Y) \arrow[r, "f^*"'] \arrow[u] & \mathscr{F}(X) \arrow[u] \end{tikzcd}$$

In this way, we get a functor $\mathscr{X} \longmapsto \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ on the category $\mathbf{Cov}(\mathscr{C}, \mathscr{T})^{\mathrm{op}}$.

[4] This is called the *fiber* of the functor $\mathbf{Cov}(\mathscr{C}, \mathscr{T}) \to \mathscr{C}$ over X.

Remark III.2.2.5. By definition of sheaves, the presheaf $\mathscr{F}$ is a sheaf (resp. a separated presheaf) if and only if, for every object X of $\mathscr{C}$ and every covering family $\mathscr{X}$ of X, the canonical morphism $\mathscr{F}(X) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ is an isomorphism (resp. an injection).

From now on, we assume the following :

- we have fixed a universe $\mathscr{U}$;
- the category $\mathscr{C}$ is a $\mathscr{U}$-category;
- $\mathbf{Set} = \mathbf{Set}_{\mathscr{U}}$, $\mathbf{Ab} = \mathbf{Ab}_{\mathscr{U}}$, ${}_R\mathbf{Mod} = {}_R\mathbf{Mod}_{\mathscr{U}}$, $\mathbf{Mod}_R = \mathbf{Mod}_{R\mathscr{U}}$;
- **for every object X of $\mathscr{C}$, the category $\mathscr{I}_X$ is essentially $\mathscr{U}$-small, that is, equivalent to a $\mathscr{U}$-small category.**

The last condition will allow us to take colimits indexed by $\mathscr{I}_X^{\mathrm{op}}$ in $\mathbf{Set}$ (or $\mathbf{Ab}$, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$). Note that we can always ensure that this conditions holds by replacing $\mathscr{U}$ with a bigger universe. However, the sheafification functor can then depend on the universe, see Waterhouse's article [14] for an example in the case of the fpqc topology.

Definition III.2.2.6. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$. If X is an object of $\mathscr{C}$, we set

$$\check{\mathrm{H}}^0(X, \mathscr{F}) = \mathscr{F}^+(X) = \varinjlim_{\mathscr{X} \in \mathrm{Ob}(\mathscr{I}_X^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}).$$

This is called 0*th Čech cohomology of* $\mathscr{F}$ *on* X.

If $f : X \to Y$ is a morphism of $\mathscr{C}$, then we have a functor $f^* : \mathscr{I}_Y \to \mathscr{I}_X$ sending a covering family $\mathscr{Y} = (Y_j)_{j\in J}$ of Y to the covering family $f^*(\mathscr{Y}) = (Y_j \times_Y X)_{j\in J}$ of X; the identity $J \to J$ and the first projections give a canonical morphism of coverings $u : f^*(\mathscr{Y}) \to \mathscr{Y}$ over f, so we get a commutative diagram:

$$\begin{tikzcd}
\check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) \arrow[r, "u^*"] & \check{\mathrm{H}}^0(f^*(\mathscr{Y}), \mathscr{F}) \\
\mathscr{F}(Y) \arrow[u] \arrow[r, "f^*"'] & \mathscr{F}(X) \arrow[u]
\end{tikzcd}$$

Going to the colimit over $\mathscr{I}_Y^{\mathrm{op}}$, we get a morphism

$$\begin{aligned}
\mathscr{F}^+(Y) = \varinjlim_{\mathscr{Y} \in \mathrm{Ob}((\mathscr{I}_Y)^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) &\to \varinjlim_{\mathscr{Y} \in \mathrm{Ob}((\mathscr{I}_Y)^{\mathrm{op}})} \check{\mathrm{H}}^0(f^*(\mathscr{Y}), \mathscr{F}) \\
&\to \varinjlim_{\mathscr{X} \in \mathrm{Ob}((\mathscr{I}_X)^{\mathrm{op}})} \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathscr{F}^+(X).
\end{aligned}$$

This makes $\mathscr{F}^+$ into a presheaf on $\mathscr{C}$. Also, the canonical maps $\mathscr{F}(X) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ (for $\mathscr{X}$ a covering family of X) induce a morphism of presheaves $\iota_0(F) : \mathscr{F} \to \mathscr{F}^+$. Finally, the formation of $\mathscr{F}^+$ is clearly functorial in $\mathscr{F}$, and it is easy to see that ι_0 is actually a morphism of functors.

Lemma III.2.2.7. *(i). Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$, $f : X \to Y$ be a morphism of $\mathscr{C}$ and $u, v : \mathscr{X} \to \mathscr{Y}$ be a morphism of covering families above f. Then the maps $u^* : \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ and $v^* : \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F}) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ are equal.*

In particular, if X is an object of $\mathscr{C}$, the functor $\mathscr{X} \longmapsto \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ on $\mathscr{I}_X^{\mathrm{op}}$ factors through $(\mathscr{I}_X^0)^{\mathrm{op}}$.

(ii). For every object X of $\mathscr{C}$, the category $(\mathscr{I}_X^0)^{\mathrm{op}}$ is filtrant.

Proof. (i). Write $\mathscr{X} = (f_i : X_i \to X)_{i \in I}$, $\mathscr{Y} = (g_j : Y_j \to Y)_{j \in J}$, $u = (\alpha, (u_i : X_i \to Y_{\alpha(i)})_{i \in I})$ and $v = (\beta, (v_i : X_i \to Y_{\beta(i)})_{i \in I})$. Let $s = (s_j)_{j \in J} \in \check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F})$. We want to show that, for every $i \in I$, we have $u_i^*(s_{\alpha(i)}) = v_i^*(s_{\beta(i)})$. As the diagrams

$$\begin{tikzcd} X_i \arrow[r, "u_i"] \arrow[d, "f_i"'] & Y_{\alpha(i)} \arrow[d, "g_{\alpha(i)}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[r, "v_i"] \arrow[d, "f_i"'] & Y_{\beta(i)} \arrow[d, "g_{\beta(i)}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commute by definition of a morphism of covering families, we have a unique morphism $h : X_i \to Y_{\alpha(i)} \times_Y Y_{\beta(i)}$ whose composition with the first (resp. second) projection p_1 (resp. p_2) is equal to u_i (resp. v_i). So

$$u_i^*(s_{\alpha(i)}) = h^* p_1^*(s_{\alpha(i)}) = h^* p_2^*(s_{\beta(i)}) = v_i^*(s_{\beta(i)})$$

(where $p_1^*(s_{\alpha(i)}) = p_2^*(s_{\beta(i)})$ because s is in $\check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F})$).

(ii). Let $\mathscr{X} = (X_i \to X)_{i \in I}$ and $\mathscr{X}' = (X'_j \to X)_{j \in J}$ be two covering families of X. We claim that the family $\mathscr{X}'' = (X_i \times_X X'_j \to X)_{(i,j) \in I \times J}$ is covering and that it has morphisms to both $\mathscr{X}$ and $\mathscr{X}'$. Indeed, by (CF1), for every $i \in I$, the family $(X_i \times_X X'_j \to X_i)_{j \in J}$ is a covering family of X_i. Then (CF2) implies that $\mathscr{X}''$ is a covering family of X. Moreover, we have a morphism $\mathscr{X}'' \to \mathscr{X}$ given by the first projection $I \times J \to I$ and by the first projections $X_i \times_X X'_j \to X_i$. We have a similar morphism $\mathscr{X}'' \to \mathscr{X}'$.

□

Remark III.2.2.8. As $(\mathscr{I}_X^0)^{\mathrm{op}}$ is a filtrant category, the set $\mathscr{F}^+(X)$ is the set of families $(s_i)_{i \in I} \in \check{\mathrm{H}}^0((X_i)_{i \in I}, \mathscr{F})$, for $(X_i)_{i \in I}$ a covering family of X, modulo the following equivalence relation : if $\mathscr{X}_1$ and $\mathscr{X}_2$ are covering families of X and $s_1 \in \check{\mathrm{H}}^0(\mathscr{X}_1, \mathscr{F})$, $s_2 \in \check{\mathrm{H}}^0(\mathscr{X}_2, \mathscr{F})$, then s_1 and s_2 are equivalent if and only if there exists a covering family $\mathscr{X}$ of X and morphisms $u_1 : \mathscr{X} \to \mathscr{X}_1$, $u_2 : \mathscr{X} \to \mathscr{X}_2$ (over id_X) such that $u_1^*(s_1) = u_2^*(s_2)$ in $\check{\mathrm{H}}^0(\mathscr{X}, F)$.

Remark III.2.2.9. Suppose that $\mathscr{F}$ is a presheaf of left R-modules on $\mathscr{C}$. As the forgetful functor ${}_R\mathbf{Mod} \to \mathbf{Set}$ commutes with filtrant colimits, we can form the presheaf $X \longmapsto \check{\mathrm{H}}^0(X, \mathscr{F})$ by taking the colimits in the category ${}_R\mathbf{Mod}$, and we will get the same result (up to unique isomorphism). Same remark for right R-modules. So $\mathscr{F}^+$ is also a presheaf of R-modules, and in fact

$\mathscr{F} \longmapsto \mathscr{F}^+$ defines an endofunctor $\mathrm{PSh}(\mathscr{C}, R)$. Note also that the morphism $\iota_0(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^+$ is a morphism of presheaves of R-modules, because it is defined using restriction maps for $\mathscr{F}$, which are R-linear.

Proposition III.2.2.10. *Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$.*

(i). The presheaf $\mathscr{F}^+$ is separated.

(ii). If $\mathscr{F}$ is a separated presheaf, then $\iota_0(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^+$ is injective.

(iii). If $\mathscr{F}$ is a separated presheaf, then $\mathscr{F}^+$ is a sheaf.

(iv). The presheaf $\mathscr{F}^{++}$ is a sheaf.

(v). If $\mathscr{F}$ is a sheaf, then the morphism $\iota_0(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

(vi). If $\mathscr{G}$ is a sheaf and $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then there exists a unique morphism of presheaves $u' : \mathscr{F}^{++} \to \mathscr{G}$ such that $u = u' \circ \iota_0(\mathscr{F}^+) \circ \iota_0(\mathscr{F})$:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "u"] \arrow[d] & \mathscr{G} \\ \mathscr{F}^{++} \arrow[ur, "u'"'] & \end{tikzcd}$$

If moreover $\mathscr{F}$ is a presheaf of R-modules and $\mathscr{G}$ is a sheaf of R-modules, then u is a morphism of presheaves of R-modules if and only if u' is.

In particular, the functor $\mathscr{F} \longmapsto \mathscr{F}^{++}$ can be seen as a functor from the category of presheaves to the category of sheaves; this is called the *sheafification functor* and denote by $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$. If we see $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$ as an endofunctor $\mathrm{PSh}(\mathscr{C})$, we have a morphism of functors $\iota : \mathrm{id}_{\mathrm{PSh}(\mathscr{C})} \to (\cdot)^{\mathrm{sh}}$ given by $\iota(\mathscr{F}) = \iota_0(\mathscr{F}^+) \circ \iota_0(\mathscr{F})$. If $\mathscr{F}$ is a presheaf of R-modules, then $\iota(\mathscr{F})$ is clearly a morphism of presheaves of R-modules (see Remark III.2.2.9).

Proof. (i). Let X be an object of $\mathscr{C}$ and $\mathscr{X} = (f_i : X_i \to X)_{i \in I}$ be a covering family. Let $s, t \in \mathscr{F}^+(X)$ such that $f_i^*(s) = f_i^*(t)$ for every $i \in I$. We choose a covering family $\mathscr{Y} = (g_j : Y_j \to X)_{j \in J}$ and elements $(s_j)_{j \in J}$, $(t_j)_{j \in J}$ of $\check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F})$ representing s and t. By Lemma III.2.2.7(ii), we may assume that there is a morphism $u = (\alpha, (u_j)) : \mathscr{Y} \to \mathscr{X}$ in $\mathscr{I}_X$. Let $j \in J$. Then we have $u_j : Y_j \to X_{\alpha(j)}$, and so the fact that $f_i^*(s) = f_i^*(t)$ in $\mathscr{F}^+(X_i)$ implies that s_j and t_j define the same element of $\mathscr{F}^+(Y_j)$; by Remark III.2.2.8, this means that there exists a covering family $(h_{jk} : Z_{jk} \to Y_j)_{k \in K_j}$ such that $h_{jk}^* g_j^*(s) = h_{jk}^* g_j^*(t)$ in in $\mathscr{F}(Z_{jk})$ for every $k \in J_j$. Consider the covering family $\mathscr{Z} = (g_j \circ h_{jk} : Z_{jk} \to X)_{j \in J, k \in J_k}$ (this is covering by (CF2)). Then we have a morphism $v : \mathscr{Z} \to \mathscr{Y}$ (given by the h_{jk}), and the images of the families (s_j) and (t_j) in $\check{\mathrm{H}}^0(\mathscr{Z}, F)$ are equal, so (s_j) and (t_j) define the same element of $\mathscr{F}^+(X)$.

(ii). This follows immediately from Remarks III.2.2.5 and III.2.2.8.

(iii). Let X be an object of $\mathscr{X}$, $\mathscr{X} = (f_i : X_i \to X)_{i\in I}$ be a covering family and $(s_i) \in \prod_{i\in I} \mathscr{F}^+(X_i)$ such that $p_{i,ij}^*(s_i) = p_{j,ij}^*(s_j)$ for all $i,j \in I$, where $p_{i,ij} : X_i \times_X X_j \to X_i$ and $p_{j,ij} : X_i \times_X X_j \to X_j$ are the projections. For every $i \in I$, choose a covering family $\mathscr{X}_i = (g_{ij} : X_{ij} \to X_i)_{j\in J_i}$ and an element $(s_{ij})_{j\in I_j}$ of $\check{\mathrm{H}}^0(\mathscr{X}_i, \mathscr{F})$ representing s_i. Let $\mathscr{Y} = (f_i \circ g_{ij} : X_{ij})_{i\in I, j\in J_i}$; this is a covering family by (CF2). We claim that the family $(s_{ij})_{i\in I, j\in J_i}$ is an element of $\check{\mathrm{H}}^0(\mathscr{Y}, \mathscr{F})$. Let $i, i' \in I$, $j \in J_i$ and $j' \in J_{i'}$; we need to show that $q_1^*(s_{ij}) = q_2^*(s_{i'j''})$ in $\mathscr{F}(X_{ij} \times_X X_{i'j'})$, where $q_1 : X_{ij} \times_X X_{i'j'} \to X_{ij}$ and $q_2 : X_{ij} \times_X X_{i'j'} \to X_{i'j'}$ are the two projections. We know that $p_{i,ii'}^*(s_i) = p_{i',ii'}^*(s_{i'})$ in $\mathscr{F}^+(X_i \times_X X_{i'})$; as the morphism $X_{ij} \times_X X_{i'j'} \to X$ factors through $X_i \times_X X_{i'} \to X$ (take the morphism $X_{ij} \times_X X_{i'j'} \to X_i \times_X X_{i'}$ induced by $(g_{ij}, g_{i'j'})$), this implies that $q_1^*(s_{ij})$ and $q_2^*(s_{i'j''})$ define the same element of $\mathscr{F}^+(X_{ij} \times_X X_{i'j'})$, and, by (ii), we get that they are equal in $\mathscr{F}(X_{ij} \times_X X_{i'j'})$.

Now let s' be the element of $\mathscr{F}^+(X)$ defined by the family $(s_{ij})_{i\in I, j\in J_i} \in \check{\mathrm{H}}^0(\mathscr{Y}, F)$. It remains to show that $f_i^*(s') = s_i$ in $\mathscr{F}^+(X_i)$ for every $i \in I$. Let $i \in I$. Then $f_i^*(s_i) \in \mathscr{F}^+(X_i)$ is represented by the family $(s_{ij})_{j\in J_i} \in \check{\mathrm{H}}^0(\mathscr{X}_i, \mathscr{F})$, so it is equal to s_i.

(iv). This follows immediately from (i) and (iii).

(v). This follows immediately from Remark III.2.2.5.

(vi). We have a commutative square

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "u"'] & \mathscr{F}^{++} \arrow[d, "u^{++}"] \\ \mathscr{G} \arrow[r] & \mathscr{G}^{++} \end{tikzcd}$$

where the horizontal morphisms are given by two applications of ι_0. By (v), the morphism $\mathrm{G} \to \mathscr{G}^{++}$ is an isomorphism, so we get the desired $u' : \mathscr{F}^{++} \to \mathscr{G}$ by composing the inverse of this isomorphism and u^{++}. As $\iota_0(\mathscr{H})$ is a morphism of presheaves of R-modules for every presheaf of R-modules $\mathscr{H}$, and as $\mathscr{H} \longmapsto \mathscr{H}^+$ also induces an endofunctor of $\mathrm{PSh}(\mathscr{C}, R)$, we see that u' is a morphism of presheaves of R-modules if u is.

Suppose that we have another morphism of sheaves $u'' : \mathscr{F}^{++} \to \mathscr{G}$ such that $u = u'' \circ \iota_0(\mathscr{F}^+) \circ \iota_0(\mathscr{F})$. Let X be an object of $\mathscr{C}$, let $s \in \mathscr{F}^{++}(X)$. We want to show that $u'(s) = u''(s)$. By Remark III.2.2.8 (applied twice), we can find a covering family $(f_i : X_i \to X)_{i\in I}$ and sections $s_i \in \mathscr{F}(X_i)$ such that $f_i^*(s)$ is equal to the image of s_i in $\mathscr{F}^{++}(X_i)$ for every $i \in I$. By the condition on u' and u'', we have, for every $i \in I$, $f_i^*(u'(s)) = u(s_i) = f_i^*(u''(s))$. As $\mathscr{G}$ is a sheaf, this implies that $u'(s) = u''(s)$. The same kind of proof shows that, if $\mathscr{F}$ is a presheaf of R-modules, $\mathscr{G}$ is a sheaf of R-modules and $u' : \mathscr{F}^{++} \to \mathscr{G}$ is a morphism of presheaves of R-modules, then $u : \mathscr{F} \to \mathscr{G}$ also respects the R-module structures on sections.

□

Corollary III.2.2.11. *The functor* $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$ *from* $\mathrm{PSh}(\mathscr{C})$ *to* $\mathrm{Sh}(\mathscr{C})$ *is right adjoint to the inclusion* $\mathrm{Sh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$.

We have a similar result for presheaves of R*-modules : the functor* $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$ *from* $\mathrm{PSh}(\mathscr{C}, R)$ *to* $\mathrm{Sh}(\mathscr{C}, R)$ *is right adjoint to the inclusion* $\mathrm{Sh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C}, R)$.

Proof. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$ and $\mathscr{G}$ be a sheaf on $\mathscr{C}$. We have a map $\mathrm{Hom}_{\mathrm{Sh}(\mathscr{C})}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \mathscr{G})$ sending a morphism of sheaves $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ to $f \circ \iota(\mathscr{F}) : \mathscr{F} \to \mathscr{G}$. This clearly defines a morphism of functors (from $\mathrm{PSh}(\mathscr{C})^{\mathrm{op}} \times \mathrm{Sh}(\mathscr{C})$ to $\mathbf{Set}$), and point (vi) of Proposition III.2.2.10 says that it is an isomorphism.

The second sentence follows from the last part of Proposition III.2.2.10(vi).

□

Example III.2.2.12. If $\mathscr{C}$ is any category and S is a set, the *constant presheaf* on $\mathscr{C}$ with value S is the functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending any object to S and any morphism to id_S. If $(\mathscr{C}, \mathscr{T})$ is a site, the *constant sheaf* on $\mathscr{C}_{\mathscr{T}}$ with value S, often denoted by $\underline{S}_{\mathscr{C}_{\mathscr{T}}}$ or just $\underline{S}$, is the sheafification of the constant presheaf on $\mathscr{C}$ with value S.

Corollary III.2.2.13. *The category* $\mathrm{Sh}(\mathscr{C})$ *has all* $\mathscr{U}$*-small limits and colimits, the inclusion functor* $\mathrm{Sh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$ *commutes with* $\mathscr{U}$*-small limits, and the sheafification functor* $\mathrm{PSh}(\mathscr{C}) \to \mathrm{Sh}(\mathscr{C})$ *commutes with all* $\mathscr{U}$*-small colimits and with finite limits.*

We have similar results for presheaves and sheaves of R*-modules.*

Moreover, the forgetful functors $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C})$ *and* $\mathrm{Sh}(\mathscr{C}, R) \to \mathrm{Sh}(\mathscr{C})$ *commute with limits and with filtrant colimits.*

Proof. We prove the existence of limits and colimits (and the commutation properties of the inclusion and sheafification functors) for presheaves and sheaves of sets. The proofs for presheaves and sheaves of R-modules are similar.

If $\mathscr{F}$ is a presheaf, then $\mathscr{F}$ is a sheaf if and only if, for every object X of $\mathscr{C}$ and every covering family $\mathscr{X}$ of X, the canonical morphism $\mathscr{F}(X) \to \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ is an isomorphism. The functor $\mathscr{F} \longmapsto \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ is constructed from functors that commute with limits (taken sections of $\mathscr{F}$ on objects of $\mathscr{C}$, products, kernels), so it commutes with limits.

This shows that, if we have a functor $\alpha : \mathscr{I} \to \mathrm{Sh}(\mathscr{C})$, with $\mathscr{I}$ a $\mathscr{U}$-small category, then its limit in the category of presheaves (that is, the limit of its composition with the inclusion functor $\mathrm{Sh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$) is a sheaf. So limits exist in $\mathrm{Sh}(\mathscr{C})$, and the inclusion $\mathrm{Sh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$ commutes with limits.

Now let α be a functor from $\mathscr{I}$ to $\mathrm{PSh}(\mathscr{C})$, and let $\mathscr{F} = \varinjlim \alpha$. Then we have an isomorphism of functors $\mathrm{Sh}(\mathscr{C}) \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathrm{Sh}(\mathscr{C})}(\mathscr{F}^{\mathrm{sh}}, \cdot) \simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \cdot) \simeq \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\alpha(i), \cdot) = \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathrm{Sh}(\mathscr{C})}(\alpha(i)^{\mathrm{sh}}, \cdot).$$

If each $\alpha(i)$ is a sheaf, that is, if α is actually a functor from $\mathscr{I}$ to $\mathrm{Sh}(\mathscr{C})$, this shows that $\mathscr{F}^{\mathrm{sh}}$ is the colimit of this functor in $(\mathrm{Sh}(\mathscr{C}))$. In general, it shows that $(.)^{\mathrm{sh}}$ commutes with colimits (which would also follow from the fact that $(.)^{\mathrm{sh}}$ is a left adjoint and Proposition I.5.4.3, once we know that colimits exist in $\mathrm{Sh}(\mathscr{C})$).

We show that the sheafification functor commutes with finite limits. In fact, the functor $\mathscr{F} \longmapsto \mathscr{F}^{+}$ commutes with finite limits. Indeed, thanks to the way limits are calculate in $\mathrm{PSh}(\mathscr{C})$ and $\mathrm{Sh}(\mathscr{C})$, it suffices to show that, for every $X \in \mathrm{Ob}(\mathscr{C})$, the functor $\mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$, $\mathscr{F} \longmapsto \check{\mathrm{H}}^0(X, \mathscr{F})$ commutes with finite limits. We know that $\mathscr{F} \longmapsto \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ commutes with $\mathscr{U}$-small limits for every covering family $\mathscr{X}$ of X, and $\check{\mathrm{H}}^0(X, \cdot)$ is a filtrant colimit of these functors, so the result follows from the fact that filtrant colimits commute with finite limits (Proposition I.5.6.4).

The fact that the forgetful functor $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C})$ commutes with $\mathscr{U}$-small limits and $\mathscr{U}$-small filtrant colimits follows from the way these limits and colimits are calculated in categories of presheaves (Proposition I.5.3.1) and from the fact that the forgetful functor ${}_R\mathbf{Mod} \to \mathbf{Set}$ commutes with $\mathscr{U}$-small limits and $\mathscr{U}$-small filtrant colimits (Subsection I.5.5.1 and Corollary I.5.6.3). This implies immediately that the forgetful functor $\mathrm{Sh}(\mathscr{C}, R) \to \mathrm{Sh}(\mathscr{C})$ commutes with $\mathscr{U}$-small limits, and we get the commutation with $\mathscr{U}$-small filtrant colimits in the same way, thanks to Remark III.2.2.9.

□

Corollary III.2.2.14. *The category* $\mathrm{Sh}(\mathscr{C}, R)$ *is abelian, the inclusion functor* $\mathrm{Sh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C}, R)$ *is left exact, and the sheafification functor* $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{Sh}(\mathscr{C})$ *is exact.*

Proof. Everything follows immediately from Corollary III.2.2.13 except for the fact that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for every morphism f of $\mathrm{Sh}(\mathscr{C}, R)$. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathrm{Sh}(\mathscr{C}, R)$, and let f_0 be f, seen as a morphism in $\mathrm{PSh}(\mathscr{C}, R)$. We have $\mathrm{Ker}(f) = \mathrm{Ker}(f_0)$, so $\mathrm{Coim}(f) = \mathrm{Coker}(\mathrm{Ker}(f) \to \mathscr{F}) = \mathrm{Coim}(f_0)^{\mathrm{sh}}$. Similar, $\mathrm{Coker}(f) = \mathrm{Coker}(f_0)^{\mathrm{sh}}$, so

$$\mathrm{Im}(f) = \mathrm{Ker}(\mathscr{G} \to \mathrm{Coker}(f_0)^{\mathrm{sh}}) = \mathrm{Ker}(\mathscr{G}^{\mathrm{sh}} \to \mathrm{Coker}(f_0)^{\mathrm{sh}}) = \mathrm{Im}(f_0)^{\mathrm{sh}},$$

by the exactness of the sheafification functor. Let $u_0 : \mathrm{Coim}(f_0) \to \mathrm{Im}(f_0)$ be the canonical morphism. Then $u = u_0^{\mathrm{sh}} : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "f"] \arrow[d] & \mathscr{G} \arrow[d] \\
\mathrm{Coim}(f_0) \arrow[r, "u_0"] \arrow[d, "\iota(\mathrm{Coim}(f_0))"'] & \mathrm{Im}(f_0) \arrow[d, "\iota(\mathrm{Im}(f_0))"] \\
\mathrm{Coim}(f) \arrow[r, "u"] & \mathrm{Im}(f)
\end{tikzcd}$$

so it is equal to the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$. But u_0 is an isomorphism because $\mathrm{PSh}(\mathscr{C}, R)$ is an abelian category, so u is also an isomorphism.

□

Corollary III.2.2.15. *Suppose that $\mathscr{C}$ is a $\mathscr{U}$-small category (and not just a $\mathscr{U}$-category).*[5] *The categories $\mathrm{PSh}(\mathscr{C}, R)$ and $\mathrm{Sh}(\mathscr{C}, R)$ are Grothendieck abelian categories.*

Proof. By the assumption on $\mathscr{C}$, the categories $\mathrm{PSh}(\mathscr{C}, R)$ and $\mathrm{Sh}(\mathscr{C}, R)$ are $\mathscr{U}$-categories. We already know that they have all $\mathscr{U}$-small colimits, and that $\mathscr{U}$-small filtrant colimits are exact in $\mathrm{PSh}(\mathscr{C}, R)$ (because of Proposition I.5.3.1 and Corollary I.5.6.5). Also, all colimits are right exact in $\mathrm{Sh}(\mathscr{C}, R)$ (because colimits commutes with colimits, see Subsection I.5.4.1), so it suffices to show that filtrant colimits are left exact in $\mathrm{Sh}(\mathscr{C}, R)$; but they are obtained by applying the forgetful functor $\mathrm{Sh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C}, R)$, which is left exact, taking the filtrant colimits in $\mathrm{PSh}(\mathscr{C}, R)$, which we have just seen is an exact operation, and applying the sheafification functor, which is exact.

To finish the proof, we need to give generators of $\mathrm{PSh}(\mathscr{C}, R)$ and $\mathrm{Sh}(\mathscr{C}, R)$. Let $\mathscr{Q} = \bigoplus_{X \in \mathrm{Ob}(\mathscr{C})} R^{(X)}$, where $R^{(X)}$ is the presheaf sending $Y \in \mathrm{Ob}(\mathscr{C})$ to the free R-module on $\mathrm{Hom}_{\mathscr{C}}(Y, X)$. We saw in Example II.3.1.2(4) that $\mathscr{Q}$ is a projective generator of $\mathrm{PSh}(\mathscr{C}, R)$, because, for every presheaf of R-modules $\mathscr{F}$ of $\mathscr{C}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(\mathscr{Q}, \mathscr{F}) \simeq \prod_{X \in \mathrm{Ob}(\mathscr{C})} \mathscr{F}(X).$$

So, if $\mathscr{F}$ is a sheaf of R-modules, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(\mathscr{C},R)}(\mathscr{Q}^{\mathrm{sh}}, \mathscr{F}) \simeq \prod_{X \in \mathrm{Ob}(\mathscr{C})} \mathscr{F}(X).$$

This shows that the functor $\mathrm{Hom}_{\mathrm{Sh}(\mathscr{C},R)}(\mathscr{Q}^{\mathrm{sh}}, \cdot)$ is conservative, so $\mathscr{Q}^{\mathrm{sh}}$ is a generator of $\mathrm{Sh}(\mathscr{C}, R)$. [6]

□

III.3 The Freyd-Mitchell embedding theorem

In this section, we put a lot of previous results together to prove the following theorem, know as the *Freyd-Mitchell embedding theorem.*

Theorem III.3.1. *Let $\mathscr{U}$ be a universe and $\mathscr{A}$ be a $\mathscr{U}$-small abelian category. Then there exists a ring R whose underlying set is an element of $\mathscr{U}$ and a fully faithul exact functor $\mathscr{A} \to {}_R\mathbf{Mod}_{\mathscr{U}}$.*

[5] We can always make this true by replacing $\mathscr{U}$ with a bigger universe.

[6] But it is not projective in general, because the functors $\mathrm{Sh}(\mathscr{C}, R) \to {}_R\mathbf{Mod}$, $\mathscr{F} \longmapsto \mathscr{F}(X)$ are not exact.

Proof. Let $\mathscr{B} = \mathrm{Sh}(\mathscr{A}_{\mathrm{can}}, \mathbf{Ab}_{\mathscr{U}})$ be the category of sheaves of abelian groups on $\mathscr{A}$ equipped with its canonical topology (see problem A.3.6). We know by question (c) of problem A.3.1 that the Yoneda functor $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ factors through $\mathscr{B}$, the Yoneda lemma (Theorem I.3.2.2) says that this functor is fully faithul, and we know by problem A.4.3 that this functor is exact. So we get an exact and fully faithful functor $h : \mathscr{A} \to \mathscr{B}$.

Next, Corollary III.2.2.15 implies that $\mathscr{B}$ is a Grothendieck abelian category, so it has an injective cogenerator by Corollary II.3.2.8. We also know that $\mathscr{B}$ has all $\mathscr{U}$-small limits by Corollary III.2.2.13. So the opposite category $\mathscr{B}^{\mathrm{op}}$ (which is also abelian) has all $\mathscr{U}$-small limits and a projective generator. By problem A.4.4(d) and the fact that $h(\mathscr{A})$ is $\mathscr{U}$-small, there exists a fully faithful exact functor $h(\mathscr{A})^{\mathrm{op}} \to \mathbf{Mod}_S = {}_{S^{\mathrm{op}}}\mathbf{Mod}$, where S is the ring of endomorphisms of an object of $\mathscr{B}^{\mathrm{op}}$; in particular, the underlying set of S is an element of $\mathscr{U}$. So we get a fully faithful exact functor $\mathscr{A}^{\mathrm{op}} \to {}_R\mathbf{Mod}$, where $R = S^{\mathrm{op}}$.

Applying this construction to the $\mathscr{U}$-small abelian category $\mathscr{A}^{\mathrm{op}}$, we get the desired result.

□

IV Complexes

IV.1 Complexes in additive categories

IV.1.1 Definitions

Let $\mathscr{C}$ be an additive category. (For many of the definitions, we only need $\mathscr{C}$ to be preadditive.)

Definition IV.1.1.1. A a functor from $\mathbb{Z}$ to $\mathscr{C}$ is also called a *differential object* in $\mathscr{C}$. We usually denote it by $(X^\bullet, d_X^\bullet)$, where $X^n \in \mathrm{Ob}(\mathscr{C})$ is the image of $n \in \mathbb{Z}$ and $d_X^n : X^n \to X^{n+1}$ is the image of the unique morphism $n \to n+1$ of $\mathbb{Z}$. (We can deduce the images of all the other morphisms from this information.) The morphisms d_X^n are called the *differentials* of the differential object; sometimes we just write $X^\bullet$ if the differential are clear from the context.

A differential object $(X^\bullet, d_X^\bullet)$ is called a *complex* (or *cohomological complex*, or *cochain complex*) of objects of $\mathscr{C}$ if $d_X^{n+1} \circ d_X^n = 0$ for every $n \in \mathbb{Z}$.

The category $\mathrm{Func}(\mathbb{Z}, \mathscr{C})$ is additive, and complexes form a full additive subcategory of $\mathrm{Func}(\mathbb{Z}, \mathscr{C})$, that we denote by $\mathcal{C}(\mathscr{C})$.

Remark IV.1.1.2. Homological complexes (or chain complexes) of objects of $\mathscr{C}$ are defined similarly, using contravariant functors $\mathbb{Z} \to \mathscr{C}$. As $\mathbb{Z} \simeq \mathbb{Z}^{\mathrm{op}}$, we get an isomorphic category, and the choice between cohomological and homological complex is mostly a matter of taste and habit. We will stick to cohomological complexes in these notes and just call them complexes.

If $X : \mathbb{Z}^2 \to \mathscr{C}$ is a functor, we can similarly write X as a triple $(X^{n,m}, d_{1,X}^{n,m}, d_{2,X}^{n,m})_{n,m\in\mathbb{Z}}$, where $X^{n,m} = X((n,m))$ is an object of $\mathscr{C}$ and $d_{1,X}^{n,m} : X^{n,m} \to X^{n+1,m}$, $d_{2,X}^{n,m} : X^{n,m+1}$ are morphisms such that, for all $n, m \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd} X^{n,m} \arrow[r, "d_{1,X}^{n,m}"] \arrow[d, "d_{2,X}^{n,m}"'] & X^{n+1,m} \arrow[d, "d_{2,X}^{n,m+1}"] \\ X^{n,m+1} \arrow[r, "d_{1,X}^{n+1,m}"'] & X^{n+1,m+1} \end{tikzcd}$$

Definition IV.1.1.3. We say that a functor $(X^{n,m}, d_{1,X}^{n,m}, d_{2,X}^{n,m})_{n,m\in\mathbb{Z}}$ from $\mathbb{Z}^2$ to $\mathscr{C}$ is a *double complex* if, for all $n, m \in \mathbb{Z}$, we have $d_{2,X}^{n,m+1} \circ d_{2,X}^{n,m} = 0$ and $d_{1,X}^{n+1,m} \circ d_{1,X}^{n,m} = 0$.

Double complexes form a full additive subcategory of $\mathrm{Func}(\mathbb{Z}^2, \mathscr{C})$, that we denote by $\mathcal{C}^2(\mathscr{C})$.

We have two isomorphisms of categories $F_I, F_{II} : \mathcal{C}^2(\mathscr{C}) \to \mathcal{C}(\mathcal{C}(\mathscr{C}))$ that send a double complex X to the complex whose objects are the rows (resp. the columns) of X.

Definition IV.1.1.4. We say that a complex $(X^\bullet, d_X^\bullet)$ is *bounded above* (resp. *bounded below*, resp. *bounded*) if $X^n = 0$ for $n >> 0$ (resp. $n << 0$, resp. $|n| >> 0$).

We denote by $\mathcal{C}^+(\mathscr{C})$ (resp. $\mathcal{C}^-(\mathscr{C})$, resp. $\mathcal{C}^b(\mathscr{C})$) the full additive subcategory of $\mathcal{C}(\mathscr{C})$ whose objects are bounded below complexes (resp. bounded above complexes, resp. bounded complexes).

More generally, we say that a complex X is *concentrated in degrees betwen a and b* if $X^n = 0$ for $n \notin [a, b]$, and we denote by $\mathcal{C}^{[a,b]}(\mathscr{C})$ the full subcategory of these complexes. We also write $\mathcal{C}^{\geq a}(\mathscr{C})$ and $C^{\leq b}(\mathscr{C})$ for $\mathcal{C}^{[a,+\infty]}(\mathscr{C})$ and $\mathcal{C}^{[-\infty,b]}(\mathscr{C})$.

Remark IV.1.1.5. There is a fully faithful additive functor $\mathscr{C} \to \mathcal{C}(\mathscr{C})$ sending $A \in \mathrm{Ob}(\mathscr{C})$ to the complex $(X^\bullet, d_X^\bullet)$ defined by $X^0 = A$, $X^n = 0$ for $n \neq 0$ and $d_X^n = 0$ for every $n \in \mathbb{Z}$. We often use this to identity $\mathscr{C}$ to a full subcategory of $\mathcal{C}(\mathscr{C})$ (that we call the category of complexes *concentrated in degree* 0).

We define two basic operations on complexes: the shift and the mapping cone.

Definition IV.1.1.6. Let $X = (X^\bullet, d_X^\bullet)$ be a complex and $n \in \mathbb{Z}$. We define a complex $X[n]$ by :

- $(X[n])^k = X^{n+k}$;
- $d^k_{X[n]} = (-1)^n d^k_{X[n]}$.

If $f : X \to Y$ is a morphism of complexes, we define $f[n] : X[n] \to Y[n]$ by $(f[n])^k = f^{n+k} : (X[n])^k = X^{n+k} \to (Y[n])^k = Y^{n+k}$.

We get an endofunctor $X \longmapsto X[n]$ of the category $\mathcal{C}(\mathscr{C})$, which is an automorphism with inverse $X \longmapsto X[-n]$. Note that this functor preserves the subcategories $\mathcal{C}^*(\mathscr{C})$, for $* \in \{+, -, b\}$.

Definition IV.1.1.7. Let $f : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{C})$. The *mapping cone* of f is the differential object $\mathrm{Mc}(f)$ defined by :

- $\mathrm{Mc}^n(f) = (X[1])^n \oplus Y^n = X^{n+1} \oplus Y^n$;
- $d^n_{\mathrm{Mc}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$.

It is easy to check that $\mathrm{Mc}(f)$ is actually a complex. We have two morphisms $\alpha(f) : Y \to \mathrm{Mc}(f)$ and $\beta(f) : \mathrm{Mc}(f) \to X[1]$ defined by $\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta(f)^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}$.

Note that $\beta(f) \circ \alpha(f) = 0$, and that all the constructions of the definition are functorial in f; that is, if we have a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g"'] & Y_1 \arrow[d, "h"] \\ X_2 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$

in $\mathcal{C}(\mathscr{C})$, then we get a morphism $\mathrm{Mc}(g,h) : \mathrm{Mc}(f_1) \to \mathrm{Mc}(f_2)$, given by $\mathrm{Mc}(g,h)^n = \begin{pmatrix} g^{n+1} & 0 \\ 0 & h^n \end{pmatrix}$ (or, in other words, $\mathrm{Mc}(g,h) = \begin{pmatrix} g[1] & 0 \\ 0 & h \end{pmatrix}$), such that the following diagram commutes:

$$\begin{tikzcd} Y_1 \arrow[r, "\alpha(f_1)"] \arrow[d, "h"'] & \mathrm{Mc}(f_1) \arrow[r, "\beta(f_1)"] \arrow[d, "\mathrm{Mc}(g,h)"'] & X_1[1] \arrow[d, "g[1]"'] \\ Y_2 \arrow[r, "\alpha(f_2)"'] & \mathrm{Mc}(f_2) \arrow[r, "\beta(f_2)"'] & X_2[1] \end{tikzcd}$$

Remark IV.1.1.8. Although $\mathrm{Mc}(f)^n = X[1]^n \oplus Y^n$ for every $n \in \mathbb{Z}$, we don't have $\mathrm{Mc}(f) \simeq X[1] \oplus Y$ in $\mathcal{C}(\mathscr{C})$ in general. For example, if $f = \mathrm{id}_X$, then $\mathrm{Mc}(f)$ is homotopic to 0, and $X \oplus X[1]$ might not be (just take X nonzero and concentrated in degre 0).

Remark IV.1.1.9. If X and Y are in $\mathcal{C}^*(\mathscr{C})$ (with $* \in \{+,-,b\}$), then the mapping cone of any morphism $X \to Y$ is also in $\mathcal{C}^*(\mathscr{C})$.

IV.1.2 Homotopy

Definition IV.1.2.1. (i). Two morphisms $f, g : X \to Y$ in $\mathcal{C}(\mathscr{C})$ are called *homotopic* if there exist morphism $s^n : X^n \to Y^{n-1}$, for every $n \in \mathbb{Z}$, such that

$$f^n - g^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n.$$

The maps s^n are shown on the following *non*-commutative diagram:

$$\begin{tikzcd} X^{n-1} \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n - g^n"] & X^{n+1} \arrow[dl, "s^{n+1}"] \\ Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \end{tikzcd}$$

The family (s^n) is called a *homotopy* between f and g.

(ii). A complex $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{C}))$ is called *homotopic to* 0 if id_X is homotopic to 0.

(iii). We say that a morphism $f : X \to Y$ in $\mathcal{C}(\mathscr{C})$ is a *homotopy equivalence* if there exists $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Example IV.1.2.2. If $\mathscr{A}$ is an abelian category and $0 \to A \to B \to C \to 0$ is a short exact sequence of objects of $\mathscr{A}$ seen as a complex (for example with A in degree 0, it doesn't matter), then this complex is homotopic to 0 if and only if the short exact sequence is split. Indeed, the existence of the homotopy is exactly condition (d) of Proposition II.2.1.11.

Definition IV.1.2.3. For all $X, Y \in \mathcal{C}(\mathscr{C})$. We denote by $Ht(X, Y)$ the set of morphisms $X \to Y$ that are homotopic to 0.

The following lemma follows immediately from the definition.

Lemma IV.1.2.4. *(i). For all $X, Y \in \mathrm{Ob}(\mathcal{C}(\mathscr{C}))$, the set $Ht(X, Y)$ is a subgroup of $\mathrm{Hom}_{\mathcal{C}(\mathscr{C})}(X, Y)$.*

(ii). Let $f : X' \to X$ and $g : Y \to Y'$ be morphisms in $\mathcal{C}(\mathscr{C})$. If $u : X \to Y$ is homotopic to 0, so are $u \circ f$ and $g \circ u$. More precisely, if (s^n) is a homotopy between u and 0, then $(s^n \circ f^n)$ (resp. $(g^n \circ s^n)$) is a homotopy between $u \circ f$ (resp. $g \circ u$) and 0.

IV.1.3 Quotient of preadditive categories

The construction in this subsection is a generalization of the construction of the quotient of a ring by a two-sided ideal.

Let $\mathscr{D}$ be a preadditive category.

Definition IV.1.3.1. An *ideal* $\mathscr{I}$ in $\mathscr{D}$ is the data, for all objects $A, B \in \mathrm{Ob}(\mathscr{D})$, of a subgroup $I_{A,B}$ of $\mathrm{Hom}_{\mathscr{D}}(A, B)$, such that the following condition hold : for all morphisms $f : A' \to A$ and $g : B \to B'$ in $\mathscr{D}$ and for every $u \in I_{A,B}$, we have $u \circ f \in I_{A',B}$ and $g \circ u \in I_{A,B'}$.

The *quotient* of $\mathscr{D}$ by the ideal $\mathscr{I}$ is the preadditive category $\mathscr{D}/\mathscr{I}$ given by :

- $\mathrm{Ob}(\mathscr{D}/\mathscr{I}) = \mathrm{Ob}(\mathscr{D})$;
- for all $A, B \in \mathrm{Ob}(\mathscr{D})$, $\mathrm{Hom}_{\mathscr{D}/\mathscr{I}}(A, B) = \mathrm{Hom}_{\mathscr{D}}(A, B)/I_{A,B}$;
- the composition law is induced by that of $\mathscr{D}$.

The functor $\mathscr{D} \to \mathscr{D}/\mathscr{I}$ that is the identity on objects and the quotient map on morphisms is called the canonical quotient functor. It is an additive functor.

The condition in the definition of an ideal is exactly what we needed to ensure that the composition law of $\mathscr{D}$ would go to the quotient and define a bilinear composition law in $\mathscr{D}/\mathscr{I}$.

The quotient category has the following obvious universal property: for every additive functor $F : \mathscr{D} \to \mathscr{D}'$ into another preadditive category, if $F(f) = 0$ for every $f \in I_{A,B}$, then there exists a unique functor $\overline{F} : \mathscr{D}/\mathscr{I} \to \mathscr{D}'$ such that $F = \overline{F} \circ q$, where q is the canonical quotient functor; this functor $\overline{F}$ is automatically additive.

Lemma IV.1.3.2. *If $\mathscr{D}$ is an additive category, then its quotient by an ideal is also an additive category.*

Proof. This follows easily from Proposition II.1.1.6. Indeed, if $A_1, \dots, A_n$ are objects of $\mathscr{D}$, then their product satisfy condition (iii) of that proposition, and then the image of this product in the quotient category also satisfies condition (iii).

□

IV.1.4 The homotopy category

We come back to complexes. Let $\mathscr{C}$ be an additive category.

Definition IV.1.4.1. Let $* \in \{\varnothing, +, -, b\}$. Then the subsets $Ht(X,Y)$ define an ideal in $\mathcal{C}^*(\mathscr{C})$ by Lemma IV.1.2.4. We denote the quotient category by $K^*(\mathscr{C})$ and call it the *homotopy category* of $\mathcal{C}^*(\mathscr{C})$. It is an additive category.

If $* \in \{+, -, b\}$, we have an obvious functor $K^*(\mathscr{C}) \to K(\mathscr{C})$, which is fully faithful.

Remark IV.1.4.2. Composing the fully faithful functor additive $\mathscr{C} \to \mathcal{C}(\mathscr{C})$ of Remark IV.1.1.5 with the quotient functor $\mathcal{C}(\mathscr{C}) \to K(\mathscr{C})$, we get a fully faithful additive functor $\mathscr{C} \to K(\mathscr{C})$.

Remark IV.1.4.3. A complex X in $\mathcal{C}(\mathscr{C})$ is homotopic to 0 if and only if its image in $K(\mathscr{C})$ is isomorphic to 0, and a morphism of $\mathcal{C}(\mathscr{C})$ is a homotopy equivalence if and only if its image in $K(\mathscr{C})$ is an isomorphism.

Remark IV.1.4.4. The shift endofunctors $[p]$ of $\mathcal{C}(\mathscr{C})$ (for $p \in \mathbb{Z}$) defines an endofunctor of $K(\mathscr{C})$, that we will still denote by $[p]$.

Remark IV.1.4.5. Let $F : \mathscr{C} \to \mathscr{C}'$ be an additive functor. It induces an additive functor $\mathcal{C}(\mathscr{C}) \to \mathcal{C}(\mathscr{C}')$ that we will denote by F. If $f, g : X \to Y$ are homotopic morphisms in $\mathcal{C}(\mathscr{C})$, then $F(f)$ and $F(g)$ are homotopic morphisms in $\mathcal{C}(\mathscr{C}')$ (indeed, if (s^n) is a homotopy between f and g, then $(F(s^n))$ is a homotopy betweem $F(f)$ and $F(g)$). So we get a functor $K(F) : K(\mathscr{C}) \to K(\mathscr{C}')$.

IV.1.5 The case of abelian categories

Let $\mathscr{A}$ be an abelian category. Then the category $\mathrm{Func}(\mathbb{Z}, \mathscr{A})$ of differential objects of $\mathscr{A}$ is an abelian subcategory, and so are its full subcategories $\mathcal{C}(\mathscr{A})$, $\mathcal{C}^+(\mathscr{A})$, $\mathcal{C}^-(\mathscr{A})$ and $\mathcal{C}^b(\mathscr{A})$ (because they are stable by kernels and cokernels in $\mathrm{Func}(\mathbb{Z}, \mathscr{A})$).

Cohomology

Definition IV.1.5.1. Let $n \in \mathbb{Z}$. We define additive functors $Z^n, B^n H^n : \mathcal{C}(\mathscr{A}) \to \mathscr{A}$ in the following way:

(1) If X is a complex of objects of $\mathscr{A}$, we set

$$Z^n(X) = \operatorname{Ker}(d_X^n) \subset X^n,$$

$$B^n(X) = \operatorname{Im}(d_X^{n-1}) \subset Z^n(X),$$
$$\mathrm{H}^n(X) = Z^n(X)/B^n(X).$$

(2) If $f : X \to Y$ is a morphism of complexes, then $f^{n+1} \circ d_X^n = d_Y^n \circ f^n$, so f^n induces a morphism from $Z^n(X)$ to $Z^n(Y)$, and we take $Z^n(f)$ to be this morphism. Also, $f^n \circ d_X^{n-1} = d_Y^{n-1} \circ f^{n-1}$, so $f^n(\operatorname{Im}(d_X^{n-1})) \subset \operatorname{Im}(d_Y^{n-1})$, and we take $B^n(f)$ to be the morphism induced by f^n. Finally, we take $H^n(f)$ to be the morphism $\mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ induced by $Z^n(f)$.

The object $\mathrm{H}^n(X)$ is called the *nth cohomology object* of the complex X.

Remark IV.1.5.2. If $p \in \mathbb{Z}$, we have $Z^n(X[p]) = Z^{n+p}(X)$, $B^n(X[p]) = B^{n+p}(X)$ and $\mathrm{H}^n(X[p]) = \mathrm{H}^{n+p}(X)$.

Proposition IV.1.5.3. *Let $f, g : X \to Y$ be two morphisms in $\mathcal{C}(\mathscr{A})$. If f and g are homotopic, then $\mathrm{H}^n(f) = \mathrm{H}^n(g)$ for every $n \in \mathbb{Z}$.*

In other words, the functor $\mathrm{H}^n : \mathcal{C}(\mathscr{A}) \to \mathscr{A}$ factor through functors $K(\mathscr{A}) \to \mathscr{A}$, that we will still denote by H^n.

Proof. Let $(s^n)_{n\in\mathbb{Z}}$ be an homotopy between f and g. Let $n \in \mathbb{Z}$. We have $f^n - g^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$, so the restriction of $f^n - g^n$ to $Z^n(X)$ is equal to the restriction of $d_Y^{n-1} \circ s^n$. As the composition of the morphism $Z^n(X) \to Z^n(Y)$ induced by $d_Y^{n-1} \circ s^n$ with the quotient morphism $Z^n(Y) \to H^n(Y)$ is zero, we get that $\mathrm{H}^n(f) = \mathrm{H}^n(g)$.

□

Definition IV.1.5.4. Let $f : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{A})$. We say that f is a *quasi-isomorphism* (qis for short) if $\mathrm{H}^n(f)$ is an isomorphism for every $n \in \mathbb{Z}$.

If there exists a third complex Z and quasi-isomorphisms $Z \to X$ and $Z \to Y$, we say that X and Y are *quasi-isomorphic*.[1]

Remark IV.1.5.5. A complex X is quasi-isomorphic to 0 if and only $H^n(X) = 0$ for every $n \in \mathbb{Z}$; if this holds, we say that the complex X is *exact* or *acyclic*. This generalizes Definition II.2.1.8.

Note that a complex that is homotopic to 0 is quasi-isomorphic to 0, but that the converse is not true. For example, if X is the complex $0 \to A \to B \to C \to 0$ (concentrated in degrees $0, 1, 2$), then X is quasi-isomorphic to 0 if and only if it is an exact sequence, and homotopic to 0 if and only if this exact sequence is split.

[1] We will see later (add reference) why we are taking this slightly strange definition.

In fact, the property for complexes of being homotopic to 0 is preserved if we apply an additive functor, but not the property of being quasi-isomorphic to 0. This is a generalization of Remark II.2.1.14, and is the reason we have derived functors.

Remark IV.1.5.6. If X and Y are quasi-isomorphic complexes, then $\mathrm{H}^n(X) \simeq \mathrm{H}^n(Y)$ for every $n \in \mathbb{Z}$. The converse is absolutely not true. [2]

IV.1.6 Total complex of a double complex

Let $\mathscr{C}$ be an additive category.

Definition IV.1.6.1. Let $X = (X^{n,m}, d_{X,1}^{n,m}, d_{2,X}^{n,m})_{n,m\in\mathbb{Z}}$ be a double complex of objects of $\mathscr{C}$ (see Definition IV.1.1.3). We suppose that all (small) coproducts exist in $\mathscr{C}$ or that, for every $n \in \mathbb{Z}$, the set of $p \in \mathbb{Z}$ such that $X^{p,n-p} \neq 0$ is finite.

The *total complex* of X is the complex $\mathrm{Tot}(X)$ such that, for every $n \in \mathbb{Z}$:

(a) $\mathrm{Tot}(X)^n = \bigoplus_{p\in\mathbb{Z}} X^{p,n-p}$;

(b) $d^n_{\mathrm{Tot}(X)}$ is equal on the component $X^{p,n-p}$ to $d_{1,X}^{p,n-p} + (-1)^p d_{2,X}^{p,n-p}$.

For every $n \in \mathbb{Z}$, the morphism $d^{n+1}_{\mathrm{Tot}(X)} \circ d^n_{\mathrm{Tot}(X)}$ on the component $X^{p,n-p}$ of $\mathrm{Tot}(X)^n$ is equal to:

$$\begin{aligned}
&d^{n+1}_{\mathrm{Tot}(X)} \circ (d_{1,X}^{p,n-p} + (-1)^p d_{2,X}^{p,n-p}) \\
&= d_{1,X}^{p+1,n-p} \circ d_{1,X}^{p,n-p} + (-1)^{p+1} d_{2,X}^{p+1,n-p} \circ d_1^{p,n-p} + (-1)^{2p} d_{2,X}^{p,n-p+1} \circ d_{2,X}^{p,n-p} + (-1)^p d_{1,X}^{p,n-p+1} \circ d_{2,X}^{p,n-p} \\
&= (-1)^p(-d_{2,X}^{p+1,n-p} \circ d_{1,X}^{p,n-p} + d_1^{p,n-p+1} \circ d_{2,X}^{p,n-p}) \\
&= 0.
\end{aligned}$$

This shows that $\mathrm{Tot}(X)$ is a complex.

Remark IV.1.6.2. Suppose that $F : \mathscr{C} \times \mathscr{C}' \to \mathscr{D}$ is an additive bifunctor. Then F defines a functor $F : \mathcal{C}(\mathscr{C}) \times \mathcal{C}(\mathscr{C}') \to \mathcal{C}^2(\mathscr{D})$ in the following way: if $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{C}))$ and $Y \in \mathrm{Ob}(\mathcal{C}(\mathscr{D}))$, then $F(X,Y)^{n,m} = F(X^n, Y^m)$, $d^{n,m}_{1,F(X,Y)} = F(d^n_X, \mathrm{id}_{Y^n})$ and $d^{n,m}_{2,F(X,Y)} = F(\mathrm{id}_{X^n}, d^n_Y)$.

Example IV.1.6.3. (1) Take $\mathscr{C} = \mathbf{Mod}_R$, $\mathscr{C}' = {}_R\mathbf{Mod}$ and $F = (\cdot) \otimes (\cdot) : \mathscr{C} \times \mathscr{C}' \to \mathbf{Ab}$. If X and Y are objects of $\mathcal{C}(\mathbf{Mod}_R)$ and $\mathcal{C}({}_R\mathbf{Mod})$ respectively, we write $X \otimes Y$ for the double complex $F(X, Y)$.

(2) Let $F = \mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Ab}$, where we see the factor $\mathscr{C}$ as the first variable. We denote by $\mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}$ the resulting functor from $\mathcal{C}(\mathscr{C}^{\mathrm{op}}) \times \mathcal{C}(\mathscr{C})$ to $\mathcal{C}^2(\mathscr{C})$.

A complex of objects of $\mathscr{C}^{\mathrm{op}}$ is a family $(X^n)_{n\in\mathbb{Z}}$ of objects of $\mathscr{C}$ together with morphisms $d^n_X : X^{n+1} \to X^n$ such that $d^{n-1}_X \circ d^n_X = 0$ for every $n \in \mathbb{Z}$. We get a complex Y of

[2] See problem set 5 or 6.

objects of $\mathscr{C}$ by setting $Y^n = X^{-n}$ and $d_Y^n = (-1)^n d_X^{-n-1}$. This defines an isomorphism of categories $\mathcal{C}(\mathscr{C}^{\mathrm{op}}) \xrightarrow{\sim} \mathcal{C}(\mathscr{C})$, so we can see $\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}$ as a functor from $\mathcal{C}(\mathscr{C}) \times \mathcal{C}(\mathscr{C})$ to $\mathcal{C}^2(\mathscr{C})$.

With this convention (and with the convention about the covariant variable being the first one), for all $X, Y \in \mathrm{Ob}(\mathcal{C}(\mathscr{C}))$, we have

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, Y^n),$$

the first differential is $d_1^{n,m} = \mathrm{Hom}_{\mathscr{C}}(\mathrm{id}_{X^{-m}}, d_Y^n)$, and the second differential is $d_2^{n,m} = \mathrm{Hom}_{\mathscr{C}}((-1)^m d_X^{-m-1}, \mathrm{id}_{Y^n})$.

Definition IV.1.6.4. If $X \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{C}))$ and $Y \in \mathrm{Ob}(\mathcal{C}^-(\mathscr{C}))$, we set

$$\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y) = \mathrm{Tot}(\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)).$$

The total complex exists because, if $n \in \mathbb{Z}$, then $X^p = 0$ for $p >> 0$ and $Y^{n+p} = 0$ for $p << 0$, so $\mathrm{Hom}_{\mathscr{C}}(X^p, Y^{n+p})$ is nonzero for only a finite number of $p \in \mathbb{Z}$.

By definition of the total complex (and using the convention of Example IV.1.6.3(2)), we have

$$\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^n = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^p, Y^{n+p}),$$

and the differential $d^n : \underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^n \to \underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^{n+1}$ sends $f = \sum_{p\in\mathbb{Z}} f^p \in \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^p, Y^{n+p})$ to $\sum_{p\in\mathbb{Z}} g^p \in \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^p, Y^{n+1+p})$, with

$$g^p = d_Y^{n+p} \circ f^p + (-1)^{n+1} f^{p+1} \circ d_X^p.$$

Proposition IV.1.6.5. *If $X \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{C}))$ and $Y \in \mathrm{Ob}(\mathcal{C}^-(\mathscr{C}))$, we have*

$$Z^0(\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)) = \mathrm{Hom}_{\mathcal{C}(\mathscr{C})}(X,Y),$$

$$B^0(\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)) = Ht(X,Y)$$

and

$$\mathrm{H}^0(\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)) = \mathrm{Hom}_{K(\mathscr{C})}(X,Y).$$

Proof. We have $\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^0 = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^n, Y^n)$. As $X^n = 0$ for $n << 0$ and $Y^n = 0$ for $n >> 0$, a morphism $f : X \to Y$ must have $f^n = 0$ for $|n| >> 0$, so we have a natural inclusion $\mathrm{Hom}_{\mathcal{C}(\mathscr{C})}(X,Y) \subset \underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^0$. If $f = \sum_{n\in\mathbb{Z}} f^n \in \underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^n$, then $d^0(f)$ is the element $\sum_{n\in\mathbb{Z}} g^n \in \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^n, Y^{n+1})$ given by $g^n = d_Y^n \circ f^n - f^{n+1} \circ d_X^n$. So $d^0(f) = 0$ if and only if $d_Y^n \circ f^n = f^{n+1} \circ d_X^n$ for every $n \in \mathbb{Z}$, that is, if and only if f is a morphism of complexes.

Let $\sum_{n\in\mathbb{Z}} s^n \in \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^n, Y^{n-1}) = \underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{C})}(X,Y)^{-1}$. Then $d^{-1}(s^n) = \sum_{n\in\mathbb{Z}} g^n$, with $g^n = d_Y^n \circ s^n + s^{n+1} \circ d_X^n : X^n \to Y^n$. So f is in the image of d^{-1} if and only if it is homotopic to 0.

The last statement follows immediately from the definition of the homotopy category.

□

IV.2 Diagram chasing lemmas

We fix an abelian category $\mathscr{A}$.

IV.2.1 The salamander lemma

The main reference for the material in this subsection is Bergman's article [2]. Anton Geraschenko's exposition in [3] is also a good source.

Consider a double complex in $\mathscr{A}$. We concentrate on what happens around one of the objects of this double complex and give names to this object and the morphisms having it as a source or target (the diagonal morphisms are by definition the composition of both sides of the corresponding squares, which are equal):

$$\begin{tikzcd}
\bullet \arrow[dr, "e"] & \bullet \arrow[d, "c"] & \\
\bullet \arrow[r, "a"] & A \arrow[r, "b"] \arrow[d, "d"] \arrow[dr, "f"] & \bullet \\
& \bullet & \bullet
\end{tikzcd}$$

We set:

- ${}_{=}A = \operatorname{Ker} b / \operatorname{Im} a$ (the *horizontal cohomology*);
- $A^{\|} = \operatorname{Ker} d / \operatorname{Im} c$ (the *vertical cohomology*);
- $A_{\square} = \operatorname{Ker} f / (\operatorname{Im} a + \operatorname{Im} c)$ (the *donor*);
- ${}^{\square}A = (\operatorname{Ker} b \cap \operatorname{Ker} d) / \operatorname{Im} e$ (the *receptor*).

Lemma IV.2.1.1. *We have four canonical morphisms induced by* id_A*, called* intramural morphisms *and forming a commutative square*

$$\begin{tikzcd}
{}^{\square}A \arrow[r] \arrow[d] & A^{\|} \arrow[d] \\
{}_{=}A \arrow[r] & A_{\square}
\end{tikzcd}$$

If B is the target of the morphism b or d, we also have a canonical morphism, called an extramural morphism*, from $A_{\square}$ to ${}^{\square}B$.*

All these morphisms are natural in the double complex.

Proof. The naturality will be clear on the construction (all the morphisms are induced from identity morphisms or from a differntial of the complex).

The first intramural morphism is induced by the inclusion $\operatorname{Ker} b \cap \operatorname{Ker} d \subset \operatorname{Ker} d$ (and the fact that $\operatorname{Im} e \subset \operatorname{Im} c$), the second from the inclusion $\operatorname{Ker} d \subset \operatorname{Ker} f$, the third from the inclusion $\operatorname{Ker} b \cap \operatorname{Ker} d \subset \operatorname{Ker} d$ (and the fact that $\operatorname{Im} e \subset \operatorname{Im} a$) and the fourth from the inclusion $\operatorname{Ker} b \subset \operatorname{Ker} f$.

For the extramural morphism, we treat the case where B is the target of b (the other case is similar). We give names to a few more morphisms:

$$\begin{tikzcd}
\bullet \arrow[dr, "e"] & \bullet \arrow[d, "c"] \arrow[dr, "w"] & & \\
\bullet \arrow[r, "a"] & A \arrow[r, "b"] \arrow[d, "d"] \arrow[dr, "f"] & B \arrow[r, "u"] \arrow[d, "v"] & \bullet \\
& \bullet & \bullet &
\end{tikzcd}$$

We have $A_{\square} = \operatorname{Ker} f/(\operatorname{Im} a + \operatorname{Im} c)$ and ${}^{\square}B = (\operatorname{Ker} u \cap \operatorname{Ker} v)/\operatorname{Im} w$. We have $b(\operatorname{Ker} f) \subset \operatorname{Ker} u$ because $u \circ b = 0$, and $b(\operatorname{Ker} f) \subset \operatorname{Ker} v$ because $f = v \circ b$. Also, $b(\operatorname{Im} a + \operatorname{Im} c) = \operatorname{Im}(b \circ a) + \operatorname{Im}(b \circ c) = \operatorname{Im}(b \circ c) = \operatorname{Im}(w)$ (because $b \circ a = 0$ and $b \circ c = w$). So b induces a morphism $({}_{\square}A) \to {}^{\square}B$.

$\square$

Remark IV.2.1.2. Suppose that the double complex locally looks like:

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "g"] \arrow[dr, "e"] & \bullet \arrow[d, "c"] & \\
\bullet \arrow[r, "a"] & A \arrow[r, "b"] \arrow[d, "d"] \arrow[dr, "f"] & \bullet \arrow[d, "u"] \\
& \bullet \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

If d and u are injective, then $A^{\|} = {}^{\square}A = 0$ and ${}_{=}A \to A_{\square}$ is an isomorphism, because $\operatorname{Ker} f = \operatorname{Ker} b$ and $\operatorname{Im} a + \operatorname{Im} c = \operatorname{Im} a$ (as $\operatorname{Im} c \subset \operatorname{Ker} d = 0$) If b and v are injective, then ${}_{=}A = {}^{\square}A = 0$ and $A^{\|} \to A_{\square}$ is an isomorphism, because $\operatorname{Ker} f = \operatorname{Ker} d$ and $\operatorname{Im} a + \operatorname{Im} c = \operatorname{Im} c$ (as $\operatorname{Im} a \subset \operatorname{Ker} b = 0$).

Dually: If c and g are surjective, then $A^{\|} = A_{\square} = A$ and ${}_{=}A \to {}^{\square}A$ is an isomorphism, because $\operatorname{Ker} b \cap \operatorname{Ker} d = \operatorname{Ker} b$ (as $\operatorname{Ker} d \supset \operatorname{Im} c = A$) and $\operatorname{Im}(e) = \operatorname{Im}(a \circ g) = \operatorname{Im}(a)$ (as g is surjective). If a and h are surjective, then ${}_{=}A = A_{\square} = A$ and $A^{\|} \to {}^{\square}A$ is an isomorphism, because $\operatorname{Ker} b \cap \operatorname{Ker} d = \operatorname{Ker} d$ (as $\operatorname{Ker} b \supset \operatorname{Im} a = A$) and $\operatorname{Im}(e) = \operatorname{Im}(c \circ h) = \operatorname{Im}(c)$ (as h is surjective).

The salamander lemma is the following theorem.

Theorem IV.2.1.3. *We fix a double complex X of objects of $\mathscr{A}$.*

(i). If we have a fragment of X of the following form

$$\begin{tikzcd} C \arrow[d] & \\ A \arrow[r] & B \arrow[d] \\ & D \end{tikzcd}$$

then we get a six-term exact sequence called the salamander sequence

$$C_\square \xrightarrow{(1)} {}_{=}A \xrightarrow{(2)} A_\square \xrightarrow{(3)} {}^\square B \xrightarrow{(4)} {}_{=}B \xrightarrow{(5)} {}^\square D,$$

where (1) is the composition of the extramural morphism $C_\square \to {}^\square A$ and of the intramural morphism ${}^\square A \to {}_{=}A$, (2) is an intramural morphism, (3) is an extramural morphism, (4) is an intramural morphisms, and (5) is the composition of the intramural morphism ${}_{=}B \to B_\square$ and of the extramural morphism $B_\square \to {}^\square D$.

(ii). If we have a fragment of X of the following form

$$\begin{tikzcd} C \arrow[r] & A \arrow[d] & \\ & B \arrow[r] & D \end{tikzcd}$$

then we get a six-term exact sequence called the salamander sequence

$$C_\square \xrightarrow{(1)} A^{\|} \xrightarrow{(2)} A_\square \xrightarrow{(3)} {}^\square B \xrightarrow{(4)} B^{\|} \xrightarrow{(5)} {}^\square D,$$

where (1) is the composition of the extramural morphism $C_\square \to {}^\square A$ and of the intramural morphism ${}^\square A \to A^{\|}$, (2) is an intramural morphism, (3) is an extramural morphism, (4) is an intramural morphisms, and (5) is the composition of the intramural morphism $B^{\|} \to B_\square$ and of the extramural morphism $B_\square \to {}^\square D$.

Proof. See Problem A.6.1.

□

Corollary IV.2.1.4. *We fix a double complex X of objects of $\mathscr{A}$. If we have a fragment of X of the following form*

$$\begin{tikzcd} C \arrow[d] & \\ A \arrow[r] & B \arrow[d] \\ & D \end{tikzcd}$$

with the row containing A and B exact at A and B, or of the following form:

$$\begin{tikzcd} C \arrow[r] & A \arrow[d] & \\ & B \arrow[r] & D \end{tikzcd}$$

with the column containing A and B exact at A and B, then the extramural morphism $A_\square \to {}^\square B$ is an isomorphism.

Proof. We prove the first case, the second is similar. The hypothesis says that ${}_=A = 0$ and ${}_=B = 0$, so the result follows immediately from the salamander sequence.

□

IV.2.2 Some applications

In this section, we will apply the salamander lemma to finite diagrams drawn on a square grid. We will always assume that these diagrams have been completed to double complexes by adding zero objects in all the positions that don't appear explicitely.

The four lemma

Corollary IV.2.2.1. *Consider a commutative diagram with exact rows in $\mathscr{A}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & B \arrow[r, "f"] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "w"'] & D \arrow[d, "t"'] \\ A' \arrow[r] & B' \arrow[r, "g"'] & C' \arrow[r] & D' \end{tikzcd}$$

Suppose that u is surjective and t is injective.

Then:

(i). $f(\operatorname{Ker} v) = \operatorname{Ker} w$*;*

(ii). $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$.

In particular:

(a) If v is injective, then w is also injective.

(b) If w is surjective, then v is also surjective.

Proof. We complete the diagram to a double complex where the columns and the middle two rows are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{Ker}\,u \arrow[r] \arrow[d] & \mathrm{Ker}\,v \arrow[r] \arrow[d] & \mathrm{Ker}\,w \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & A \arrow[r] \arrow[d, "u"'] & B \arrow[r, "f"] \arrow[d, "v"'] & C \arrow[r, "h"] \arrow[d, "w"'] & D \arrow[r] \arrow[d, "t"'] & \mathrm{Coker}\,h \\
\mathrm{Ker}\,k \arrow[r] & A' \arrow[r, "k"'] \arrow[d] & B' \arrow[r, "g"'] \arrow[d] & C' \arrow[r] \arrow[d] & D' \arrow[d] & \\
 & 0 \arrow[r] & \mathrm{Coker}\,v \arrow[r] \arrow[d] & \mathrm{Coker}\,w \arrow[r] \arrow[d] & \mathrm{Coker}\,t & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Point (i) says that ${}_{=}(\mathrm{Ker}(w)) = 0$. By Remark IV.2.1.2, the intramural morphism ${}_{=}(\mathrm{Ker}\,w) \to (\mathrm{Ker}\,w)_{\square}$ is an isomorphism. By Corollary IV.2.1.4 and the exactness properties of the diagram, we have isomorphisms $(\mathrm{Ker}\,w)_{\square} \simeq {}^{\square}C \simeq B_{\square} \simeq {}^{\square}B' \simeq A'_{\square} \simeq {}^{\square}0 = 0$.

Point (ii) says that ${}_{=}(\mathrm{Coker}\,v) = 0$. The intramural morphism ${}_{=}(\mathrm{Coker}\,v) \to {}^{\square}(\mathrm{Coker}\,v)$ is clearly an isomorphism, and Corollary IV.2.1.4 gives isomorphisms ${}^{\square}(\mathrm{Coker}\,v) \simeq B'_{\square} \simeq {}^{\square}C' \simeq C_{\square} \simeq {}^{\square}D \simeq 0_{\square} = 0$.

□

As an immediate corollary, we get the five lemma:

Corollary IV.2.2.2. *Consider a commutative diagram with exact rows in $\mathscr{A}$:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & D \arrow[r] \arrow[d, "d"'] & E \arrow[d, "e"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

If a is surjective, b and d are isomorphisms and e is injective, then c is an isomorphism.

The snake lemma

Corollary IV.2.2.3. *Suppose that we have a commutative diagram with exact rows:*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

Then we get an exact sequence, functorial in the diagram:

$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{\delta} \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h,$$

where the morphisms between the kernels are induced by the morphisms $A \to B$ and $B \to C$, and the morphisms between the cokernels are induced by the morphisms $A' \to B'$, $B' \to C'$.

Moreover, if $A \to B$ is injective (resp. if $B' \to C'$ is surjective), then so is $\operatorname{Ker} f \to \ker g$ *(resp.* $\operatorname{Coker} g \to \operatorname{Coker} h$*).*

The morphism δ is sometimes called the *connecting morphism.*

Proof. We complete the diagram to a double complex

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} g \arrow[r] \arrow[d] & \operatorname{Ker} h \arrow[d] & \\
\operatorname{Ker} u \arrow[r] \arrow[d] & A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \arrow[d] \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r] \arrow[d] & \operatorname{Coker} v' \\
& \operatorname{Coker} f \arrow[r] \arrow[d] & \operatorname{Coker} g \arrow[r] \arrow[d] & \operatorname{Coker} h \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where the new morphisms are the obvious ones, and where the columns are the two middle rows are exact.

It is easy to see that ${}_{=}(\operatorname{Ker} g) = 0$ and ${}_{=}(\operatorname{Coker} g) = 0$: By Remark IV.2.1.2, we have ${}_{=}(\operatorname{Ker} g) = (\operatorname{Ker} g)_{\square}$, and Corollary IV.2.1.4 gives isomorphisms $(\operatorname{Ker} g)_{\square} \simeq {}^{\square}B \simeq A_{\square} \simeq {}^{\square}A'0_{\square} = 0$. Similarly, Remark IV.2.1.2 gives ${}_{=}(\operatorname{Coker} g) = {}^{\square}(\operatorname{Coker} g)$, and Corollary IV.2.1.4 gives isomorphisms ${}^{\square}(\operatorname{Coker} g) \simeq B'_{\square} \simeq {}^{\square}C' \simeq C_{\square} \simeq {}^{\square}0 = 0$.

The last sentence of the statement also is easy to prove: if u is injective, so is its restriction to $\operatorname{Ker} f$, and if v' is surjective, so is the morphism $\operatorname{Coker} g \to \operatorname{Coker} h$ that it induces (because $C' \to \operatorname{Coker} h$ is surjective).

We must now construct the connecting morphism $\delta : \operatorname{Ker} h \to \operatorname{Coker} f$, functorially in the double complex above, such that the sequence of the statement is exact. This means that we must construct an isomorphism $\operatorname{Coker}(\operatorname{Ker} g \to \operatorname{Ker} h) \xrightarrow{\sim} \operatorname{Ker}(\operatorname{Coker} f \to \operatorname{Coker} g)$. Note that

$\operatorname{Coker}(\operatorname{Ker} g \to \operatorname{Ker} h) = \operatorname{Ker} h_\square$ and $\operatorname{Ker}(\operatorname{Coker} f \to \operatorname{Coker} g) = {}^\square \operatorname{Coker} f$. The corollary of the salamder lemme (Corollary IV.2.1.4) gives extramural isomorphisms

$$\begin{tikzcd}
 & & \operatorname{Ker} h_\square \arrow[d, "\wr"] \\
 & B_\square \arrow[r, "\sim"] \arrow[d, "\wr"] & {}^\square C \\
A'_\square \arrow[r, "\sim"] \arrow[d, "\wr"] & {}^\square B' & \\
{}^\square \operatorname{Coker} f & &
\end{tikzcd}$$

and we get the morphism δ by composing these isomorphisms.

$\square$

The $\infty \times \infty$ lemma

This is a generalization of the 3×3 lemma (which we will get as an immediate corollary).

Corollary IV.2.2.4. *Suppose that we have a double complex $X = (X^{n,m}, d_1^{n,m}, d_2^{n,m})$ such that $X^{n,m} = 0$ if $n < 0$ or $m < 0$:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X^{0,0} \arrow[r] \arrow[d] & X^{1,0} \arrow[r] \arrow[d] & X^{2,0} \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & X^{0,1} \arrow[r] \arrow[d] & X^{1,1} \arrow[r] \arrow[d] & X^{2,1} \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & X^{0,2} \arrow[r] \arrow[d] & X^{1,2} \arrow[r] \arrow[d] & X^{2,2} \arrow[r] \arrow[d] & \dots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Suppose also that all the rows and columns in the diagram above are exact except maybe for the first row and the first column; that is, assume that the complex $(X^{n,i}, d_1^{n,i})_{n\in\mathbb{Z}}$ is exact for every $i \geq 1$, and that the complex $(X^{i,n}, d_2^{n,i})_{n\in\mathbb{Z}}$ is exact for every $i \geq 1$.

Then the complexes $(X^{n,0}, d_1^{n,0})_{n\in\mathbb{Z}}$ and $(X^{0,n}, d_2^{0,n})_{n\in\mathbb{Z}}$ have canonically isomorphic cohomologies. That is, we have canonical isomorphisms ${}_{=}(X^{n,0}) \simeq (X^{0,n})^{\|}$ for every $n \in \mathbb{Z}$. In particular, the first row of the diagram is exact if and only if its first column is exact.

Proof. As the morphisms $X^{1,0} \to X^{1,1}$ and $X^{0,1} \to X^{1,1}$ are injective, the intramural morphisms ${}_{=}(X^{0,0}) \to (X^{0,0})_{\square}$ and $(X^{0,0})^{\|} \to (X^{0,0})_{\square}$ are isomorphisms, so we get an isomorphism ${}_{=}(X^{0,0}) \simeq (X^{0,0})^{\|}$.

Let $n \geq 1$. By Remark IV.2.1.2, we have intramural isomorphisms ${}_{=}(X^{n,0}) \xrightarrow{\sim} (X^{n,0})_{\square}$ and $(X^{0,n})^{\|} \xrightarrow{\sim} (X^{0,n})_{\square}$. Also, by Corollary IV.2.1.4, we get a sequence of extramural isomorphisms:

$$(X^{n,0})_{\square} \simeq {}^{\square}(X^{n,1}) \simeq (X^{n-1,1})_{\square} \simeq {}^{\square}(X^{n-1,2}) \simeq \ldots \simeq (X^{1,n-1})_{\square} \simeq {}^{\square}(X^{1,n}) \simeq (X^{0,n})_{\square}.$$

So we get an isomorphism ${}_{=}(X^{n,0}) \simeq (X^{0,n})^{\|}$.

Here is the $n = 2$ case:

$$\begin{tikzcd}
 & & (X^{2,0})_{\square} \arrow[d, "\wr"] \\
 & (X^{1,1})_{\square} \arrow[r, "\sim"] \arrow[d, "\wr"] & {}^{\square}(X^{2,1}) \\
(X^{0,2})_{\square} \arrow[r, "\sim"] & {}^{\square}(X^{1,2}) &
\end{tikzcd}$$

$\square$

We give a version of the 3×3 lemma as a corollary (there are several versions, all follow immediately from the $\infty \times \infty$ lemma).

Corollary IV.2.2.5. *Consider a commutative diagram:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If all three columns and the last two rows are exact, then the first row is also exact.

The long exact sequence of cohomology

Corollary IV.2.2.6. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of complexes of objects of $\mathscr{A}$. Then we get a canonical long exact sequence in $\mathscr{A}$ (functorial in the exact sequence of complexes):*

$$\dots \xrightarrow{\delta^{n-1}} \mathrm{H}^n(X) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(Y) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(Z) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(X) \xrightarrow{\mathrm{H}^{n+1}(f)} \mathrm{H}^{n+1}(Y) \xrightarrow{\mathrm{H}^{n+1}(g)} \mathrm{H}^{n+1}(Z) \xrightarrow{\delta^{n+1}} \dots$$

The morphisms $\delta^n : \mathrm{H}^n(Z) \to \mathrm{H}^{n+1}(X)$ are called *connecting morphisms*.

Proof. We complete the exact sequence of complexes to a double complex with exact rows (of which we only show a slice):

$$\begin{tikzcd}
0 \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & Y^{n-1} \arrow[r] \arrow[d] & Z^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{n} \arrow[r] \arrow[d] & Y^{n} \arrow[r] \arrow[d] & Z^{n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{n+1} \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & Z^{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{n+2} \arrow[r] & Y^{n+2} \arrow[r] & Z^{n+2} \arrow[r] & 0
\end{tikzcd}$$

We want to show that there is an exact sequence

$$(X^n)^{\Vert} \to (Y^n)^{\Vert} \to (Z^n)^{\Vert} \to (X^{n+1})^{\Vert} \to (Y^{n+1})^{\Vert} \to (Z^{n+1})^{\Vert}$$

(we get the exact sequence of the statement by putting all these together).

The salamander sequence centered at $Y^n \to Y^{n+1}$ gives an exact sequence

$$(X^n)_{\Box} \to (Y^n)^{\Vert} \to (Y^n)_{\Box} \to {}^{\Box}(Y^{n+1}) \to (Y^{n+1})^{\Vert} \to {}^{\Box}(Z^{n+1}).$$

Using Remark IV.2.1.2 and Corollary IV.2.1.4, we get isomorphisms $(X^n)_{\Box} \simeq (X^n)^{\Vert}$, $(Y^n)_{\Box} \xrightarrow{\sim} {}^{\Box}(Z^n) \simeq (Z^n)^{\Vert}$, $(X^{n+1})^{\Vert} \simeq (X^{n+1})_{\Box} \xrightarrow{\sim} {}^{\Box}(Y^{n+1})$ and ${}^{\Box}(X^{n+1}) \simeq (Z^{n+1})^{\Vert}$. Using these and the salamander sequence, we get the desired exact sequence. (The morphisms are the correct ones, because intramural morphisms are always induced by identity morphisms and extramural morphisms by the morphisms of the double complex.)

□

Corollary IV.2.2.7. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of complexes of objects of $\mathscr{A}$. If two of the complexes X, Y and Z are acyclic, then so is the third.*

Proof. Indeed, if two of the three complexes are acyclic, then, in the long exact sequence of cohomology, two out of three of the entries will be zero, which forces the other entries to be zero too.

□

Corollary IV.2.2.8. *Let $f : X \to Y$ be a morphism of complexes of objects of $\mathscr{A}$. Then the sequence (*) $0 \to Y \overset{\alpha(f)}{\to} \mathrm{Mc}(f) \overset{\beta(f)}{\to} X[1] \to 0$ is exact, so we get a long exact sequence:*

$$\ldots \to \mathrm{H}^{n-1}(\mathrm{Mc}(f)) \to \mathrm{H}^n(X) \overset{f^n}{\to} \mathrm{H}^n(Y) \to \mathrm{H}^n(\mathrm{Mc}(f)) \to \ldots$$

Proof. In degree n, the sequence (*) is the sequence

$$0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$$

which is exact and even split. (Note however that (*) is not split as a sequence of complexes.) So Corollary IV.2.2.6 gives a long exact sequence of cohomology objects, and we just need to check that the connecting morphism $\delta^{n-1} : \mathrm{H}^{n-1}(X[1]) = \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ is equal to f^n for every $n \in \mathbb{Z}$. By the construction of the long exact sequence in the proof of Corollary IV.2.2.6, this morphism is the composition of the following sequence of morphisms

$$(X^n)^{\|} \xleftarrow[(1)]{\sim} {}^{\square}(X^n) \xleftarrow[(2)]{\sim} (\mathrm{Mc}(f)^{n-1})_{\square} \xrightarrow[(3)]{} {}^{\square}(\mathrm{Mc}(f)^n) \xleftarrow[(4)]{\sim} (Y^n)_{\square} \xleftarrow[(5)]{\sim} (Y^n)^{\|}$$

Morphism (1) is induced by the identity of X^n, morphism (2) by the morphism $\mathrm{Mc}(f)^{n-1} = X^n \oplus Y^{n-1} \to X^n$ with matrix $\begin{pmatrix} \mathrm{id}_{X^n} & 0 \end{pmatrix}$, morphism (3) by the morphism $\mathrm{Mc}(f)^{n-1} \to \mathrm{Mc}(f)^n$ with matrix $\begin{pmatrix} -d_X^{n+1} & 0 \\ f^n & d_Y^{n-1} \end{pmatrix}$, morphism (4) by the morphism $Y^n \to \mathrm{Mc}(f)^n = X^{n+1} \oplus Y^n$ with matrix $\begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and morphism (5) by the identity of Y^n. So the composition is induced by f^n.

□

IV.3 Derived functors

We fix an abelian category $\mathscr{A}$.

IV.3.1 Resolutions

Definition IV.3.1.1. Let $\mathscr{I}$ be a full additive subcategory of $\mathscr{A}$.

(i). A *right $\mathscr{I}$-resolution* of an object A of $\mathscr{I}$ is an exact complex

$$0 \to A \to B_0 \to B_1 \to B_2 \to \dots$$

such that $B_n \in \mathrm{Ob}(\mathscr{I})$ for every $n \in \mathbb{N}$.

(ii). A *left $\mathscr{I}$-resolution* of an object A of $\mathscr{I}$ is an exact complex

$$\dots \to B_2 \to B_1 \to B_0 \to A \to 0$$

such that $B_n \in \mathrm{Ob}(\mathscr{I})$ for every $n \in \mathbb{N}$.

If $\mathscr{I}$ is the category of injective (resp. projective) objects of $\mathscr{A}$, the a right (resp. left) $\mathscr{I}$-resolution of $A \in \mathrm{Ob}(\mathscr{A})$ is also called an *injective resolution* (resp. *projective resolution*) of A.

If $\mathscr{I} = \mathscr{A}$, we just talk about right and left resolutions of objects of $\mathscr{A}$.

Lemma IV.3.1.2. *Let $\mathscr{I}$ be a full additive subcategory of $\mathscr{A}$.*

(i). The following are equivalent:

(a) Every object of $\mathscr{A}$ has a right $\mathscr{I}$-resolution.

(b) For every object A of $\mathscr{A}$, there exists an injective morphism $A \to B$ with $B \in \mathrm{Ob}(\mathscr{I})$.

(ii). The following are equivalent:

(a) Every object of $\mathscr{A}$ has a left $\mathscr{I}$-resolution.

(b) For every object A of $\mathscr{A}$, there exists an surjective morphism $B \to A$ with $B \in \mathrm{Ob}(\mathscr{I})$.

Proof. It suffices to prove (i). If (a) holds, let $A \in \mathrm{Ob}(\mathscr{A})$, and let $0 \to A \to B_0 \to B_1 \to \dots$ be a right $\mathscr{I}$-reolution of A; then $A \to B_0$ is injective and $B_0 \in \mathrm{Ob}(\mathscr{I})$.

Conversely, suppose that (b) holds, and let A be an object of $\mathscr{A}$. We choose an injective morphism $f_0 : A_0 = A \to B_0$ with $B_0 \in \mathrm{Ob}(\mathscr{I})$. We construct by induction on $n \geq 0$ a sequence of injective morphisms $f_n : A_n \to B_n$ with $B_n \in \mathrm{Ob}(\mathscr{I})$ such that $A_n = \mathrm{Coker}(f_{n-1})$ for $n \geq 1$ in the following way:

(1) We already have $f_0 : A_0 \to B_0$.

(2) If $n \geq 1$ and we have constructed f_{n-1}, let $A_n = \mathrm{Coker}(f_{n-1})$, and let $f_n : A_n \to B_n$ be an injective morphism with $B_n \in \mathrm{Ob}(\mathscr{I})$.

We define $d^n : B_n \to B_{n+1}$ as the composition of the cokernel morphism $B_n \to A_{n+1}$ and of $f_{n+1} : A_{n+1} \to B_{n+1}$. Then $0 \to A \to B_0 \xrightarrow{d^0} B_1 \xrightarrow{d^1} \dots$ is a right $\mathscr{I}$-resolution of A.

$\square$

Remark IV.3.1.3. If $\mathscr{A}$ is a Grothendieck abelian category, then, if we use the functorial injective embedding of Theorem II.3.2.4, the proof of Lemma IV.3.1.2 will give a functorial injective resolution of objects of $\mathscr{A}$. This is nice, although it is very useful to know that the later constructions will not depend on the injective resolution, because all injective resolutions of the same object are isomorphic in the homotopy category $K^+(\mathscr{C})$ (see Corollary IV.3.2.2).

Corollary IV.3.1.4. *If $\mathscr{A}$ has enough injectives (resp. enough projectives), then every object of $\mathscr{A}$ has an injective (resp. projective) resolution.*

We can actually construct injective resolutions of any bounded below complex (if $\mathscr{A}$ has enough injective objects), see Lemma IV.4.1.9 and Corollary IV.4.1.11.

Proposition IV.3.1.5 (The horseshoe lemma)**.** *Let $0 \to A \to B \to C \to 0$ be short exact sequence in $\mathscr{A}$, and let $A \to I^\bullet$ an injective resolution of A and $C \to J^\bullet$ be a right resolution of C. Then there exist morphisms $d^{-1} : B \to I^0 \oplus J^0$ and $d^n : I^n \oplus J^n \to I^{n+1} \oplus J^{n+1}$, for $n \in \mathbb{N}$, such that, if $K^\bullet = (I^n \oplus J^n, d^n)_{n \geq 0}$ and if $i : I^\bullet \to K^\bullet$ and $p : K^\bullet \to J^\bullet$ are the morphism of complexes such that $i^n : I^n \to I^n \oplus J^n$ is the morphism $\begin{pmatrix} \mathrm{id}_{I^n} \\ 0 \end{pmatrix}$ and $p^n : I^n \oplus J^n \to J^n$ is the morphism $\begin{pmatrix} 0 & \mathrm{id}_{J^n} \end{pmatrix}$, then $B \stackrel{d^{-1}}{\to} K^\bullet$ is an injective resolution of B and $0 \to I^\bullet \stackrel{i}{\to} K^\bullet \stackrel{p^\bullet}{\to} J^\bullet \to 0$ is a short exact sequence in $\mathcal{C}(\mathscr{A})$.*

Proof. We know that $0 \to I^n \stackrel{i^n}{\to} K^n = I^n \oplus J^n \stackrel{p^n}{\to} J^n \to 0$ is a short exact sequence for every $n \in \mathbb{N}$ (it is even split). So we just need to construct morphisms $d^{-1} : K^{-1} := B \to K^0$ and $d^n : K^n \to K^{n+1}$ such that $(K^n, d^n)_{n \geq 0}$ is a complex and an injective resolution of B, and such that i and p are morphisms of complexes. To harmonize the notation, we denote the morphisms $I^{-1} := A \to I^0$ and $J^{-1} := C \to J^0$ by d_I^{-1} and d_J^{-1} and the morphisms $A \to B$ and $B \to C$ by i^{-1} and p^{-1}. We show by induction on $n \geq -1$ that there exists a morphism $d^n : K^n \to K^{n+1}$ such that $d^n \circ i^n = i^{n+1} \circ d_I^n$, $d_J^n \circ p^n = p^{n+1} \circ d^n$ and that the sequence

$$0 \to \operatorname{Coker} d_I^n \to \operatorname{Coker} d^n \to \operatorname{Coker} d_J^n \to 0$$

is exact.

If $n = -1$, we are trying to fill in the following diagram (whose rows and columns are exact):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "d_I^{-1}"'] & B \arrow[r] \arrow[d, dashed, "d^{-1}"'] & C \arrow[r] \arrow[d, "d_J^{-1}"] & 0 \\
0 \arrow[r] & I^0 \arrow[r, "i^0"'] & I^0 \oplus J^0 \arrow[r, "p^0"'] & J^0 \arrow[r] & 0
\end{tikzcd}$$

This is possible by Lemma IV.3.1.6; also, the same lemma says that d^{-1} is injective, and the snake lemma implies that the sequence

$$0 \to \operatorname{Coker} d_I^{-1} \to \operatorname{Coker} d^{-1} \to \operatorname{Coker} d_J^{-1} \to 0$$

is exact.

Suppose that we have constructed d^n, and consider the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Coker} d_I^n \arrow[r] \arrow[d, "d_I^{n+1}"'] & \operatorname{Coker} d^n \arrow[r] \arrow[d, dashed, "d^{n+1}"'] & \operatorname{Coker} d_J^n \arrow[r] \arrow[d, "d_J^{n+1}"] & 0 \\
0 \arrow[r] & I^{n+2} \arrow[r, "i^{n+2}"'] & I^{n+2} \oplus J^{n+2} \arrow[r, "p^{n+2}"'] & J^{n+2} \arrow[r] & 0
\end{tikzcd}$$

The first row of this diagram is exact by the induction hypothesis, and the first and third column are exact because $A \to I^\bullet$ and $C \to J^\bullet$ are resolutions. By Lemma IV.3.1.6, there exists d^{n+1} such that the diagram commutes and that the second column is exact, and then the snake lemma implies that the sequence

$$0 \to \operatorname{Coker} d_I^{n+1} \to \operatorname{Coker} d^{n+1} \to \operatorname{Coker} d_J^{n+1} \to 0$$

is exact.

□

Lemma IV.3.1.6. *Consider a diagram in $\mathscr{A}$:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, dashed, "h"'] & C \arrow[r] \arrow[d, "g"] & 0 \\
 & I \arrow[r, "i"'] & I \oplus J \arrow[r, "p"'] & J &
\end{tikzcd}$$

where the row is exact, I is an injective object and $i : I \to I \oplus J$ and $p : I \oplus J \to J$ are the morphisms with matrices $\begin{pmatrix} \mathrm{id}_I \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \mathrm{id}_J \end{pmatrix}$ respectively.

Then there exists $h : B \to I \oplus J$ such that $h \circ u = i \circ f$ and $g \circ v = p \circ h$. Moreover, h is injective if and f and g are.

Proof. The last sentence follows from the snake lemma. We prove the existence of h. As I is an injective object and u is a monomorphism, there exists $h_1 : B \to I$ such that $h_1 \circ u = f$. Let $h : B \to I \oplus J$ be the morphism with matrix $\begin{pmatrix} h_1 \\ g \circ v \end{pmatrix}$.

□

IV.3.2 Complexes of injective objects

The main result of this subsection is the following theorem. We state it for injective objects, but it has an obvious dual version for projective objects (where all the morphisms go in the other direction).

Theorem IV.3.2.1. *We denote by $\mathscr{I}$ the additive subcategory of injective objects of $\mathscr{A}$.*

(i). Let $f^\bullet : X^\bullet \to I^\bullet$ be a morphism of complexes, with $X^\bullet \in \operatorname{Ob} \mathcal{C}(\mathscr{A})$ and $I^\bullet \in \operatorname{Ob} \mathcal{C}^+(\mathscr{I})$. Suppose that $X^\bullet$ is acyclic. Then $f^\bullet$ is homotopic to 0.

(ii). Any acyclic complex in $\mathcal{C}^+(I^\bullet)$ is homotopic to 0.

(iii). Let $f : I^\bullet \to J^\bullet$ be a morphism between objects of $\mathcal{C}^+(\mathscr{I})$. If f is a quasi-isomorphism, then it is a homotopy equivalence.

(iv). Let $f : A \to B$ be a morphism of $\mathscr{A}$, let $A \to X^\bullet$ be a right resolution of A, and let $B \to I^\bullet$ be a morphism of complexes with $I^\bullet \in \mathcal{C}^{\geq 0}(\mathscr{I})$. Then there exists a morphism of complexes $f^\bullet : X^\bullet \to I^\bullet$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X^\bullet \arrow[d, "f^\bullet"] \\ B \arrow[r] & I^\bullet \end{tikzcd}$$

Moreover, this morphism $f^\bullet$ is unique up to homotopy.

Proof. (i). We construct by induction on $n \in \mathbb{Z}$ morphisms $s^n : X^n \to I^{n-1}$ such that $f^n = s^{n+1} \circ d_X^n + d_I^{n-1} \circ s^n$. As $I^\bullet$ is in $\mathcal{C}^+(\mathscr{A})$, we have $I^n = 0$ for $n << 0$, so we have $f^n = 0$ and we can take $s^n = 0$ for $n << 0$. Suppose that we have constructed the s^m for all $m \leq n$; we show how to construct s^{n+1}. Let $g^n = f^n - d_I^{n-1} \circ s^n$. Then

$$\begin{aligned} g^n \circ d_X^{n-1} = f^n \circ d_X^{n-1} - d_I^{n-1} \circ s^n \circ d_X^{n-1} &= f^n \circ d_X^{n-1} - d_I^{n-1} \circ (f^{n-1} - d_I^{n-2} \circ s^{n-1}) \\ &= f^n \circ d_X^{n-1} - d_I^{n-1} \circ f^{n-1} = 0, \end{aligned}$$

so g^n induces a morphism $\overline{g}_n : X^n / \operatorname{Im}(d_X^{n-1}) \to I^n$. As $X^\bullet$ is exact, we have $\operatorname{Im}(d_X^{n-1}) = \operatorname{Ker}(d_X^n)$, so the morphism $X^n / \operatorname{Im}(d_X^{n-1}) \to X^{n+1}$ induced by d_X^n is a monomorphism. So, as I^n is an injective object, there exists a morphism $s^{n+1} : X^{n+1} \to I^n$ extending $\overline{g}_n$, and then we have $s^{n+1} \circ d_X^n = g^n = f^n - d_I^{n-1} \circ s^n$, as desired.

(ii). Apply (i) to $\mathrm{id}_{I^\bullet}$.

(iii). By Corollary IV.2.2.8 and the hypothesis on f, the mapping cone $\mathrm{Mc}(f)$ is acyclic. Note also that $\mathrm{Mc}(f)$ is an object of $\mathcal{C}^+(\mathscr{I})$ by definition, so, by (ii), it is homotopic to 0. Let $(S^n : \mathrm{Mc}(f)^n \to \mathrm{Mc}(f)^{n-1})_{n \in \mathbb{Z}}$ be a homotopy between $\mathrm{id}_{\mathrm{Mc}(f)}$ and 0, and write $S^n = \begin{pmatrix} s^{n+1} & g^n \\ h^n & t^n \end{pmatrix}$, with $s^{n+1} : X^{n+1} \to X^n$, $g^n : Y^n \to X^n$, $t^n : Y^n \to Y^{n-1}$ and $h^n : X^{n+1} \to Y^{n-1}$. The identity $d_{\mathrm{Mc}(f)}^{n-1} \circ S^n + S^{n+1} \circ d_{\mathrm{Mc}(f)}^n = \mathrm{id}_{\mathrm{Mc}(f)^n}$ implies the following three identities:

(1) $-d_X^n \circ s^{n+1} - s^{n+2} \circ d_X^{n+1} + g^{n+1} \circ f^{n+1} = \mathrm{id}_{X^{n+1}}$;

(2) $-d_X^n \circ g^n + g^{n+1} \circ d_Y^n = 0$;

(3) $f^n \circ g^n + d_Y^{n-1} \circ t^n + t^{n+1} \circ d_Y^n = \mathrm{id}_{Y^n}$.

The identities (2) imply that the family $(g^n : Y^n \to X^n)_{n\in\mathbb{Z}}$ defines a morphism of complexes $g : Y \to X$, and then (1) says that $(s^n)_{n\in\mathbb{Z}}$ is a homotopy between $g \circ f$ and id_X and (3) says that $(t^n)_{n\in\mathbb{Z}}$ is a homotopy between $f \circ g$ and id_Y. So we have proved that f is a homotopy equivalence.

(iv). We denote the morphisms $A \to X^0$ and $B \to I^0$ by d_X^{-1} and d_I^{-1} respectively (for convenience). As both $X^\bullet$ and $I^\bullet$ are concentrated in degree ≥ 0, it suffices to construct f^n for $n \geq 0$. We do this by induction on n:

(a) Suppose that $n = 0$. The morphism $d_X^{-1} : A \to X^0$ is injective by assumption, so, as I^0 is an injective object, there exists a morphism $f^0 : X^0 \to I^0$ such that $f^0 \circ d_X^{-1} = d_I^{-1} \circ f$.

(b) Suppose that $n \geq 0$, and that we have constructed $f^0, \dots, f^n$ making the obvious squares commute. Let $g^n = d_I^n \circ f^n$. Then $g^n \circ d_X^{n-1} = f^{n-1} \circ d_X^n \circ d_X^{n-1} = 0$, so g^n induces a morphism $\overline{g}_n : X^n/\operatorname{Im}(d_X^{n-1}) \to I^{n+1}$. As $A \to X^\bullet$ is a resolution, the morphism $\overline{d_X^n} : X^n/\operatorname{Im} d_X^{n-1} \to X^{n+1}$ induced by d_X^n is a monomorphism, so, as I^{n+1} is an injective object, there exists $f^{n+1} : X^{n+1} \to I^{n+1}$ such that $f^{n+1} \circ \overline{d_X^n} = \overline{g}^n$, which means that $f^{n+1} \circ d_X^n = g^n = d_I^n \circ f^n$.

Let $g^\bullet : X^\bullet \to I^\bullet$ be another morphism satisfying the condition of the statement. We want to show that $f^\bullet - g^\bullet$ is homotopic to 0. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "0"'] & X^\bullet \arrow[d, "f^\bullet - g^\bullet"] \\
0 \arrow[r] & 0 \arrow[r] & I^\bullet
\end{tikzcd}$$

In other words, $f^\bullet - g^\bullet$ gives a morphism from the acyclic complex $0 \to A \to X^0 \to X^1 \to \dots$ to the complex of injective objects $0 \to 0 \to I^0 \to I^1 \to \dots$. By (i), this morphism is homotopic to 0.

□

Remember that we have a fully faithful additive functor $\iota : \mathscr{A} \to K^b(\mathscr{A})$ that sends an object A to the complex $\dots \to 0 \to A \to 0 \to \dots$ concentrated in degree 0. (See Remark IV.1.4.2.)

Corollary IV.3.2.2. *(i). Let $\mathscr{I}$ be the full additive subcategory of injective objects of $\mathscr{A}$, and suppose that $\mathscr{A}$ has enough injective objects. Let $\mathscr{B}$ be the full subcategory of $K^+(\mathscr{I})$ whose objects are complexes $I^\bullet$ such that $\mathrm{H}^n(I^\bullet) = 0$ for $n \neq 0$. Then the functor H^0 induces an equivalence of categories from $\mathscr{B}$ to $\mathscr{A}$.*

In particular, taking a quasi-inverse of this equivalence, we get a functor $\lambda : \mathscr{A} \to K^+(\mathscr{I})$ and a morphism of functors $\iota \to \lambda$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, the morphism $\iota(A) \to \lambda(A)$ is a quasi-isomorphism; moreover, we can choose λ such that $\lambda(A)$ is concentrated in degree ≥ 0 for every $A \in \mathrm{Ob}(\mathscr{A})$.

(ii). Let $\mathscr{P}$ be the full additive subcategory of projective objects of $\mathscr{A}$. Suppose that $\mathscr{A}$ has enough projective objects. Let $\mathscr{B}$ be the full subcategory of $K^-(\mathscr{P})$ whose objects are complexes $P^\bullet$ such that $\mathrm{H}^n(P^\bullet) = 0$ for $n \neq 0$. Then the functor H^0 induces an equivalence of categories from $\mathscr{B}$ to $\mathscr{A}$.

In particular, taking a quasi-inverse of this equivalence, we get a functor $\lambda : \mathscr{A} \to K^-(\mathscr{P}))$ and a morphism of functors $\lambda \to \iota$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, the morphism $\lambda(A) \to \iota(A)$ is a quasi-isomorphism; moreover, we can choose λ such that $\lambda(A)$ is concentrated in degree ≤ 0 for every $A \in \mathrm{Ob}(\mathscr{A})$.

We will see in the chapter on derived categories[3] how to extend the functor λ to $\mathcal{C}^+(\mathscr{A})$ or $\mathcal{C}^-(\mathscr{A})$ (depending on whether we are in the situation of (i) or of (ii).)

Proof. We only prove (i). (Point (ii) follows by applying (i) to the opposite category.)

Let $\mathscr{B}'$ be the full subcategory of $\mathscr{B}$ whose objects are complexes concentrated in degree ≥ 0. An object $I^\bullet$ Of $\mathscr{B}'$ is simply an injective resolution of $\mathrm{Ker}\, d_I^0 = \mathrm{H}^0(I^\bullet)$. So the functor $\mathrm{H}^0 : \mathscr{B}' \to \mathscr{A}$ is essentially surjective because every object of $\mathscr{A}$ has an injective resolution by Lemma IV.3.1.2), it is full because, if $f : A \to B$ is a morphism between objects of $\mathscr{A}$ and $A \to I^\bullet$ and $B \to J^\bullet$ are injective resolutions, then f extends to $f^\bullet : I^\bullet \to J^\bullet$ by Theorem IV.3.2.1(iv), and it is faithful because this extenstion is unique up to homotopy (by the same reference).

To finish the proof, it suffices to show that every object of $\mathscr{B}$ is isomorphic to an object of the subcategory $\mathscr{B}'$. Let $I^\bullet$ be an object of $\mathscr{B}$. We prove by induction on n that $\mathrm{Coker}\, d_I^n$ is an injective object for every $n \leq -1$. Indeed, if n is small enough, then $\mathrm{Coker}\, d_I^n = 0$, so the claim holds. Suppose that $n \leq -1$ and that we have proved the claim for $n-1$. We have an exact sequence $0 \to I^n/\mathrm{Ker}\, d_I^n \to I^{n+1} \to \mathrm{Coker}\, d_I^n \to 0$. As, as $\mathrm{H}^n(I^\bullet) = 0$, the canonical morphism $\mathrm{Coker}\, d_I^{n-1} = I^n/\mathrm{Im}\, d_I^{n-1} \to I^n/\mathrm{Ker}\, d_I^n$ is an isomorphism, so $I^n/\mathrm{Im}\, d_I^{n-1}$ is an injective object by the induction hypothesis. By Corollary II.2.4.6, this implies that $\mathrm{Coker}\, d_I^n$ is also injective. Now consider the canonical morphism $u : I^\bullet \to \tau^{\geq 0}(I^\bullet)$ (see Definition V.4.2.1); this is a quasi-isomorphism because $\mathrm{H}^n(I^\bullet) = 0$ for $n \leq -1$. Also, by what we just proved, the complex $\tau^{\geq 0}(I^\bullet)$ is in $\mathcal{C}^+(\mathscr{I})$, hence it is an object of $\mathscr{B}'$; so Theorem IV.3.2.1(iii) implies that u is a homotopy equivalence, that is, that its image in $K^+(\mathscr{I})$ is an isomorphism. So we have found an isomorphism in $K^+(\mathscr{I})$ from $I^\bullet$ to an object of $\mathscr{B}'$, and we are done.

$\square$

IV.3.3 Derived functors

Definition IV.3.3.1. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between abelian categories.

[3] Add precise reference.

(i). Suppose that F is left exact and that $\mathscr{A}$ has enough injective objects. Let $\mathscr{I}$ be the full subcategory of injective objects of $\mathscr{A}$, and $\lambda : \mathscr{A} \to K^+(\mathscr{I})$ be the functor of Corollary IV.3.2.2. Then, for every $n \in \mathbb{Z}$, the *nth left derived functor* of F is the functor $R^nF : \mathscr{A} \xrightarrow{\lambda} K^+(\mathscr{I}) \xrightarrow{K(F)} K^+(\mathscr{B}) \xrightarrow{\mathrm{H}^n} \mathscr{B}$.

(ii). Suppose that F is right exact and that $\mathscr{A}$ has enough projective objects. Let $\mathscr{P}$ be the full subcategory of projective objects of $\mathscr{A}$, and $\lambda : \mathscr{A} \to K^-(\mathscr{P})$ be the functor of Corollary IV.3.2.2. Then, for every $n \in \mathbb{Z}$, the *nth left derived functor* of F is the functor $L^nF : \mathscr{A} \xrightarrow{\lambda} K^-(\mathscr{P}) \xrightarrow{K(F)} K^-(\mathscr{B}) \xrightarrow{\mathrm{H}^{-n}} \mathscr{B}$.

We will mostly state results for right derived functors, but every such result has a dual version for left derived functors that follows from it by looking at opposite categories.

Remark IV.3.3.2. In the situation of Definition IV.3.3.1(i):

(a) $R^nF : \mathscr{A} \to \mathscr{B}$ is an additive functor for every $n \in \mathbb{Z}$ (because it is a composition of additive functors).

(b) If we want to calculate $R^nF(A)$ for some object A of $\mathscr{A}$, what the definition tells us is that we should pick an injective resolution $0 \to A \to I^0 \to I^1 \to \ldots$ of A, delete A to get a complex $\ldots \to 0 \to I^0 \to I^1 \to \ldots$, apply F to it, and then the nth cohomology of the resulting complex will be isomorphic to $R^nF(A)$.

(c) In particular, $R^nF = 0$ for $n \leq -1$, and, if I is an injective object of $\mathscr{A}$, we have $R^nF(I) = 0$ for $n \geq 1$. Also, as F is left exact, we have a canonical isomorphism $F \overset{\sim}{\to} R^0F$.

Theorem IV.3.3.3. *Suppose that we are in the situation of Definition IV.3.3.1(i). Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathscr{A}$. Then we get a long exact sequence in $\mathscr{B}$, functorial in the original short exact sequence:*

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{\delta^0} R^1F(A) \to R^1F(B) \to \ldots$$

$$\ldots \to R^{n-1}F(C) \xrightarrow{\delta^{n-1}} R^nF(A) \to R^nF(B) \to R^nF(C) \xrightarrow{\delta^n} R^{n+1}F(A) \to \ldots$$

In fact, $(R^nF)_{n\geq 0}$ is universal among all families of functors having that property.

Proof. By Proposition IV.3.1.5 (and Corollary IV.3.2.2), we have a diagram in $\mathcal{C}^+(\mathscr{I})$:

$$\begin{tikzcd} & \lambda(A) \arrow[r] \arrow[d] & \lambda(B) \arrow[r] \arrow[d] & \lambda(C) \arrow[d] & \\ 0 \arrow[r] & I^\bullet \arrow[r] & J^\bullet \arrow[r] & K^\bullet \arrow[r] & 0 \end{tikzcd}$$

such that the vertical morphisms are homotopy equivalences, the bottom row is exact, and the diagram becomes commutative in $K^+(\mathscr{I})$. So the existence of the long exact sequence follows from Corollary IV.2.2.6. We also have to show that the connecting morphisms δ^n don't depend on the choices we made, but this follows from Corollary IV.2.2.6, as all the possible choices are unique up to a unique homotopy equivalence.

$\square$

Definition IV.3.3.4. (i). Suppose that we are in the situation of Definition IV.3.3.1(i). An object A of $\mathscr{A}$ such that $R^nF(A) = 0$ for every $n \geq 1$ is called *F-acyclic.*

(ii). Suppose that we are in the situation of Definition IV.3.3.1(ii). An object A of $\mathscr{A}$ such that $L^nF(A) = 0$ for every $n \geq 1$ is called *F-acyclic.*

In particular, injective object are acyclic for left exact functors, and projective objects are acyclic for right exact functors.

Lemma IV.3.3.5. *Suppose that $\mathscr{A}$ has enough injective objects, and let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. We denote by $\mathscr{I}_F$ the full subcategory of $\mathscr{A}$ whose objects are the F-acyclic objects.*

(i). If $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathscr{A}$ such that $A, B \in \mathscr{I}_F$, then C is also in $\mathscr{I}_F$, and the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.

(ii). If $X^\bullet \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{I}_F))$ is acyclic, then $F(X^\bullet)$ is also acyclic.

Proof. Point (i) follows immediately from Theorem IV.3.3.3: as $R^nF(A) = 0$ and $R^nF(B) = 0$ for every $n \geq 1$, the long exact sequence of that theorem reduces to a short exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ and to isomorphisms $R^nF(C) \simeq 0$ for every $n \geq 1$.

We prove point (ii). As $X^\bullet$ is acyclic, the canonical morphism $\operatorname{Coker} d_X^n = X^{n+1}/\operatorname{Im}(d_X^n) \to X^{n+1}/\operatorname{Ker} d_X^n$ is an isomorphism for every $n \in \mathbb{Z}$. We prove by induction on n that, for every $n \in \mathbb{Z}$, the object $\operatorname{Coker} d_X^n$ is F-acyclic and the sequence

$$0 \to F(X^n/\operatorname{Ker} d_X^n) \to F(X^{n+1}) \to F(\operatorname{Coker} d_X^n) \to 0$$

is exact. If n is small enough, then $X^n = 0$, $X^{n+1} = 0$ and $\operatorname{Coker} d_X^n = 0$, so the claim holds. Suppose that we know the claim for some $n \in \mathbb{Z}$. We have an exact sequence

$$0 \to \operatorname{Coker} d_X^n \xrightarrow{\sim} X^{n+1}/\operatorname{Ker} d_X^{n+1} \to X^{n+2} \to \operatorname{Coker} d_X^{n+1} \to 0,$$

so the claim for $n+1$ follows immediately from (i).

Let $n \in \mathbb{Z}$. We have just shown that the morphism $F(d_X^n) : F(X^n) \to F(X^{n+1})$ factors as the composition of a surjection $F(X^n) \to F(\operatorname{Coker} d_X^{n-1})$ and an injection $F(\operatorname{Coker} d_X^{n-1}) \xrightarrow{\sim} F(X^n/\operatorname{Ker} d_X^n) \to F(X^{n+1})$, so

$$\operatorname{Ker}(F(d_X^n)) = \operatorname{Ker}(F(X^n) \to F(\operatorname{Coker} d_X^{n-1})) = \operatorname{Im}(F(\operatorname{Coker} d_X^{n-2}) \to F(X^n)).$$

As the morphism $F(X^{n-1}) \to F(\operatorname{Coker} d_X^{n-2})$ is surjective, we finally get $\operatorname{Ker}(F(d_X^n)) = \operatorname{Im}(d_X^{n-1})$, so that $\mathrm{H}^n(F(X^\bullet)) = 0$.

□

Theorem IV.3.3.6. *Suppose that $\mathscr{A}$ has enough injective objects, and let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. We denote by $\mathscr{I}_F$ the full subcategory of $\mathscr{A}$ whose objects are the F-acyclic objects.*

Let $A \in \operatorname{Ob}(\mathscr{C})$ and let $A \to X^\bullet$ be a right $\mathscr{I}_F$-resolution of A. Then we have canonical isomorphisms $\mathrm{H}^n(F(X^\bullet)) \xrightarrow{\sim} R^n F(A)$.

Proof. Let $A \to I^\bullet$ be an injective resolution of A. By Theorem IV.3.2.1(iv), there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & X^\bullet \arrow[d, "f^\bullet"] \\ A \arrow[r] & I^\bullet \end{tikzcd}$$

where $f^\bullet$ is uniquely determined up to homotopy. We claim that $F(f^\bullet) : F(X^\bullet) \to F(I^\bullet)$ is a quasi-isomorphism, which immediately implies the theorem. Let's prove this claim. Let $Y^\bullet = \mathrm{Mc}(f^\bullet)$. Then $Y^\bullet \in \mathcal{C}^+(\mathscr{I}_F)$ and $Y^\bullet$ is acyclic by Corollary IV.2.2.8; by Lemma IV.3.3.5, the complex $F(\mathrm{Mc}(f^\bullet))$ is also acyclic. As $F(\mathrm{Mc}(f^\bullet)) = \mathrm{Mc}(F(f^\bullet))$ (by definition of the mapping cone), another application of Corollary IV.2.2.8 shows that $F(f^\bullet)$ is a quasi-isomorphism.

□

It is natural to ask how to calculate the derived functors of a composition $G \circ F$ in term of the derived functors of F and G. We will give a partial answer here. The full answer requires spectral sequences or derived categories, so we will defer for a while.

Proposition IV.3.3.7. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left exact additive functors with $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ abelian categories. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects.*

(i). If G is exact, we have canonical isomorphisms of functors $R^n(G \circ F) \simeq G \circ R^n F$.

(ii). Assume that F is exact. For every $n \in \mathbb{N}$, there is a canonical morphism of functors $R^n(G \circ F) \to (R^n G) \circ F$.

(iii). Suppose that F sends the injective objects of $\mathscr{A}$ to G-acyclic objects of $\mathscr{B}$ and that $A \in \operatorname{Ob}(\mathscr{A})$ is F-acyclic. Then there is a canonical isomorphism $R^n(G \circ F)(A) \simeq (R^n G)(F(A))$ for every $n \in \mathbb{N}$.

(iv). If F is exact and sends the injective objects of $\mathscr{A}$ to G-acyclic objects of $\mathscr{B}$, then the morphisms $R^n(G \circ F) \to (R^n G) \circ F$ are isomorphisms.

Proof. Let A be an object of $\mathscr{A}$, and let $A \to I^\bullet$ be an injective resolution of A. Then $R^n(G \circ F)(A) = \mathrm{H}^n(G(F(I^\bullet)))$.

If G is exact, then it commutes with kernels and cokernels, so we have isomorphisms $\mathrm{H}^n(G(F(I^\bullet))) \simeq G(\mathrm{H}^n(F(I^\bullet))) = G(R^nF(A))$. This gives (i).

To prove (ii), assume that F is exact. choose an injective resolution $F(A) \to J^\bullet$ of $F(A)$. As F is exact, the morphism $F(A) \to F(I^\bullet)$ is a resolution of $F(A)$. So, by Theorem IV.3.2.1(iv), there exists a morphism of complexes $f^\bullet : F(I^\bullet) \to J^\bullet$ extending $\mathrm{id}_{F(A)}$, and such a $f^\bullet$ is unique up to homotopy. Applying G and taking the nth cohomology object, we get the desired morphism

$$R^n(G \circ F)(A) = \mathrm{H}^n(G(F(I^\bullet))) \to \mathrm{H}^n(J^\bullet) = (R^nG)(F(A)).$$

We now assume that A is F-acyclic and that F sends the injective objects of $\mathscr{A}$ to G-acyclic objects of $\mathscr{B}$. For every $n \geq 1$, we have $\mathrm{H}^n(F(I^\bullet)) = R^nF(A) = 0$, so $F(A) \to F(I^\bullet)$ is a resolution of $F(A)$ by G-acyclic objects. By Theorem IV.3.3.6, we have canonical isomorphisms $(R^nG)(F(A)) \simeq \mathrm{H}^n(G(F(I^\bullet))) = R^n(G \circ F)(A)$.

Point (iv) follows immediately from (ii) and (iii).

□

IV.3.4 The case of bifunctors

Let $F : A \times \mathscr{B} \to \mathscr{C}$ be a left exact additive bifunctor, with $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ abelian categories. This means that F is additive and left exact in each variable.

We assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects. For every $A \in \mathrm{Ob}(\mathscr{A})$ (resp. $B \in \mathrm{Ob}(\mathscr{B})$), we denote by $R^nF(A, \cdot)$ (resp. $R^nF(\cdot, B)$) the nth right derived functor of the left exact additive functor $F(A, \cdot) : \mathscr{B} \to \mathscr{C}$ (resp. $F(\cdot, B) : \mathscr{A} \to \mathscr{C}$).

Theorem IV.3.4.1. *Suppose that, for any injective objects I of $\mathscr{A}$ and J of $\mathscr{B}$, the functors $F(I, \cdot) : \mathscr{B} \to \mathscr{C}$ and $F(\cdot, J) : \mathscr{A} \to \mathscr{C}$ are exact. Then, for any $n \in \mathbb{N}$, $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$, we have an isomorphism, functorial in A and B and compatible with the connecting morphisms in the long exact sequence of derived functosr:*

$$(R^nF(A, \cdot))(B) \simeq (R^nF(\cdot, B))(A).$$

We simply write $R^nF(A, B)$ for $(R^nF(A, \cdot))(B)$.

The theorem has an obvious variant for left derived functors of bifunctors that are additive and right exact.

Proof. Let $A \to I^\bullet$ and $B \to J^\bullet$ be injective resolutions of A and B. We consider the following double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F(I^0, B) \arrow[r] \arrow[d] & F(I^1, B) \arrow[r] \arrow[d] & F(I^2, B) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(A, J^0) \arrow[r] \arrow[d] & F(I^0, J^0) \arrow[r] \arrow[d] & F(I^1, J^0) \arrow[r] \arrow[d] & F(I^2, J^0) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(A, J^1) \arrow[r] \arrow[d] & F(I^0, J^1) \arrow[r] \arrow[d] & F(I^1, J^1) \arrow[r] \arrow[d] & F(I^2, J^1) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(A, J^2) \arrow[r] \arrow[d] & F(I^0, J^2) \arrow[r] \arrow[d] & F(I^1, J^2) \arrow[r] \arrow[d] & F(I^2, J^2) \arrow[r] \arrow[d] & \dots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the hypothesis, all the rows and columns of this complex are exact except possibly for the first row and the first column. By Corollary IV.2.2.4 (the $\infty \times \infty$ lemma), the first row and the first column have canonically isomorphism cohomology objects. This proves the statement.

□

Example IV.3.4.2. (1) If $\mathscr{A}$ has enough injective and projective objects, then the theorem applies to the bifunctor $\mathrm{Hom}_{\mathscr{A}}(\cdot, \cdot) : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Ab}$. The nth right derived functor of $\mathrm{Hom}_{\mathscr{A}}$ is denoted by $\mathrm{Ext}^n_{\mathscr{A}}(\cdot, \cdot) : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Ab}$.

(2) Let R be a ring. We denote by $\mathrm{Tor}^R_n(\cdot, \cdot)$ the nth left derived functor of the bifunctor $(\cdot) \otimes_R (\cdot) : \mathbf{Mod}_R \times {}_R\mathbf{Mod} \to \mathbf{Ab}$. If R is commutative, we can see $(\cdot) \otimes_R (\cdot)$ and the Tor^R_n as bifunctors ${}_R\mathbf{Mod} \times {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$.

Definition IV.3.4.3. Suppose that $\mathscr{A}$ has enough injective (resp. projective) objects. For every $A \in \mathrm{Ob}(\mathscr{A})$ (resp. $B \in \mathrm{Ob}(\mathscr{A})$), we denote by $\mathrm{Ext}^n_{\mathscr{A}}(A, \cdot)$ (resp. $\mathrm{Ext}^{\mathscr{A}}(\cdot, B)$) the nth derived functor of the left exact additive functor $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$ (resp. $\mathrm{Hom}_{\mathscr{A}}(\cdot, B)$). If $\mathscr{A}$ is the category of left or right R-modules, we write Ext^n_R instead of $\mathrm{Ext}^n_{\mathscr{A}}$.

By Example IV.3.4.2(1), if $\mathscr{A}$ has enough injective and projective objects, the two definitions of $\mathrm{Ext}^n_{\mathscr{A}}(A, B)$ agree.

IV.3.5 More examples of derived functors

Group homology and cohomology

Let G be an abstract group, R be a ring and $\mathscr{A}$ be the category ${}_{R[G]}\mathbf{Mod}$, that is, the category of representations of G on left R-modules. By Corollary II.3.2.9, the category $\mathscr{A}$ has enough injective objects, so we can derived any left or right exact additive functor on it.

Consider the functors $\mathrm{H}_0(G,\cdot) = (\cdot)_G : \mathscr{A} \to {}_R\mathbf{Mod}$ ("G-coinvariants") and $\mathrm{H}^0(G,\cdot) = (\cdot)^G : \mathscr{A} \to {}_R\mathbf{Mod}$ ("G-invariants") defined by

$$M_G = M/\operatorname{Span}\{g\cdot m - m,\ g\in G,\ m\in M\}$$

and

$$M^G = \{m\in M \mid \forall g\in G,\ g\cdot m = m\},$$

for every $R[G]$-module M.

The functor $\mathrm{H}_0(G,\cdot)$ (resp. $\mathrm{H}^0(G,\cdot)$) is right (resp. left) exact, and its nth left (resp. right) derived functor is denote by $\mathrm{H}_n(G,\cdot)$ (resp. $\mathrm{H}^n(G,\cdot)$). If M is a $R[G]$-module, the R-module $\mathrm{H}_n(G,M)$ is called the *nth homology module* of M, and $\mathrm{H}^n(G,M)$ is called the *nth cohomology module* of M.

We write R for the R-module R with the trivial action of G. Note that this makes R a left and right $R[G]$-module. For every $R[G]$-module M, we have isomorphisms, functorial in M,

$$R\otimes_{R[G]} M \xrightarrow{\sim} M_G,\ 1\otimes m \longmapsto m$$

and

$$\operatorname{Hom}_G(R,M)\to M^G,\ f\longmapsto f(1).$$

So $\mathrm{H}_n(G,M) = \operatorname{Tor}_n^{R[G]}(R,M)$ and $\mathrm{H}^n(G,M) = \operatorname{Ext}^n_{R[G]}(R,M)$.

Remark IV.3.5.1. If M is a left $R[G]$-module, we can calculate $\mathrm{H}^n(G,M)$ and $\mathrm{H}_n(G,M)$ as derievd functors in the category ${}_{R[G]}\mathbf{Mod}$, or in the category ${}_{\mathbb{Z}[G]}\mathbf{Mod}$ (by forgetting the R-module structure on M). Let us show that this gives the same result.

The forgetful functor ${}_{R[G]}\mathbf{Mod} \to {}_{\mathbb{Z}[G]}\mathbf{Mod}$ and has a left adjoint $M\longmapsto R\otimes_k M$, however it is not clear that this left adjoint sends injective (resp. projective) $R[G]$-modules to $\mathrm{H}^0(G,\cdot)$-acyclic (resp. $\mathrm{H}_0(G,\cdot)$-acyclic) objects,[4] so we cannot apply Proposition IV.3.3.7 directly. Instead, we use the description of $\mathrm{H}^n(G,\cdot)$ and $\mathrm{H}_n(G,\cdot)$ as Ext^n and Tor_n functors. Let $P^\bullet\to\mathbb{Z}$ (resp. $Q^\bullet\to\mathbb{Z}$) be a projective resolution of $\mathbb{Z}$ as a right (resp. left) $\mathbb{Z}[G]$-module (with trivial G-action); this means that $P^n = 0$ (resp. $Q^n = 0$) if $n > 0$, that P^n (resp. Q^n) is a projective right (resp. left) $\mathbb{Z}[G]$-module for $n\leq 0$ and that $\dots P^{-2}\to P^{-1}\to P^0\to\mathbb{Z}$ is an exact sequence in $\mathbf{Mod}_{\mathbb{Z}[G]}$ (resp. $\dots Q^{-2}\to Q^{-1}\to Q^0\to\mathbb{Z}$ is an exact sequence in ${}_{\mathbb{Z}[G]}\mathbf{Mod}$).

[4] Of course, this follows from what we are trying to prove.

Let M be a left $R[G]$-module; we use the same notation for M seen as a left $\mathbb{Z}[G]$-module. Then

$$\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M) = \mathrm{H}^{-n}(P^\bullet \otimes_{\mathbb{Z}[G]} M)$$

and

$$\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M) = \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q^\bullet, M)).$$

Let $P'^\bullet = P^\bullet \otimes_{\mathbb{Z}} R$ and $Q'^\bullet = R \otimes_{\mathbb{Z}} Q^\bullet$. Then all P^n (resp. Q^n) are projective right (resp. left) $R[G]$-modules, by Example II.2.4.7(1) and because the functor $(\cdot) \otimes_{\mathbb{Z}} R$ (resp. $R \otimes_{\mathbb{Z}} (\cdot)$) sends free right (resp. left) $\mathbb{Z}[G]$-modules to free $R[G]$-modules. Also, as a projective module is flat (by Example II.2.4.7(1) again), the sequences $\ldots \to P'^{-2} \to P'^{-1} \to P'^0 \to R$ and $\ldots \to Q'^{-2} \to Q'^{-1} \to Q'^0 \to R$ are still exact. So

$$\mathrm{Tor}_n^{R[G]}(R, M) = \mathrm{H}^{-n}(P'^\bullet \otimes_{R[G]} M) = \mathrm{H}^{-n}(P^\bullet \otimes_{\mathbb{Z}[G]} M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

and

$$\mathrm{Ext}^n_{R[G]}(R, M) = \mathrm{H}^n(\mathrm{Hom}_{R[G]}(Q'^\bullet, M)) = \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q^\bullet, M)) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

In problem A.6.2(b), we have constructed two right resolutions of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. Applying the functor $R \otimes_{\mathbb{Z}} (\cdot)$ to these resolutions, we get two right resolutions of R by free $R[G]$-modules, that we can use to calculate the functors $\mathrm{H}_n(G, \cdot)$ and $\mathrm{H}^n(G, \cdot)$, thanks to Theorem IV.3.4.1.

Consider for example the unnormalized bar resolution $X^\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ defined in problem A.6.2(b). We have $X^n = 0$ if $n \geq 1$ and $X^{-n} = \mathbb{Z}^{(G^{n+1})}$ if $n \geq 0$. For every $n \geq 0$ and every $R[G]$-module M, we set $C^n(G, M) = \mathrm{Hom}_{R[G]}(R \otimes_{\mathbb{Z}} X^{-n}, M)$; this is called the group of *n-cochains* on G with values in M. The differentials of the complex $X^\bullet$ induce morphisms $d^n : C^n(G, M) \to C^{n+1}(G, M)$, so we get a complex $C^\bullet(G, M)$, and $\mathrm{H}^n(G, M)$ is the nth cohomology group of this complex. We write $Z^n(G, M) = \mathrm{Ker}(d^n)$ (resp. $B^n(G, M) = \mathrm{Im}(d^{n-1})$) and call it the group of *n-cocyles* (resp. *n-coboundaries*) on G with values in M.

We have seen in problem A.6.2(b) that X^{-n} is the free $\mathbb{Z}[G]$-module with basis $(e_{(1,g_1,g_1g_2,\ldots,g_1g_2\ldots g_n)})_{(g_1,\ldots,g_n)\in G^n}$, so we have a R-linear isomorphism $C^n(G, M) \overset{\sim}{\to} \mathscr{F}(G^n, M)$ (where $\mathscr{F}(G^n, M)$ is the set of functions from G^n to M) sending $u : R \otimes_{\mathbb{Z}} X^{-n} \to M$ to the function $(g_1, \ldots, g_n) \longmapsto u(e_{(1,g_1,g_1g_2,\ldots,g_1g_2\ldots g_n)})$.

It is easy to calculate the differentials: if $c : G^n \to M$ is a function, then, for all $g_1, \ldots, g_{n+1} \in G$, we have

$$\begin{aligned}(d^n c)(g_1, \ldots, g_{n+1}) = g_1 \cdot c(g_2, \ldots, g_{n+1}) + &\sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} c(g_1, \ldots, g_n).\end{aligned}$$

So, for example, a 1-cocycle is a function $c : G \to M$ such that, for all $g, h \in G$, we have $c(gh) = g \cdot c(h) + c(g)$, and a 1-coboundary is a function $c : G \to M$ of the form $g \longmapsto g \cdot m - m$, for some $m \in M$. If the action of G on M is trivial, 1-cocycles are simply group morphisms $G \to M$ and 1-coboundaries are all 0, so $\mathrm{H}^1(G, M) = \mathrm{Hom}_{\mathbf{Grp}}(G, M)$.

If moreover we have a group H acting on G and on M such that the action of H on M is R-linear and that $h \cdot (g \cdot x) = (h \cdot g)(h \cdot x)$ for all $h \in H$, $g \in G$ and $x \in M$, then H acts on $\mathrm{H}^n(G, M)$ for every $n \in \mathbb{N}$, and this action comes from the action on n-cochains given by $(h \cdot c)(g_1, \ldots, g_n) = h \cdot c(h^{-1} \cdot g_1, \ldots, h \cdot g_n)$. This occurs for example if G is a normal subgroup of H on which H acts by conjugation and M is a $R[H]$-module.

Suppose that we have a normal subgroup $K \subset G$. Then $\mathrm{H}^0(G, \cdot)$ is isomorphic to the composition $\mathrm{H}^0(G/K, \cdot) \circ \mathrm{H}^0(K, \cdot)$, but we cannot apply Proposition IV.3.3.7 to reduce the calculation of $\mathrm{H}^0(G, \cdot)$ to that of $\mathrm{H}^0(K, \cdot)$ and $\mathrm{H}^0(G/K, \cdot)$, because neither of these last two functors is exact in general.

Sheaf cohomology

Let $\mathscr{C}$ be a category having all fiber products and $\mathscr{T}$ be a Grothendieck pretopology on $\mathscr{C}$. Let R be a ring. We know that $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, R)$ is a Grothendieck abelian category by Corollary III.2.2.15, so it has enough injective objects by Theorem II.3.2.4.

If X is an object of $\mathscr{C}$, then the functor $\mathrm{H}^0(X, \cdot) : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, R) \to {}_R\mathbf{Mod}$, $\mathscr{F} \longmapsto \mathscr{F}(X)$ is left exact. We denote by $\mathrm{H}^n(X, \cdot)$ its nth right derived functor; the R-module $\mathrm{H}^n(X, \mathscr{F})$ is called the *nth cohomology module of $\mathscr{F}$ on X*. These are difficult to calculate in general.

The inclusion from sheaves to presheaves

We keep the setting of the beginning of the previous example, and denote the inclusion functor $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \to \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ by Φ. This functor is left exact but not exact, so it should have nontrivial right derived functors. We calculate these functors.

Let $\mathscr{F}$ be an abelian sheaf on $\mathscr{C}_{\mathscr{T}}$, and let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z})$. Then the presheaf $R^n\Phi(\mathscr{F})$ is equal to $\mathrm{H}^n(\Phi(\mathscr{I}^\bullet))$. More concretely, for every object X of $\mathscr{C}$, the group $(R^n\Phi(\mathscr{F}))(X)$ is equal to the nth cohomology group of the complex $\mathscr{I}^\bullet(X)$, that is, to $\mathrm{H}^n(\mathscr{X}, \mathscr{F})$. In other word, for every $n \in \mathbb{N}$, the value at $\mathscr{F}$ of the nth derived functor $R^n\Phi$ is the presheaf $X \longmapsto \mathrm{H}^n(X, \mathscr{F})$.

Čech cohomology

Let $\mathscr{C}$ be a category, and let $\mathscr{X} = (X_i \to X)_{i \in I}$ be a family of morphisms of $\mathscr{C}$. We have defined in problem A.6.3 the Čech cohomology functors $\check{\mathrm{H}}^n(\mathscr{X}, \cdot) : \mathrm{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$ as the cohomology of an explicit complex-valued functor on $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, and we have shown that $\check{\mathrm{H}}^n(\mathscr{X}, \cdot)$ s the nth right derived functor of $\check{\mathrm{H}}^0(\mathscr{X}, \cdot) : \mathrm{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$.

Suppose that $\mathscr{C}$ has fiber products, that we have a Grothendieck pretopology $\mathscr{T}$ on $\mathscr{C}$, and that $\mathscr{X}$ is a covering family for this pretopology. Let $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \to \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ be the inclusion functor. Then the functors $\check{\mathrm{H}}^0(\mathscr{X}, \cdot) \circ \Phi$ and $\mathrm{H}^0(X, \cdot)$ are isomorphic (this is just saying that $\mathscr{F}(X) \overset{\sim}{\to} \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ if $\mathscr{F}$ isa sheaf), but the inclusion functor Φ is not exact (it is only left exact), so we cannot use Proposition IV.3.3.7 to calculate $\mathrm{H}^n(X, \cdot)$.

IV.4 Spectral sequences

Spectral sequences are a useful tool in homological algebra, but they can get a bit messy. In the first subsection, we will give the definition of a spectral sequence, state the theorem asserting the existence of the two spectral sequences of a double complex, and show how it implies the Grothendieck spectral sequence, that relates the right derived functors of a composition $G \circ F$ to the right dertived functors of F and G. In the second subsection, we will present a general method to construct spectral sequences, and in particular prove the results of the first subsection.

In this whole section, we fix an abelian category $\mathscr{A}$, and we assume that it (and the other abelian categories that may appear) has all small limits and colimits, and that direct sums and filtrant colimits are exact; for example, every category of sheaves of R-modules has these properties. This is not strictly necessary, but it will allow us to simplify the exposition somewhat.

IV.4.1 Definition and some theorems

We will only consider cohomology spectral sequence. There is a dual theory of homology spectral sequences, where the differentials go in the other direction; it is equivalent up to some reindexing.

Definition IV.4.1.1. A *spectral sequence* starting at the page $r_0 \in \mathbb{N}$ is the following data:

(a) for every integer $r \geq r_0$, a family $(E_r^{pq}, d_r^{pq})_{p,q\in\mathbb{Z}}$ of objects of $\mathscr{A}$ and of morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{pq} = 0$;

(b) for every $r \geq r_0$, isomorphisms $E_{r+1}^{pq} \simeq \mathrm{Ker}(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1})$ (which we will often write as equalities).

Spectral sequences form a category in the obvious way. (Morphisms of spectral sequences must be compatible with all the differentials and with the isomorphisms between each page and the cohomology of the previous one.)

We call the family $E_r = (E_r^{pq}, d_r^{pq})_{p,q\in\mathbb{Z}}$ the *rth page* of the spectral sequence, and often abbreviate (b) to "an isomorphism between E_{r+1} and the cohomology of E_r". (Note however that the differentials d_{r+1}^{pq} are not determined by E_r, only the objects E_{r+1}^{pq}.)

Remark IV.4.1.2. To make the notation less cumbersome, the following point of view is useful: we think of E_r as in the bigraded object $\bigoplus_{p,q\in\mathbb{Z}} E_r^{pq}$ of $\mathscr{A}$, and of $d_r = \sum_{p,q} d_r^{pq}$ as an endomorphism of E_r of bidegree $(r, -r+1)$ satisfying $d_r^2 = 0$. Then (b) becomes an isomorphism $E_{r+1} \simeq \operatorname{Ker}(d_r)/\operatorname{Im}(d_r)$.

Suppose that we are given a spectral sequence $(E_r)_{r\geq r_0}$ starting at the r_0th page. Fix $p, q \in \mathbb{Z}$. For every $r \geq r_0$, the object E_r^{pq} is isomorphic (by data (b) in the definition of a spectral sequence) to a subquotient of $E_{r_0}^{pq}$, so we have subobjects $B_r^{pq} \subset Z_r^{pq} \subset E_{r_0}^{pq}$ such that $E_r^{pq} \simeq Z_r^{pq}/B_r^{pq}$. As E_r^{pq} is also identified to a subobject of E_s^{pq} for $r_0 \leq s \leq r$, we have $Z_s^{pq} \supset Z_r^{pq}$ and $B_s^{pq} \subset B_r^{pq}$ if $s \leq r$.

Definition IV.4.1.3. We set

$$B_\infty^{pq} = \bigcap_{r\geq r_0} B_r^{pq} := \varprojlim_{r\geq r_0} B_r^{pq},$$

where the transition morphisms are the injections $B_r^{pq} \to B_s^{pq}$ for $r \geq s \geq r_0$,

$$Z_\infty^{pq} = \bigcup_{r\geq r_0} Z_r^{pq},$$

and

$$E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}.$$

In general, E_∞^{pq} could be a strict subquotient of every E_r^{pq}. However, there are interesting cases where E_r^{pq} eventually stabilizes.

Example IV.4.1.4. (1) We say that the spectral sequence *degenerates at the page* E_r if $d_{r'}^{pq} = 0$ for every $r' \geq r$ and all $p, q \in \mathbb{Z}$. If this is the case, then $E_{r'}^{pq} = E_r^{pq}$ for every $r' \geq r$, so $E_\infty^{pq} = E_r^{pq}$.

(2) We say that the spectral sequence is a *first quadrant spectral sequence* if, for every $r \geq r_0$, we have $E_r^{pq} = 0$ as soon as $p < 0$ or $q < 0$. Suppose that this is the case, and let $p, q \in \mathbb{Z}$. Then $d_r^{pq} = 0$ if $r > q+1$ and $d_r^{p-r,q+r-1} = 0$ if $r > p$, so $E_r^{pq} = E_{r+1}^{pq}$ for $r > \max(q+1, p)$, and we get $E_\infty^{pq} = E_r^{pq}$ for $r > \max(q+1, p)$.

Definition IV.4.1.5. We say that the spectral sequence *converges* to a graded object $H^* = \bigoplus_{n\in\mathbb{Z}} H^n$ of $\mathscr{A}$ if we are given a decreasing filtration $(\operatorname{Fil}^p H^n)_{p\in\mathbb{Z}}$ on each H^n (i.e. a sequence of subobjects such that $\operatorname{Fil}^p H^n \supset \operatorname{Fil}^{p+1} H^n$) such that $\bigcap_{p\in\mathbb{Z}} \operatorname{Fil}^p H^n = 0$ and $\bigcup_{p\in\mathbb{Z}} \operatorname{Fil}^p H^n = H^n$ (in other words, the filtration is separated and exhaustive), and isomorphisms $E_\infty^{pq} \simeq \operatorname{Fil}^p H^{p+q}/\operatorname{Fil}^{p+1} H^{p+q}$.

If the spectral sequence starts at the r_0th page and converges to H^*, we often write this as "$E_{r_0}^{pq} \Rightarrow H^{p+q}$".

Remark IV.4.1.6. If a first quadrant spectral sequence converges to H^*, then, for every $n \in \mathbb{Z}$, the filtration on H^n has finitely many nonzero quotients, and $H^n = 0$ if $n < 0$. Indeed, we have $\operatorname{Fil}^p H^n/\operatorname{Fil}^{p+1}\mathrm{H}^n \simeq E_\infty^{p,n-p}$, and $E_\infty^{p,n-p} = 0$ if $p < 0$ or $p > n$; in particular, if $n = 0$, then $E_\infty^{p,n-p} = 0$ for every $p \in \mathbb{Z}$.

One of the simplest ways we can get a spectral sequence if from a double complex. We will prove the following theorem in the next subsection (see subsection IV.4.2.4).

Theorem IV.4.1.7. *Let X be a double complex of objects of $\mathscr{A}$. Then we have two spectral sequences ${}^{I}E$ and ${}^{II}E$ starting at the 0th page, such that:*

(1) ${}^{I}E_0^{pq} = X^{p,q}$, ${}^{I}E_1^{pq} = \mathrm{H}^q(X^{p,\bullet}, d_{2,X}^{p,\bullet})$, ${}^{I}d_1^{pq}$ *is induced by* $(-1)^p d_{1,X}^{pq}$*;*

(2) ${}^{II}E_0^{pq} = X^{q,p}$, ${}^{II}E_1^{pq} = \mathrm{H}^q(X^{\bullet,p}, d_{1,X}^{\bullet,p})$, ${}^{II}d_1^{pq}$ *is induced by* $d_{2,X}^{pq}$.

These spectral sequences are functorial in X.

Moreover:

(i). if $X^{pq} = 0$ for all $p > 0$ or for all $q < 0$, then ${}^{I}E$ converges to $\mathrm{H}^*(\mathrm{Tot}(X))$;

(ii). if $X^{pq} = 0$ for all $p < 0$ or for all $q > 0$, then ${}^{II}E$ converges to $\mathrm{H}^*(\mathrm{Tot}(X))$;

(iii). if, for every $n \in \mathbb{Z}$, there are only finitely many $p \in \mathbb{Z}$ such that $X^{p,n-p} \neq 0$, then both spectral sequences converge to $\mathrm{H}^*(\mathrm{Tot}(X))$.

In particular, if X is a *first quadrant double complex* (i.e. if $X^{pq} = 0$ as soon as $p < 0$ or $q < 0$) or if X is a *third quadrant double complex* (i.e. if $X^{pq} = 0$ as soon as $p > 0$ or $q > 0$), then both spectral sequences converge to $\mathrm{H}^*(\mathrm{Tot}(X))$.

As a corollary, we get the Grothendieck spectral sequence.

Corollary IV.4.1.8. *Suppose that $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are left exact additive functors, that $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects, and that F takes all injective objects of $\mathscr{A}$ to G-acyclic objects of $\mathscr{B}$. Then, for every $A \in \mathrm{Ob}(\mathscr{A})$, we have a spectral sequence:*

$$E_2^{pq} = (R^p G) \circ (R^q F)(A) \Rightarrow R^{p+q}(G \circ F)(A),$$

and this spectral sequence is functorial in A.

Proof. Let A be an object of $\mathscr{A}$, and let $A \to J^\bullet$ be an injective resolution of A. We choose a Cartan-Eilenberg resolution $I^{\bullet,\bullet}$ of $F(J^\bullet)$, and we apply Lemma IV.4.1.10 to this resolution and to the left functor $G : \mathscr{B} \to \mathscr{C}$. By the hypothesis on F, we know that $F(J^n)$ is G-acyclic for every $n \geq 0$. So the second spectral sequence of the double complex $G(I^{\bullet,\bullet})$ converges to $\mathrm{H}^n(G(F(J^\bullet))) = R^n(G \circ F)(A)$, and we have

$${}^{II}E_2^{pq} = (R^p G)(\mathrm{H}^q(F(J^\bullet))) = (R^p G)(R^q F(A)).$$

□

Lemma IV.4.1.9. *[Cartan-Eilenberg resolution] Let $X^\bullet$ be a complex in $\mathcal{C}^+(\mathscr{A})$, and assume that $\mathscr{A}$ has enough injective objects. Then there exist a double complex $I^{\bullet,\bullet}$ and a morphism $X^\bullet \to I^{\bullet,0}$ such that: such that:*

(a) $I^{n,m} = 0$ *for* $m < 0$*, and, if* $N \in \mathbb{Z}$ *is such that* $X^n = 0$ *for every* $n < N$*, then* $I^{n,m} = 0$ *for* $n < N$*.*

(b) *For every* $n \in \mathbb{Z}$*, the complex* $(I^{n,\bullet}, d_{2,I}^{n,\bullet})$ *is an injective resolution of* X^n*.*

(c) *For all* $n \in \mathbb{Z}$*, the complex* $\mathrm{Ker}(d_{1,I}^{n,\bullet})$ *(with the differentials induced by the morphisms* $d_{2,I}^{n,m}$*) is an injective resolution of* $\mathrm{Ker}(d_X^n)$*.*

(d) *For all* $n \in \mathbb{Z}$*, the complex* $\mathrm{Im}(d_{1,I}^{n,\bullet})$ *(with the differentials induced by the morphisms* $d_{2,I}^{n,m}$*) is an injective resolution of* $\mathrm{Im}(d_X^n)$*.*

(e) *For all* $n \in \mathbb{Z}$*, the complex* $\mathrm{H}^n(I^{n,\bullet}, d_{1,I}^{n,\bullet}) = (\mathrm{Ker}(d_{1,I}^{n,\bullet})/\,\mathrm{Im}(d_{1,I}^{n-1,\bullet}))$ *(with the differentials induced by the morphisms* $d_{2,I}^{n,m}$*) is an injective resolution of* $\mathrm{H}^n(X)$*.*

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & I^{0,2} \arrow[u] \arrow[r, "d_{1,I}^{0,2}"] & I^{1,2} \arrow[u] \arrow[r, "d_{1,I}^{1,2}"] & I^{2,2} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & I^{0,1} \arrow[u, "d_{0,I}^{0,1}"] \arrow[r, "d_{1,I}^{0,1}"] & I^{1,1} \arrow[u] \arrow[r, "d_{1,I}^{1,1}"] & I^{2,1} \arrow[u, "d_{2,I}^{2,1}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & I^{0,0} \arrow[u, "d_{2,I}^{0,1}"] \arrow[r, "d_{1,I}^{0,0}"] & I^{1,0} \arrow[u] \arrow[r, "d_{1,I}^{1,0}"] & I^{2,0} \arrow[u, "d_{2,I}^{2,0}"'] & \cdots \\
\cdots \arrow[r] & X^0 \arrow[u] \arrow[r, "d_X^0"'] & X^1 \arrow[u] \arrow[r, "d_X^1"'] & X^2 \arrow[u] \arrow[r, "d_X^2"'] & \cdots
\end{tikzcd}
$$

Such a double complex is called a *Cartan-Eilenberg resolution* of $X^\bullet$.

Proof. As in the statement, fix $N \in \mathbb{Z}$ such that $X^n = 0$ for $n < N$. We set $I^{n,m} = 0$ for $n < N$ or $m < 0$. For every $n \in \mathbb{Z}$, we set $Z^n = \mathrm{Ker}(d_X^n)$, $B^n = \mathrm{Im}(d_X^{n-1})$ and $H^n = \mathrm{H}^n(X^\bullet) = Z^n/B^n$. We have short exact sequences:

$$0 \to Z^n \to X^n \to B^{n+1} \to 0$$

and

$$0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$$

for every $n \in \mathbb{Z}$. We define injective resolutions $Z^n \to I_Z^{n,\bullet}$, $B^n \to I_B^{n,\bullet}$, $H^n \to I_H^{n,\bullet}$ and $X^n \to I^{n,\bullet}$ and exact sequences $0 \to I_B^{n,\bullet} \to I_Z^{n,\bullet} \to I_H^{n,\bullet} \to 0$ and $0 \to I_Z^{n,\bullet} \to I^{n,\bullet} \to I_B^{n+1,\bullet} \to 0$ by induction on n in the following way:

(1) If $n < N$, take all these resolutions to be 0.

(2) Suppose that you have an injective resolution $Z^n \to I_Z^{n,\bullet}$ coming from the induction hypothesis. Choose an injective resolution $B^{n+1} \to I_B^{n+1,\bullet}$. By the horseshoe lemma (Proposition IV.3.1.5), we can find an injective resolution $X^n \to I^{n,\bullet}$ and an exact sequence $0 \to I_Z^{n,\bullet} \to I^{n,\bullet} \to I_B^{n+1,\bullet} \to 0$.

(3) Choose an injective resolution $H^{n+1} \to I_H^{n+1,\bullet}$. Using the horseshoe lemma again, we get a resolution $Z^{n+1} \to I_Z^{n+1,\bullet}$ and an exact sequence $0 \to I_B^{n+1,\bullet} \to I_Z^{n+1,\bullet} \to I_H^{n+1,\bullet} \to 0$.

(4) Replace n by $n+1$ and go to step (2).

We now get the double complex $I^{\bullet,\bullet}$ by taking $d_{2,I}^{n,m}$ equal to $d_I^{n,m} : I^{n,m} \to I^{n,m+1}$ and $d_{1,I}^{n,m} : I^{n,m} \to I^{n+1,m}$ equal to the composition $I^{n,m} \to B^{n+1,m} \to Z^{n+1,m} \to I^{n+1,m}$. All the required properties are clear.

□

Lemma IV.4.1.10. *Suppose that $\mathscr{A}$ has enough injective objects, and let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Let $X^\bullet$ be a complex of objects of $\mathscr{A}$ such that $X^n = 0$ for $n < 0$, and let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $X^\bullet$. We consider the two spectral sequences ${}^IE$ and ${}^{II}E$ associated to the double complex $F(I^{\bullet,\bullet})$. Then:*

(i). Both spectral sequences converge.

(ii). We have ${}^IE_1^{p,q} = R^qF(X^p)$.

(iii). We have ${}^{II}E_1^{p,q} = F(\mathrm{H}^q(I^{\bullet,p}, d_{1,I}^{\bullet,p}))$ and ${}^{II}E_2^{p,q} = R^pF(\mathrm{H}^q(X^\bullet))$.

In particular, if each X^n is F-acyclic, then ${}^IE_2^{p,q} = \begin{cases} \mathrm{H}^p(F(X^\bullet)) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$, so the spectral sequence ${}^IE$ degenerates at E_2, and $\mathrm{H}^n(\mathrm{Tot}(F(I^{\bullet,\bullet}))) \simeq \mathrm{H}^n(F(X^\bullet))$ for every $n \in \mathbb{Z}$.

Proof. The spectral sequences converge because $F(I^{\bullet,\bullet})$ is a first quadrant complex. We calculate their first pages.

We have ${}^IE_0^{pq} = F(I^{p,q})$ and ${}^IE_1^{pq} = \mathrm{H}^q(F(I^{p,\bullet}), F(d_{2,I}^{p,\bullet}))$. As $I^{p,\bullet}$ is an injective resolution of X^p, we have ${}^IE_1^{pq} = R^qF(X^p)$ by the definition of the derived functor.

On the other hand, we have ${}^{II}E_0^{pq} = F(I^{q,p})$ and ${}^{II}E_1^{pq} = \mathrm{H}^q(F(I^{\bullet,p}), F(d_{1,I}^{\bullet,p}))$. Fix $p \in \mathbb{Z}$. The short exact sequences

$$0 \to \mathrm{Ker}(d_{1,I}^{q,p}) \to I^{q,p} \to \mathrm{Im}(d_{1,I}^{q,p}) \to 0$$

and

$$0 \to \mathrm{Im}(d_{1,I}^{q-1,p}) \to \mathrm{Ker}(d_{1,I}^{q,p}) \to \mathrm{H}^q(I^{\bullet,p}) \to 0$$

are exact sequences of injective objects (by definition of a Cartan-Eilenberg resolution), so they are split, so they remain exact after we apply the functor F. This shows that ${}^{II}E_1^{pq} = F(\mathrm{H}^q(I^{\bullet,p}, d_{1,I}^{\bullet,p}))$. Also, the differential $d_1^{pq} : {}^{II}E_1^{pq} \to {}^{II}E_1^{p+1,q}$ is the image by F of the morphism $\mathrm{H}^q(I^{\bullet,p}, d_{1,I}^{\bullet,p}) \to \mathrm{H}^q(I^{\bullet,p+1}, d_{1,I}^{\bullet,p+1})$ induced by $d_{2,I}^{p,q}$. By the definition of a Cartan-Eilenberg resolution again, the complex $\mathrm{H}^q(I^{\bullet,p}, d_{1,I}^{\bullet,p})$ is an injective resolution of $\mathrm{H}^q(X^\bullet)$, so ${}^{II}E_2^{pq} = R^pF(\mathrm{H}^q(X^\bullet))$.

Suppose that X^n is F-acyclic for every $n \in \mathbb{Z}$. By the calculation of ${}^IE_1^{pq}$ above, we immediately get that ${}^IE^{pq} = 0$ for every $q \geq 1$ and that ${}^IE_1^{p,0} = F(X^p)$. The formula for ${}^IE_2^{pq}$ follows. We show by induction on $r \geq 2$ that ${}^IE_r^{pq} = 0$ for $q \neq 0$ and that all the differentials of ${}^IE_r$ are 0. For $r = 2$, we already know the first statement, and the second follows because d_2^{pq} goes from ${}^IE_2^{pq} \to {}^IE_2^{p+2,q-1}$, so either its source its target is 0 for all $p, q \in \mathbb{Z}$. Suppose that we know the result for some $r \geq 2$. Then ${}^IE_{r+1}^{pq} = {}^IE_r^{pq}$, so ${}^IE_{r+1}^{pq} = 0$ for $q \neq 0$. Again, the second part follows immediately from this, because d_{r+1}^{pq} goes from E_{r+1}^{pq} to $E_{r+1}^{p+r+1,q-r}$, and $q \neq q-r$, so at least one of the source or target of d_{r+1}^{pq} has to be 0.

As the spectral sequence ${}^IE$ degenerates at the second page, we have ${}^IE_\infty^{pq} = {}^IE_2^{pq}$. In particular, ${}^IE_\infty^{pq} = 0$ for $q \neq 0$. So, for every $n \geq 0$, the filtration on $\mathrm{H}^n(\mathrm{Tot}(F(I^{\bullet,\bullet})))$ corresponding to ${}^IE$ has only one nonzero quotient, which is ${}^IE_\infty^{n,0}$; in other words, we get an isomorphism

$$\mathrm{H}^n(\mathrm{Tot}(F(I^{\bullet,\bullet}))) \simeq {}^IE_\infty^{n,0} = {}^IE_2^{n,0} = \mathrm{H}^n(F(X^\bullet)).$$

□

Corollary IV.4.1.11. *Suppose that $\mathscr{A}$ has enough injective objects, and let $X^\bullet \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{A}))$. Then there exists an object $J^\bullet$ of $\mathcal{C}^+(\mathscr{A})$ such that J^n is injective for every $n \in \mathbb{Z}$ and a quasi-isomorphism $X^\bullet \to J^\bullet$.*

Proof. After shifting $X^\bullet$, we may assume that $X^n = 0$ for $n < 0$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $X^\bullet$, let $J^\bullet = \mathrm{Tot}(I^{\bullet,\bullet})$. We have an obvious morphism $X^\bullet \to J^\bullet$ (induced by the morphism $X^\bullet \to I^{\bullet,0}$. By Lemma IV.4.1.10 for the functor $F = \mathrm{id}_{\mathscr{A}}$, this morphism is a quasi-isomorphism.

□

In fact, we can construct resolutions of objects of $\mathcal{C}^+(\mathscr{A})$ by more general objects.

Corollary IV.4.1.12. *Let $\mathscr{C}$ be a full additive subcategory of $\mathscr{A}$, and suppose that, for every $X \in \mathrm{Ob}(\mathscr{A})$, there exists a monomorphism $X \to Y$ with $Y \in \mathrm{Ob}(\mathscr{C})$. Then, for every $X^\bullet \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to Y^\bullet$ with $Y^\bullet \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{C}))$.*

Proof. Fix $N \in \mathbb{Z}$ such that $X^i = 0$ for $i < N$. We first find a double complex $Z^{\bullet,\bullet}$ of objects of $\mathscr{C}$ such that $Z^{n,m} = 0$ if $n < N$ or $m < 0$ and a morphism of complexes $d^{-1} : X^\bullet \to (Z^{\bullet,0}, d_{1,X}^{\bullet,0})$ such that, for every $n \in \mathbb{Z}$, the complex $(Z^{n,\bullet}, d_{2,Z}^{n,\bullet})$ is a resolution of X^n. We construct $(Z^{n,\bullet}, d_{2,Z}^{n,\bullet})$ and the morphism of complexes $d_{1,Z}^{n-1,\bullet} : (Z^{n-1,\bullet}, d_{2,Z}^{n-1,\bullet}) \to (Z^{n,\bullet}, d_{2,Z}^{n,\bullet})$ by induction on $n \in \mathbb{Z}$. If $n < N$, we take $(Z^{n,\bullet}, d_{2,Z}^{n,\bullet}) = 0$. Suppose that we have constructed $(Z^{n,\bullet}, d_{2,Z}^{n,\bullet})$; then we get a morphism of complexes $d_{1,Z}^{n-1,\bullet} : (Z^{n-1,\bullet}, d_{2,Z}^{n-1,\bullet}) \to (Z^{n,\bullet}, d_{2,Z}^{n,\bullet})$

extending $d_X^n : X^n \to X^{n+1}$ by Lemma IV.4.1.13. Now we take $Y^\bullet = \mathrm{Tot}(Z^{\bullet,\bullet})$, with the morphisms of complexes $X^\bullet \to Y^\bullet$ induced by d^{-1}. We have $Y^i = 0$ if $i < N$, so $Y^\bullet \in \mathcal{C}^+(\mathscr{C})$. We use the first spectral sequence ${}^I E$ of the double complex $Z^{\bullet,\bullet}$ to calculate the cohomology of $Y^\bullet$; this spectral sequence converges because $Z^{n,m} = 0$ for $m < 0$. (See Theorem IV.4.1.7.) As the columns of $Y^{\bullet,\bullet}$ are resolutions of the X^n, we have

$$ {}^I E_1^{pq} = \begin{cases} X^p & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} $$

So

$$ {}^I E_2^{pq} = \begin{cases} \mathrm{H}^p(X^\bullet) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} $$

This implies that the spectral sequence ${}^I E$ degenerates at the second page and that ${}^I E_\infty^{pq} = {}^I E_2^{pq}$. So we get $\mathrm{H}^n(Y^\bullet) \simeq {}^I E_\infty^{n,0} = \mathrm{H}^n(X^\bullet)$.

□

Lemma IV.4.1.13. *In the situation of Corollary IV.4.1.12, if $f : X \to X'$ is a morphism of $\mathscr{A}$ and $X \overset{d_Y^{-1}}{\to} Y^0 \overset{d_Y^0}{\to} Y^1 \overset{d_Y^1}{\to} \dots$ is a resolution of X by objects of $\mathscr{C}$, then there exists a resolution $X' \overset{d_{Y'}^{-1}}{\to} Y'^0 \overset{d_{Y'}^0}{\to} Y'^1 \overset{d_{Y'}^1}{\to} \dots$ of X' by objects of $\mathscr{C}$ and a commutative diagram*

$$\begin{tikzcd} Y^\bullet \arrow[r, "f^\bullet"] & Y'^\bullet \\ X \arrow[u] \arrow[r, "f"'] & X' \arrow[u] \end{tikzcd}$$

Proof. We write $Y^{-1} = X$, $Y'^{-1} = X'$ and $f^{-1} = f$. We construct $d_{Y'}^{n-1} : Y'^{n-1} \to Y'^n$ and $f^n : Y^n \to Y'^n$ by induction on $n \geq 0$:

- Suppose that $n = 0$. Choose a monomorphism $u : Y^0 \oplus_X X' \to Y'^0$ with $Y'^0 \in \mathrm{Ob}(\mathscr{C})$, and let $f'^0 : Y^0 \to Y'^0$ and $d_{Y'}^{-1} : X' \to Y'^0$ be the composition of u and of the two canonical morphisms from Y^0 and X' to $Y^0 \oplus_X X'$.

$$\begin{tikzcd} Y^0 \arrow[r] \arrow[rr, bend left, "f^0"] & Y^0 \oplus_X X' \arrow[r, hook] & Y'^0 \\ X \arrow[u] \arrow[r] & X' \arrow[u] \arrow[ur, "d_{Y'}^{-1}"'] & \end{tikzcd}$$

  By Corollary II.2.1.16, the morphism $X' \to Y^0 \oplus_X X'$ is injective, so $d_{Y'}^{-1}$ is injective.

- Suppose that $n \geq 0$ andthat we have constructed $d_{Y'}^{i-1}$, Y'^i and f^i for $i \leq n$. We have $\mathrm{Ker}(d_Y^n) = \mathrm{Im}(d_Y^{n-1})$ (because $\mathrm{H}^n(Y^\bullet) = 0$ if $n \geq 1$, and because $\mathrm{Ker}(d_Y^0) = X$ if $n = 0$), so the canonical morphism $\operatorname{Coker} d_Y^{n-1} \to Y^{n+1}$ is injective. We choose a monomorphism $Y^{n+1} \oplus_{\operatorname{Coker} d_Y^{n-1}} \operatorname{Coker} d_{Y'}^{n-1} \to Y'^{n+1}$ with $Y'^{n+1} \in \operatorname{Ob}(\mathscr{C})$; we have obvious morphisms $f^{n+1} : Y^{n+1} \to Y'^{n+1}$ and $\operatorname{Coker} d_{Y'}^{n-1} \to Y'^{n+1}$, and composing the second one with the canonical surjection $Y'^n \to \operatorname{Coker} d_{Y'}^{n-1}$ gives a morphism $d_{Y'}^n : Y'^n \to Y'^{n+1}$.

$$\begin{tikzcd}
Y^{n+1} \arrow[r] \arrow[rr, bend left, "f^{n+1}"] & Y^{n+1} \oplus_{\operatorname{Coker} d_Y^{n-1}} \operatorname{Coker} d_{Y'}^{n-1} \arrow[r, hook] & Y'^{n+1} \\
\operatorname{Coker} d_Y^{n-1} \arrow[u, hook] \arrow[r] & \operatorname{Coker} d_{Y'}^{n-1} \arrow[u] & \\
Y^n \arrow[u, two heads] \arrow[r, "f^n"'] & Y'^n \arrow[u, two heads] \arrow[uur, "d_{Y'}^n"'] &
\end{tikzcd}$$

By Corollary II.2.1.16 again, the morphism $\operatorname{Coker} d_{Y'}^{n-1} \to Y^{n+1} \oplus_{\operatorname{Coker} d_Y^{n-1}} \operatorname{Coker} d_{Y'}^{n-1}$ is injective, so $\operatorname{Ker}(d_{Y'}^n) = \operatorname{Im}(d_{Y'}^{n-1})$.

$\square$

Example IV.4.1.14. (1) **The Hochschild-Serre spectral sequence:** Let G be a group and K be a normal subgroup of G. We write $\mathscr{A} = {}_{\mathbb{Z}[G]}\mathbf{Mod}$ and $\mathscr{B} = {}_{\mathbb{Z}[G/K]}\mathbf{Mod}$. We want to apply the Grothendieck spectral sequence to the functors $\mathrm{H}^0(K, \cdot) : \mathscr{A} \to \mathscr{B}$ and $\mathrm{H}^0(G/K, \cdot) : \mathscr{B} \to \mathbf{Ab}$. We need to check that $\mathrm{H}^0(K, \cdot)$ sends injective objects to $\mathrm{H}^0(G/K, \cdot)$-acyclic objects. In fact, the functor $\mathrm{H}^0(K, \cdot)$ even sends injective objects to injective objects, because it is right adjoint to the forgetful functor $\mathscr{B} \to \mathscr{A}$, which is exact. (Use Lemma II.2.4.4.)

(2) **The Čech cohomology to cohomology spectral sequence:** Let $\mathscr{C}$ be a category with fiber products, let $\mathscr{T}$ be a Grothendieck pretopology on $\mathscr{C}$, and let $\mathscr{X} = (X_i \to X)_{i \in I}$ be a covering family for this pretopology. We consider the inclusion functor $\Phi : \operatorname{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \to \operatorname{PSh}(\mathscr{C}, \mathbb{Z})$ and the functor $\check{\mathrm{H}}^0(\mathscr{X}, \cdot) : \operatorname{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$. They are both left exact, and their composition is the functor $\mathrm{H}^0(X, \cdot) : \operatorname{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \to \mathbf{Ab}$. Moreover, the functor Φ sends injective objects to injective objects, because it has an exact left adjoint (sheafification). So we get a Grothendieck spectral sequence:

$$E_2^{pq} = \check{\mathrm{H}}^p(\mathscr{X}, R^q\Phi(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F}),$$

for every abelian sheaf $\mathscr{F}$ on $\mathscr{C}_{\mathscr{T}}$. Remember from Subsection IV.3.5 that $R^q\Phi(\mathscr{F})$ is the presheaf $Y \longmapsto \mathrm{H}^q(Y, \mathscr{F})$.

In particular, we get canonical morphisms $\check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^n(X, \mathscr{F})$: As we have a first quadrant spectral sequence, $E_\infty^{n,0}$ is isomorphic to a subobject of the limit $\mathrm{H}^n(X, \mathscr{F})$, and is a quotient of $E_2^{n,0} = \check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F})$.

IV.4.2 Construction

IV.4.2.1 Exact couples

Exact couples are the most general method for construction spectral sequences. We will not use their full power, but we will start with them as a way to make the notation less complicated.

Notation IV.4.2.1. If A is an object of $\mathscr{A}$ and $d \in \operatorname{End}_{\mathscr{A}}(A)$ is such that $d \circ d = 0$, we write $\mathrm{H}(A, d) = \operatorname{Ker} d / \operatorname{Im}(d)$; if d is clear from the context, we write $\mathrm{H}(A)$ instead of $\mathrm{H}(A, d)$.

Definition IV.4.2.2. (i). A *graded object* of $\mathscr{A}$ is an object A of $\mathscr{A}$ of the form $A = \bigoplus_{n\in\mathbb{Z}} A^n$; the subobject A^n is called the homogeneous part of degree n in A. If $f : A \to B$ is a morphism between graded objects, we say that f is *of degree* d if $f(A^n) \subset B^{n+d}$ for every $n \in \mathbb{Z}$.

(ii). A *bigraded object* of $\mathscr{A}$ is an object A of $\mathscr{A}$ of the form $A = \bigoplus_{n,m\in\mathbb{Z}} A^{n,m}$; the subobject $A^{n,m}$ is called the homogeneous part of bidegree (n, m) in A. If $f : A \to B$ is a morphism between bigraded objects, we say that f is *of bidegree* (d, e) if $f(A^{n,m}) \subset B^{n+d,m+e}$ for all $n, m \in \mathbb{Z}$.

Example IV.4.2.3. We can see a complex $A^\bullet$ of objects of $\mathscr{A}$ as the graded object $A = \bigoplus_{n\in\mathbb{Z}} A^n$ with an endomrophism $d = \sum_{n\in\mathbb{Z}} d_A^n$ of degree 1 such that $d \circ d = 0$. Then $H(A, d) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(A^\bullet)$.

We can also see a double complex $A^{\bullet,\bullet}$ as the bigraded object $A = \bigoplus_{n,m\in\mathbb{Z}} A^{n,m}$, together with two endomorphisms $d_1 = \sum_{n,m} d_{1,A}^{n,m}$ and $d_2 = \sum_{n,m} d_{2,A}^{n,m}$ of bidegrees $(1, 0)$ and $(0, 1)$ respectively, and such that $d_1 \circ d_2 = d_2 \circ d_1$, $d_1 \circ d_1 = 0$ and $d_2 \circ d_2 = 0$.

We can generalize slightly the definition of a spectral sequence.

Definition IV.4.2.4. A *spectral sequence* starting at the page $r_0 \in \mathbb{N}$ is the following data:

(a) for every integer $r \geq r_0$, an object E_r of $\mathscr{A}$ and an endomorphism d_r of E_r such that $d_r \circ d_r = 0$;

(b) for every $r \geq r_0$, an isomorphism

$$E_{r+1} \simeq \mathrm{H}(E_r, d_r)$$

(which we will often write as an equality).

Definition IV.4.1.1 corresponds to the particular case where each E_r is bigraded, the endomorphism d_r is of bidegree $(r, -r + 1)$, and the isomorphism $E_{r+1} \simeq \mathrm{H}(E_r, d_r)$ is of bidegree $(0, 0)$.

Definition IV.4.2.5. An *exact couple* is a triangle of morphisms of $\mathscr{A}$:

(*)
$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & A \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

such that $\operatorname{Ker}\alpha = \operatorname{Im}\gamma$, $\operatorname{Ker}\gamma = \operatorname{Im}\beta$ and $\operatorname{Ker}\beta = \operatorname{Im}\alpha$.

Another way to think of the exact couple (*) is as the periodic acyclic complex:

$$\dots \to A \overset{\alpha}{\to} A \overset{\beta}{\to} E \overset{\gamma}{\to} A \overset{\alpha}{\to} A \overset{\beta}{\to} E \overset{\gamma}{\to} A \overset{\alpha}{\to} A \to \dots$$

Proposition IV.4.2.6. *We consider an exact couple (*). Let $d = \beta \circ \gamma \in \operatorname{End}_{\mathscr{A}}(E)$, and let $H(E) = H(E,d)$. Then the following triangle is an exact couple:*

(**)
$$\begin{tikzcd} \alpha(A) \arrow[rr, "\alpha'"] & & \alpha(A) \arrow[dl, "\beta'"] \\ & H(E) \arrow[ul, "\gamma'"] & \end{tikzcd}$$

where the morphism α' is the restriction of α, the morphism β' is the unique morphism such that $\beta' \circ \alpha : A \to H(E)$ is the morphism induced by β, and the morphism γ' is induced by $\gamma_{|\operatorname{Ker} d}$.

This is called the derived exact couple *of (*).*

Proof. As $\gamma \circ \beta = 0$, we have $d \circ \beta = 0$ and $\gamma \circ d = 0$. In particular, $\operatorname{Im}\beta \subset \operatorname{Ker} d$, so β does induce a morphism $A \to H(E)$. Also, the restriction of this morphism to $\operatorname{Im}\gamma$ is 0, because $\beta(\operatorname{Im}\gamma) = \operatorname{Im}(\beta \circ \gamma) = \operatorname{Im} d$; as $\operatorname{Im}\gamma = \operatorname{Ker}\alpha$, there is unique morphism $\beta' : \alpha(A) \to \mathrm{H}(E)$ such that $\beta' \circ \alpha$ is the morphism $A \to \mathrm{H}(E)$ induced by β. On the other hand, the morphism γ sends $\operatorname{Ker} d$ to $\operatorname{Ker}\beta = \operatorname{Im}\alpha$ (because $\beta \circ \gamma = d$ is 0 on $\operatorname{Ker} d$); as $\gamma \circ d = 0$, we have $\gamma(\operatorname{Im} d) = 0$, so $\gamma_{|\operatorname{Ker} d}$ does induce a morphism $\gamma' : \mathrm{H}(E) \to \alpha(A)$.

Consider the commutative diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \alpha(A) \arrow[r, "\alpha'"] \arrow[d] & \operatorname{Ker}\beta' \arrow[d] & \\ 0 \arrow[r] & \operatorname{Im}\gamma \arrow[r] \arrow[d, "\beta"'] & A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \alpha(A) \arrow[r] \arrow[d, "\beta'"] & 0 \\ 0 \arrow[r] & \operatorname{Im}(d) \arrow[r] & \operatorname{Ker}(d) \arrow[r] & \mathrm{H}(E) \arrow[r] & 0 \end{tikzcd}$$

As $d = \beta \circ \gamma$, the morphism $\operatorname{Im}\gamma \to \operatorname{Im} d$ induced by β is surjective, so the snake lemma implies that the morphism $\alpha(A) \to \operatorname{Ker}\beta'$ induced by α' is surjective, that is, that $\operatorname{Im}\alpha' = \operatorname{Ker}\beta'$. Another application of the snake lemma shows that $\operatorname{Ker}\alpha' = \operatorname{Ker}(\beta : \operatorname{Im}\gamma \to \operatorname{Im} d) = \operatorname{Ker}\beta \cap \operatorname{Im}\gamma$; on the other hand, $\operatorname{Im}(\gamma') = \gamma(\operatorname{Ker} d) = \gamma(\operatorname{Ker}(\beta \circ \gamma)) = \operatorname{Ker}\beta \cap \operatorname{Im}(\gamma)$, so $\operatorname{Ker}\alpha' = \operatorname{Im}\gamma'$. Finally, we have:

$$\operatorname{Ker}\gamma' = (\operatorname{Ker}\gamma \cap \operatorname{Ker} d)/\operatorname{Im} d = \operatorname{Ker}\gamma/\operatorname{Im} d = \operatorname{Im}\beta/\operatorname{Im} d = \operatorname{Im}\beta'.$$

□

Definition IV.4.2.7. We consider an exact couple (*) as in Definition IV.4.2.5. The spectral sequence of this exact couple is the spectral sequence starting at the first page defined inductively as follows:

(a) $E_1 = E$, $d_1 = \beta \circ \gamma$;

(b) $E_2 = \mathrm{H}(E)$, $d_2 = \beta' \circ \gamma'$;

(c) $E_3 = \mathrm{H}(\mathrm{H}(E))$, $d_3 = \beta'' \circ \gamma''$;

(d) etc.

Definition IV.4.2.8. Suppose that $(E_r, d_r)_{r\geq 1}$ is a spectral sequence starting at the first page. We define subobjects $Z_r \supset B_r$ of E_1 such that $Z_r/B_r \simeq E_r$ inductively on r llows:

(a) $Z_1 = E_1$, $B_1 = 0$;

(b) for every $r \geq 1$,

$$Z_{r+1} = \operatorname{Ker}(Z_r \to Z_r/B_r \simeq E_r \stackrel{d_r}{\to} E_r \simeq Z_r/B_r);$$

$$B_{r+1}/B_r = \operatorname{Im}(Z_r \to Z_r/B_r \simeq E_r \stackrel{d_r}{\to} E_r \simeq Z_r/B_r).$$

Note that $Z_{r+1}/Z_r \simeq \operatorname{Ker}(d_r)$ and $B_{r+1}/B_r \simeq \operatorname{Im}(d_r)$, so we do get an isomorphism $Z_{r+1}/B_{r+1} \simeq H(E_r, d_r) \simeq E_{r+1}$.

We also set $Z_\infty = \bigcap_{r\geq 1} Z_r$ (the product of all the Z_r over E_1), $B_\infty = \bigcup_{r\geq 1} B_r$ and $E_\infty = Z_\infty/B_\infty$.

Proposition IV.4.2.9. *Suppose that we are in the situation of Definition IV.4.2.7. Then, for every $r \geq 0$, and we have:*

$$Z_{r+1} = \gamma^{-1}(\operatorname{Im}(\alpha^r))$$

and

$$B_{r+1} = \beta(\operatorname{Ker}(\alpha^r)),$$

and the $(r+1)$st derived couple of (*) is

(*)
$$\begin{tikzcd} \alpha^r(A) \arrow[rr, "\alpha_r"] & & \alpha^r(A) \arrow[dl, "\beta_r"] \\ & E_{r+1} \arrow[ul, "\gamma_r"] & \end{tikzcd}$$

where:

- α_r *is the restriction of* α*;*
- γ_r *is induced by* $\gamma : Z_r = \gamma^{-1}(\alpha^r(A)) \to \alpha^r(A)$*;*
- β_r *is the unique morphism such that* $\beta_r \circ \alpha^r$ *is the morphism* $A \to E_{r+1}$ *induced by* $\beta : A \to Z_{r+1}$ *(we have* $\operatorname{Im}\beta \subset Z_{r+1} = \gamma^{-1}(\operatorname{Im}(\alpha^r))$ *because* $\gamma \circ \beta = 0$*).*

Proof. We prove the statement by induction on r. If $r = 0$, then $\alpha^0 = \mathrm{id}_A$, so $\gamma^{-1}(\operatorname{Im}(\alpha^0)) = \gamma^{-1}(A) = E = Z_1$, and $\beta(\ker(\alpha^0)) = \beta(0) = 0 = B_1$. The description of E_1 is the definition of the exact couple (*).

Suppose that we have proved the statement for some $r \geq 0$. Then we have

$$Z_{r+2} = \operatorname{Ker}(Z_{r+1} \to E_{r+1} \stackrel{\beta_r \circ \gamma_r}{\to} E_{r+1}).$$

Unpacking the definitions (and using the Freyd-Mitchell embedding theorem, i.e. Theorem III.3.1), to pretend that our objects have elements, we get

$$\begin{aligned} Z_{r+2} &= \{e \in E \mid \exists a \in A \text{ and } b \in \operatorname{Ker}(\alpha^r),\ \gamma(e) = \alpha^r(a) \text{ and } \beta(a) = \beta(b)\} \\ &= \{e \in E \mid \exists a \in A,\ \gamma(e) = \alpha^r(a) \text{ and } \beta(a) = 0\} \quad (\text{replace } a \text{ by } a - b) \\ &= \{e \in E \mid \exists a \in \operatorname{Im}(\alpha),\ \gamma(e) = \alpha^r(a)\} \quad (\operatorname{Ker}\beta = \operatorname{Im}\alpha) \\ &= \gamma^{-1}(\operatorname{Im}(\alpha^{r+1})). \end{aligned}$$

Also, still pretending that our objects have elements, we see that $e \in E$ is in B_{r+2} if and only if there exist $f \in E$, $a \in A$ and $b \in \operatorname{Ker}(\alpha^r)$ such that $\gamma(f) = \alpha^r(a)$ and $\beta(a) = e + \beta(b)$; replacing a by $a - b$, we may assume that $b = 0$; also, the condition that there exist $f \in E$ such that $\alpha^r(a) = \gamma(f)$ means that $\alpha^r(a) \in \operatorname{Im}(\gamma) = \operatorname{Ker}(\alpha)$. So

$$\begin{aligned} B_{r+2} &= \{e \in E \mid \exists a \in A,\ \alpha^r(a) \in \operatorname{Ker}\alpha \text{ and } \beta(a) = e\} \\ &= \beta(\operatorname{Ker}(\alpha^{r+1})). \end{aligned}$$

The formulas for the $(r + 2)$th derived exact couple follow immediately from the inductino hypothesis and the definition of a derived exact couple.

□

IV.4.2.2 Construction of exact couples

Definition IV.4.2.10. Let F be an object of $\mathscr{A}$ and $d, \alpha \in \operatorname{End}_{\mathscr{A}}(F)$ such that:

(a) $d \circ d = 0$;

(b) α is injective;

(c) $d \circ \alpha = \alpha \circ d$.

We write $\overline{F} = F/\alpha(F)$; by condition (c), the morphism d induces $\overline{d} : \overline{F} \to \overline{F}$ such that $\overline{d} \circ \overline{d} = 0$, and α induces an endomorphism of $H(F, d)$, still denoted by α. We write $\mathrm{H}(F) = \mathrm{H}(F, d)$ and $\mathrm{H}(\overline{F}) = \mathrm{H}(\overline{F}, \overline{d})$. Then we have an exact sequence of complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "d"'] & F \arrow[r] \arrow[d, "d"'] & \overline{F} \arrow[r] \arrow[d, "\overline{d}"] & 0 \\
0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "d"'] & F \arrow[r] \arrow[d, "d"'] & \overline{F} \arrow[r] \arrow[d, "\overline{d}"] & 0 \\
0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "d"'] & F \arrow[r] \arrow[d, "d"'] & \overline{F} \arrow[r] \arrow[d, "\overline{d}"] & 0 \\
0 \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "d"'] & F \arrow[r] \arrow[d, "d"'] & \overline{F} \arrow[r] \arrow[d, "\overline{d}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and the corresponding long exact sequence of cohomology is 3-periodic, hence is an exact couple:

$$\begin{tikzcd}
\mathrm{H}(F) \arrow[rr, "\alpha"] & & \mathrm{H}(F) \arrow[dl, "\beta"] \\
 & \mathrm{H}(\overline{F}) \arrow[ul, "\gamma"] &
\end{tikzcd}$$

The corresponding spectral sequence is called the spectral sequence of α on (F, d). We can even start it on the 0th page: we have $E_0 = \overline{F}$, $E_1 = \mathrm{H}(\overline{F})$ etc.

We now see a way to construct a triple (F, d, α) as above. We consider an object G of $\mathscr{A}$ with a decreasing filtration $\mathrm{Fil}^\bullet G$; this means that we have subobjects $\mathrm{Fil}^p G$, for $p \in \mathbb{Z}$, such that $\mathrm{Fil}^{p+1} G \subset \mathrm{Fil}^p G$. [5] Let $d \in \mathrm{End}_{\mathscr{A}}(G)$ be such that $d \circ d = 0$ and $d(\mathrm{Fil}^p G) \subset \mathrm{Fil}^p G$ for every $p \in \mathbb{Z}$. We take $F = \bigoplus_{p\in\mathbb{Z}} \mathrm{Fil}^p G$, with the endomorphism d equal to $d : \mathrm{Fil}^p G \to \mathrm{Fil}^p G$ on each component. Let $\alpha : F \to F$ be the sum of the inclusions $\mathrm{Fil}^p G \subset \mathrm{Fil}^{p-1} G$. Then $\overline{F} = F/\alpha(F) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Gr}^p G$, where $\mathrm{Gr}^p G = \mathrm{Fil}^p G/\mathrm{Fil}^{p-1} G$.

The triple (F, d, α) clearly satisfies the conditions of Definition IV.4.2.10. The spectral sequence of α on F is also called the spectral sequence of $(G, \mathrm{Fil}^\bullet G, d)$.

[5] Note that we are using superscripts for decreasing filtrations and subscripts for increasing filtration. This is a somewhat standard convention.

Proposition IV.4.2.11. *For every $r \geq 0$ and every $p \in \mathbb{Z}$, let*

$$Z_r^p = \frac{(\mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G)) + \mathrm{Fil}^{p+1} G}{\mathrm{Fil}^{p+1} G} \subset \mathrm{Gr}^p G \subset \overline{F}$$

and

$$B_r^p = \frac{(\mathrm{Fil}^p G \cap d(\mathrm{Fil}^{p-r+1} G)) + \mathrm{Fil}^{p+1} G}{\mathrm{Fil}^{p+1} G} \subset \mathrm{Gr}^p G \subset \overline{F}.$$

We write $E_r^p = Z_r^p / B_r^p$, and we denote by d_r^p the morphism $E_r^p \to E_r^{p+r}$ induced by $d : \mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G) \to \mathrm{Fil}^{p+r} G \cap d^{-1}(\mathrm{Fil}^{p+2r} G)$.

Then the rth page of the spectral sequence of $(G, \mathrm{Fil}^\bullet G, d)$ is isomorphic to $(\bigoplus_{p \in \mathbb{Z}} E_r^p, \sum_{p \in \mathbb{Z}} d_r^p)$; if $r \geq 1$, Z_r is isomorphic to the image of $\bigoplus_{p \in \mathbb{Z}} Z_r^p$ in $\mathrm{H}(\overline{F})$ and B_r is isomorphic to the image of $\bigoplus_{p \in \mathbb{Z}} B_r^p$ in $\mathrm{H}(\overline{F})$.

Proof. For $r = 0$, the statement is just the definition of $\overline{F}$ and of $d_0 = d$.

Let $r \geq 1$. We denote by Z'_r and B'_r the inverse images of $Z_r, B_r \subset E_1 = H(\overline{F})$ in $E_0 = \overline{F}$. We use the descriptions of Z_r, B_r and d_r from Proposition IV.4.2.9, and we use the Freyd-Mitchell embedding theorem to pretend that we are in a category of modules. By the definition of the long exact sequence of cohomology (see the proof of Corollary IV.2.2.6), the morphism $\gamma : H(\overline{F}) \to H(F)$ is given by the following procedure: take $z \in H(\overline{F})$, lift it to $z' \in \overline{F}$, lift z' to $z'' \in F$, and take the image of $d(z'') \in \operatorname{Ker} d$ in $H(F)$. So, by the formula of Proposition IV.4.2.9, the subobject Z'_r of $\overline{F}$ is the set of $z' \in \overline{F}$ that have a lift $z \in F$ such that $d(z) \in \alpha^{r-1}(F) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Fil}^{p+r} G$; this gives the identity $Z'_r = \bigoplus_{p \in \mathbb{Z}} Z_r^p$. As for B_r, the formula of Proposition IV.4.2.9 says that an element z of $H(\overline{F})$ is in B_r if and only if it the image by $\beta : \mathrm{H}(F) \to \mathrm{H}(\overline{F})$ of an element of $z' \in \mathrm{H}(F)$ such that $\alpha^{r-1}(z') = 0$ in $\mathrm{H}(F)$; this means that there exists $z'' \in F$ (lifting z' and hence also z) such that $d(z'') = 0$ and $\alpha^{r-1}(z'') \in d(F)$. If we write $z = \sum_{p \in \mathbb{Z}} z^p$ where $z^p \in \mathrm{H}(\mathrm{Gr}^p G)$ for every $p \in \mathbb{Z}$ and $z'' = \sum_{p \in \mathbb{Z}} z''^p$ with $z'' \in \mathrm{Fil}^p G$ for every $p \in \mathbb{Z}$, then z''^p lifts z^p and the condition on z'' is that $\alpha^{r-1} z''^p$, which is z''^p seen as an element of $\mathrm{Fil}^{p-r+1} G$, is in $d(\mathrm{Fil}^{p-r+1} G)$. So we get $B'_r = \bigoplus_{p \in \mathbb{Z}} B_r^p$. The last statement follows immediately from the formula for d_r in Proposition IV.4.2.9 and from the description of $\gamma : \mathrm{H}(\overline{F}) \to \mathrm{H}(F)$ given above.

□

Corollary IV.4.2.12. *For every $p \in \mathbb{Z}$, let $M^p = \bigcap_{r \geq 0} ((\mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G)) + \mathrm{Fil}^{p+1} G) \subset \mathrm{Fil}^p G$ and $N^p = \bigcup_{r \geq 0} ((\mathrm{Fil}^p G \cap d(\mathrm{Fil}^{p-r+1} G)) + \mathrm{Fil}^{p+1} G) \subset \mathrm{Fil}^p G$. Then $\bigcap_{r \geq 0} Z_r^p = M^p / \mathrm{Fil}^{p+1} G$, $\bigcup_{r \geq 0} B_r^p = N^p / \mathrm{Fil}^{p+1} G$, and E_∞ is isomorphic to $\bigoplus_{p \in \mathbb{Z}} E_\infty^p$, where $E_\infty^p = Z_\infty^p / B_\infty^p = M^p / N^p$.*

Definition IV.4.2.13. We define a decreasing filtration $\mathrm{Fil}^\bullet \mathrm{H}(G)$ on $\mathrm{H}(G) = \mathrm{H}(G, d)$ by

$$\mathrm{Fil}^p \mathrm{H}(G) = \mathrm{Im}(\mathrm{H}(\mathrm{Fil}^p G, d) \to \mathrm{H}(G, d)),$$

where the morphism is the image by H of the inclusion $\mathrm{Fil}^p G \subset G$. For every $p \in \mathbb{Z}$, we also write $\mathrm{Gr}^p \mathrm{H}(G) = \mathrm{Fil}^p \mathrm{H}(G) / \mathrm{Fil}^{p+1} \mathrm{H}(G)$.

For every $p \in \mathbb{Z}$, let $K^p = \mathrm{Fil}^p G \cap \mathrm{Ker}\, d$. Then $\mathrm{Fil}^p \mathrm{H}(G)$ is the image of $K^p \subset \mathrm{Ker}\, d$ in $\mathrm{H}(G)$, so $\mathrm{Fil}^p \mathrm{H}(G) = K^p/(d(G) \cap \mathrm{Fil}^p G)$, and

$$\mathrm{Gr}^p \mathrm{H}(G) = K^p/(K^{p+1} + (d(G) \cap \mathrm{Fil}^p G)) = (K^p + \mathrm{Fil}^{p+1} G)/((\mathrm{Fil}^p G \cap d(G)) + \mathrm{Fil}^{p+1} G)$$

(where the second equality holds because $K^{p+1} + (d(G) \cap \mathrm{Fil}^p G) = K^p \cap ((\mathrm{Fil}^p G + d(G)) + \mathrm{Fil}^{p+1} G)$). Observing that $K^p + \mathrm{Fil}^{p+1} G \subset M^p$ and $N^p \subset \mathrm{Fil}^{p+1} G + (\mathrm{Fil}^p G \cap d(G))$, we see that $\mathrm{Gr}^p \mathrm{H}(G)$ is canonically identified to a subquotient of E^p_∞.

Corollary IV.4.2.14. *Suppose that $\bigcap_{p\in\mathbb{Z}} \mathrm{Fil}^p G = 0$ and $\bigcup_{p\in\mathbb{Z}} \mathrm{Fil}^p G = G$ (in other words, the filtration on G is separated and exhaustive). Then the spectral sequence of $(G, \mathrm{Fil}^\bullet G, d)$ converges to $\bigoplus_{p\in\mathbb{Z}} \mathrm{Gr}^p G$ if and only if*

$$\bigcap_{p\in\mathbb{Z}} ((\mathrm{Fil}^p G \cap \mathrm{Ker}(d)) + d(G)) = d(G)$$

and, for every $p \in \mathbb{Z}$,

(*) $$\bigcap_{r\geq 0} ((\mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G)) + \mathrm{Fil}^{p+1} G) = (\mathrm{Fil}^p G \cap \mathrm{Ker}\, d) + \mathrm{Fil}^{p+1} G.$$

Proof. As the filtration on G exhaustive, so is the filtration on $\mathrm{H}(G)$. The first identity says exactly that the filtration on $\mathrm{H}(G)$ is separated.

By the discussion above, we have $E^p_\infty = \mathrm{Gr}^p \mathrm{H}(G)$ and if and only (*) and

(**) $$\bigcup_{r\geq 0} ((\mathrm{Fil}^p G \cap d(\mathrm{Fil}^{p-r+1} G)) + \mathrm{Fil}^{p+1} G) = (\mathrm{Fil}^p G \cap d(G)) + \mathrm{Fil}^{p+1} G.$$

hold, and (**) is an easy consequence of the fact that $G = \bigcup_{n\in\mathbb{Z}} \mathrm{Fil}^n G$.

□

IV.4.2.3 The spectral sequence of a filtered complex

We go one step further: we still assume that we have a triple $(G, \mathrm{Fil}^\bullet G, d)$ as in the previous subsection, but we now also assume that $G = \bigoplus_{n\in\mathbb{Z}} G^n$ is a complex and that d is its differential (see Example IV.4.2.3); we also assume that $\mathrm{Fil}^p G = \bigoplus_{n\in\mathbb{Z}} \mathrm{Fil}^p G^n$ for every $p \in \mathbb{Z}$, where $\mathrm{Fil}^p G^n = (\mathrm{Fil}^p G) \cap G^n$. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence of $(G, \mathrm{Fil}^\bullet G, d)$.

All the formulas of the last subsection have obvious bigraded versions:

Proposition IV.4.2.15. *We have $Z^p_r = \bigoplus_{q\in\mathbb{Z}} Z^{pq}_r$, $B^p_r = \bigoplus_{q\in\mathbb{Z}} B^{pq}_r$ and $E^p_r = \bigoplus_{q\in\mathbb{Z}} E^{pq}_r$, where*

$$Z^{pq}_r = \frac{(\mathrm{Fil}^p G^{p+q} \cap d^{-1}(\mathrm{Fil}^{p+r} G^{p+q+1})) + \mathrm{Fil}^{p+1} G^{p+q}}{\mathrm{Fil}^{p+1} G^{p+q}} \subset \mathrm{Gr}^p G^{p+q},$$

$$Z_r^{pq} = \frac{(\mathrm{Fil}^p G^{p+q} \cap d(\mathrm{Fil}^{p-r+1} G^{p+q+1})) + \mathrm{Fil}^{p+1} G^{p+q}}{\mathrm{Fil}^{p+1} G^{p+q}} \subset \mathrm{Gr}^p \, G^{p+q}$$

and

$$E_r^{pq} = Z_r^{pq} / B_r^{pq}.$$

Moreover, the morphism $d_r : E_r \to E_r$ is the sum of the morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ induced by $d : \mathrm{Fil}^p G^{p+q} \cap d^{-1}(\mathrm{Fil}^{p+r} G^{p+q+1}) \to \mathrm{Fil}^{p+r} G^{p+q+1} \cap d^{-1}(\mathrm{Fil}^{p+2r} G^{p+q+1})$ (note that $p+q+1 = (p+r)+(q-r+1)$).

We also have $E_\infty^p = \bigoplus_{q\in\mathbb{Z}} E_\infty^{p,q}$, where

$$E_\infty^{p,q} = \frac{\bigcap_{r\geq 0}((\mathrm{Fil}^p G^{p+q} \cap d^{-1}(\mathrm{Fil}^{p+r} G^{p+q+1})) + \mathrm{Fil}^{p+1} G^{p+q})}{\bigcup_{r\geq 0}((\mathrm{Fil}^p G^{p+q} \cap d(\mathrm{Fil}^{p-r+1} G)^{p+q-1}) + \mathrm{Fil}^{p+1} G^{p+q})}.$$

Finally, for every $p \in \mathbb{Z}$, we have $\mathrm{Gr}^p \, \mathrm{H}(G) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Gr}^p \, \mathrm{H}^n(G)$, with

$$\mathrm{Gr}^p \, \mathrm{H}^n(G) = \frac{(\mathrm{Fil}^p G^n \cap \operatorname{Ker} d) + (G^n \cap \operatorname{Im} d)}{G^n \cap \operatorname{Im} d}.$$

Corollary IV.4.2.16. *The spectral sequence of $(G, \mathrm{Fil}^\bullet G, d)$ is functorial in the data, that is, every morphism of complexes $f : (G,d) \to (G',d')$ such that $f(\mathrm{Fil}^p G) \subset \mathrm{Fil}^p G'$ for every $p \in \mathbb{Z}$ induces a morphism of spectral sequences.*

We finally address the problem of convergence.

Definition IV.4.2.17. Let A be an object of $\mathscr{A}$ and $\mathrm{Fil}^\bullet A$ be a decreasing filtration on A. We say that this filtration is *finite* if $\mathrm{Fil}^p A = A$ for $p << 0$ and $\mathrm{Fil}^p A = 0$ for $p >> 0$.

A finite filtration is automatically separated and exhaustive.

Proposition IV.4.2.18. *Let $(G, \mathrm{Fil}^\bullet G, d)$ be as before, and let (E_r, d_r) be the associated spectral sequence.*

(i). Suppose that, for every $n \in \mathbb{Z}$, we have $\mathrm{Fil}^p G^n = 0$ for $p >> 0$ and $G^n = \bigcup_{p\in\mathbb{Z}} \mathrm{Fil}^p G^n$. Then the spectral sequence converges to $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(G)$.

(ii). Suppose that the filtration $\mathrm{Fil}^\bullet G^n$ is finite for every $n \in \mathbb{Z}$. Then:

- *a) For every $n \in \mathbb{Z}$, there are only finitely many nonzero $E_0^{p,n-p}$ (we say that the spectral sequence is* bounded*).*
- *b) The filtration $\mathrm{Fil}^\bullet \mathrm{H}^n(G)$ is finite for every $n \in \mathbb{Z}$.*
- *c) The spectral sequence converges to $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(G)$.*

Remark IV.4.2.19. The boundedness of the spectral sequence in (ii) implies that, for all $p, q \in \mathbb{Z}$, there exists $s \in \mathbb{N}$ such that $d_r^{p,q} = 0$ and $d_r^{p-r,q+r-1} = 0$ for $r \geq s$, hence $E_\infty^{pq} = E_r^{pq} = E_s^{pq}$ for $r \geq s$.

Indeed, for $(p,q) \in \mathbb{Z}$, let $n = p+q$, and let $I \subset \mathbb{Z}^2$ be a finite set such that $E_0^{a,b} = 0$ if $a+b \in \{n-1, n, n+1\}$ and $(a,b) \notin I$. As $E_r^{a,b}$ is isomorphic to a subquotient of $E_0^{a,b}$, we also have $E_r^{a,b} = 0$ if $a+b \in \{n-1, n, n+1\}$ and $(a,b) \notin I$, for every $r \geq 0$. In particular, there exists $s \in \mathbb{N}$ such that, if $r \geq s$, then $(p+r, q-r+1)$ and $(p-r, q+r-1)$ are not in I, and so $d_r^{p,q} = 0$ and $d_r^{p-r,q+r-1} = 0$.

Proof of the proposition. To prove (i), we check the conditions of Corollary IV.4.2.14. The hypothesis implies immediately that the filtration $\mathrm{Fil}^\bullet G$ is separated and exhaustive. We have

$$\bigcap_{p\in\mathbb{Z}}((\mathrm{Fil}^p G \cap \operatorname{Ker} d) + d(G)) = \bigoplus_{n\in\mathbb{Z}} \bigcap_{p\in\mathbb{Z}}((\mathrm{Fil}^p G^n \cap \operatorname{Ker} d) + d(G^{n-1})).$$

If $n \in \mathbb{Z}$ is fixed, then $\mathrm{Fil}^p G^n = 0$ for p big enough, so $\bigcap_{p\in\mathbb{Z}}((\mathrm{Fil}^p G^n \cap \operatorname{Ker} d) + d(G^{n-1})) = d(G^{n-1})$. Hence

$$\bigcap_{p\in\mathbb{Z}}((\mathrm{Fil}^p G \cap \operatorname{Ker} d) + d(G)) = d(G).$$

We finally check identity (*) of Corollary IV.4.2.14. Let $p \in \mathbb{Z}$. Then

$$\bigcap_{r\geq 0}((\mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G)) + \mathrm{Fil}^{p+1} G) = \bigoplus_{n\in\mathbb{Z}} \bigcap_{r\geq 0}((\mathrm{Fil}^p G^n \cap d^{-1}(\mathrm{Fil}^{p+r} G^{n+1})) + \mathrm{Fil}^{p+1} G^n).$$

If we fix $n \in \mathbb{Z}$, then $\mathrm{Fil}^{p+r} G^{n-1} = 0$ for r big enough, so

$$\bigcap_{r\geq 0}((\mathrm{Fil}^p G^n \cap d^{-1}(\mathrm{Fil}^{p+r} G^{n+1})) + \mathrm{Fil}^{p+1} G^n) = (\mathrm{Fil}^p G^n \cap \operatorname{Ker} d) + \mathrm{Fil}^{p+1} G^n.$$

Summing over all $n \in \mathbb{Z}$ gives

$$\bigcap_{r\geq 0}((\mathrm{Fil}^p G \cap d^{-1}(\mathrm{Fil}^{p+r} G)) + \mathrm{Fil}^{p+1} G) = (\mathrm{Fil}^p G \cap \operatorname{Ker} d) + \mathrm{Fil}^{p+1} G,$$

which is (*).

We prove (ii). The convergence follows from (i) We have $E_0^{p,n-p} = \mathrm{Gr}^p G^n$. If n is fixed, only a finite number of $\mathrm{Gr}^p G^n$ are nonzero by assumption, so we get (a). Point (b) is immediate from the last formula of Proposition IV.4.2.15.

□

IV.4.2.4 The spectral sequences of a double complex

We are finally ready to construct the two spectral sequences of a double complex. Let X be a double complex of objects of $\mathscr{A}$. We consider the complex $G = \operatorname{Tot}(X)$, with the two filtrations ${}^I\mathrm{Fil}^\bullet G$ and ${}^{II}\mathrm{Fil}^\bullet G$ given by

$${}^I\mathrm{Fil}^p G^n \quad \bigoplus_{a+b=n,\ a\geq p} X^{a,b}$$

and

$$^I\mathrm{Fil}^p G^n \bigoplus_{a+b=n,\ b\geq p} X^{a,b}.$$

These filtrations are separated and exhaustive. Also, it is clear that $d_G^n(^I\mathrm{Fil}^p G^n) \subset {}^I\mathrm{Fil}^p G^{n+1}$ and $d_G^n(^{II}\mathrm{Fil}^p G^n) \subset {}^{II}\mathrm{Fil}^p G^{n+1}$. So we get two spectral sequences, which we denote by $^I E$ and ^{II}E. We calculate their first pages using Proposition IV.4.2.15. For example, this proposition gives

$$^I E_0^{p,q} = \frac{(\mathrm{Fil}^p G^{p+q} \cap d^{-1}(\mathrm{Fil}^p G^{p+q+1})) + \mathrm{Fil}^{p+1} G^{p+q}}{\mathrm{Fil}^{p+1} G^{p+q}} = \mathrm{Fil}^p G^{p+q}/\mathrm{Fil}^{p+1} G^{p+q} = X^{p,q}.$$

The differential of E_0 is the morphism $E_0^{pq} \to E_0^{p,q+1}$ induce by d, that it, $d_{2,X}^{p,q}$. The same proposition gives

$$^I Z_1^{p,q} = \frac{(\mathrm{Fil}^p G^{p+q} \cap d^{-1}(\mathrm{Fil}^{p+1} G^{p+q+1})) + \mathrm{Fil}^{p+1} G^{p+q}}{\mathrm{Fil}^{p+1} G^{p+q}}$$

and

$$^I B_1^{p,q} = \frac{(\mathrm{Fil}^p G^{p+q} \cap d(\mathrm{Fil}^p G^{p+q-1})) + \mathrm{Fil}^{p+1} G^{p+q}}{\mathrm{Fil}^{p+1} G^{p+q}}.$$

So

$$^I Z_1^{p,q} = X^{p,q} \cap d^{-1} X^{p+1,q} = \{x \in X^{p,q} \mid d_{1,X}^{p,q}(x) + (-1)^p d_{2,X}^{p,q}(x) \in X^{p+1,q}\} = \mathrm{Ker}(d_{2,X}^{p,q}),$$

and

$$^I B_1^{p,q} = X^{p,q} \cap d(X^{p,q-1}) = X^{p,q} \cap d_{2,X}^{p,q-1}(X^{p,q-1}) = \mathrm{Im}(d_{2,X}^{p,q-1}),$$

and finally

$$^I E_1^{p,q} = \mathrm{H}^q(X^{p,\bullet}, d_{2,X}^{p,\bullet}).$$

Also, the differential of $^I E_1$ is induced by d, so $d_1^{pq} : {}^I E_1^{pq} \to {}^I E_1^{p+1,q}$ is $(-1)^p d_{1,X}^{p,q}$. This gives the formulas of Theorem IV.4.1.7 for $^I E_0$ and $^I E_1$. The proof of the formulas for $^{II}E_0$ and $^{II}E_1$ is similar, we just invert the roles of p and q.

We still need to prove the convergence results of Theorem IV.4.1.7. The proofs of points (i) and (ii), so we just prove (i). Suppose that $X^{p,q} = 0$ for all $p > 0$ or for all $q < 0$. Let $n \in \mathbb{Z}$. Then $X^{p,n-p} = 0$ for p big enough (in the first case, take $p > 0$, in the second case, take $p > n$), so the filtration $^I\mathrm{Fil}^\bullet G$ satisfies the hypothesis of part (i) of Proposition IV.4.2.18, which implies that the spectral sequence $^I E$ converges. We now assume that we are in acse (iii) of Theorem IV.4.1.7. Then the filtrations $^I\mathrm{Fil}^\bullet G^n$ and $^{II}\mathrm{Fil}^\bullet G^n$ of G^n are finite for every $n \in \mathbb{Z}$, so we can apply part (ii) of Proposition IV.4.2.18.

V Derived categories

V.1 Triangulated categories

V.1.1 Definition

In this subsection, we fix an additive category $\mathscr{D}$ and an auto-equivalence T of $\mathscr{D}$.

Definition V.1.1.1. A *triangle* in $\mathscr{D}$ is a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$. A *morphism of triangles* is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & T(X) \arrow[d, "T(u)"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

Remark V.1.1.2. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a triangle. Then it is isomorphic to $X \xrightarrow{f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$, but not to $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ in general.

Example V.1.1.3. Let $\mathscr{C}$ be an additive category and $* \in \{+, -, b, \varnothing\}$. If we take $\mathscr{D} = K^*(\mathscr{A})$ and $T = [1]$, then a triangle is often written $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$.

Definition V.1.1.4. A *triangulated category* is an additive category $\mathscr{D}$ with an auto-equivalence T and a family of triangles called *distinguished triangles* or *exact triangles*, satisfying the following axioms:

(TR0) Any triangle that is isomorphic to a distinguished triangle is distinguished.

(TR1) For every $X \in \mathrm{Ob}(\mathscr{D})$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is distinguished.

(TR2) For every morphism $f : X \to Y$ in $\mathscr{D}$, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$.

(TR3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is distinguished if and only if the triangle $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is.

(TR4) Given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ and two morphisms $u : X \to X'$ and $v : Y \to Y'$ such that $v \circ f = f' \circ u'$, there exists a morphism $w : Z \to Z'$ such that the following diagram commutes (hence is a morphism of triangles):

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"'] & T(X) \arrow[d, "T(u)"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X')
\end{tikzcd}$$

(TR5) (Octahedral axiom.) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \xrightarrow{h} T(X),$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \xrightarrow{h'} T(Y)$$

and

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \xrightarrow{h} T(X),$$

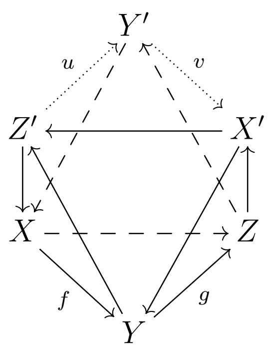
there exists a distinguished triangle $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \to T(Z')$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r, "h"] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d, dashed, "u"'] & T(X) \arrow[d, "\mathrm{id}_{T(X)}"] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \arrow[r, "l"] \arrow[d, "\mathrm{id}_Z"'] & Y' \arrow[r] \arrow[d, dashed, "v"'] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d, "h"'] & Z \arrow[r, "k"] \arrow[d, "l"'] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"'] & T(Y) \arrow[d, "T(h)"] \\
Z' \arrow[r, dashed, "u"] & Y' \arrow[r, dashed, "v"] & X' \arrow[r, dashed, "w"] & T(Z')
\end{tikzcd}$$

Remark V.1.1.5. (1) The morphism w in (TR4) is not unique, and neither are the distinguished triangle $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \to T(Z')$ in (TR5). This causes many problems.

(2) The object Z in (TR2) is unique up to isomorphism by Corollary V.1.1.12, but that isomorphism is not necessarily unique. This also causes many problems.

(3) Condition (TR5) is called the octahedral axiom because we can think of it as asserting the existence of the top face in the following octahedron:



Remark V.1.1.6. If $\mathscr{D}$ is a triangulated category, then $\mathscr{D}^{\mathrm{op}}$ also is, for the auto-equivalence T^{-1} (where T^{-1}) is s quasi-inverse of T) and the class of distinguished triangles $Z\mathrm{op} \xrightarrow{g^{\mathrm{op}}} Y^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} X \xrightarrow{T^{-1}(h^{\mathrm{op}})} T^{-1}(Z)$ and where $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and is a distinguished triangle in $\mathscr{D}$. [1]

Example V.1.1.7. The main example of a triangulated category is the homotopy category $K^*(\mathscr{C})$ of an additive category, with $* \in \{+, -, b, \varnothing\}$, with $T = [1]$ and distinguished triangles being the triangles isomorphic to $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1]$, for $f : X \to Y$ a morphism of $K^*(\mathscr{C})$. This is a nontrivial theorem and will be proved later. (See Theorem V.1.2.1.)

Before we show that our main example is indeed an example, we see some more definitions and general results about triangulated categories.

Definition V.1.1.8. Let $(\mathscr{D}, T)$ be a triangulated category.

(i). Let $(\mathscr{D}', T')$ be another triangulated category. A *triangulated functor* (or *exact functor*) is an additive functor $F : \mathscr{D} \to \mathscr{D}'$ such that there exists an isomorphism $F \circ T \simeq T' \circ F$ and that F sends distinguished triangles in $\mathscr{D}$ to distinguished triangles in $\mathscr{D}'$.

(ii). A *triangulated subcategory* of $\mathscr{D}$ is a subcategory $\mathscr{D}'$ of $\mathscr{D}$ that is triangulated and such that the inclusion functor $\mathscr{D}' \to \mathscr{D}$ is triangulated.

(iii). If $\mathscr{A}$ is an abelian category, a *cohomological functor* from $\mathscr{D}$ to $\mathscr{A}$ is an additive functor $\mathrm{H} : \mathscr{D} \to \mathscr{A}$ such that, for every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ in $\mathscr{D}$, the sequence $\mathrm{H}(X) \to \mathrm{H}(Y) \to \mathrm{H}(Z)$ is exact.

Remark V.1.1.9. By (TR3), if $\mathrm{H} : \mathscr{D} \to \mathscr{A}$ is a cohomological functor, then we get a long exact sequence

$$\ldots \to \mathrm{H}(X) \to \mathrm{H}(Y) \to \mathrm{H}(Z) \to \mathrm{H}(T(X)) \to \mathrm{H}(T(Y)) \to \mathrm{H}(T(Z)) \to \mathrm{H}(T(T(X))) \to \ldots$$

(The sequence continues on the left too because T is an equivalence.)

Example V.1.1.10. Let $* \in \{+, -, b, \varnothing\}$.

(1) If $F : \mathscr{C} \to \mathscr{C}'$ is an additive functor between two additive categories, then the functor $K(F) : K^*(\mathscr{C}) \to K^*(\mathscr{C}')$ is triangulated.

(2) If $\mathscr{A}$ is an abelian category, then the functor $\mathrm{H}^0 : K^*(\mathscr{A}) \to \mathscr{A}$ is triangulated by Corollary IV.2.2.8.

We now fix a triangulated category $(\mathscr{D}, T)$.

Proposition V.1.1.11. *(i). If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a distinguished triangle, then $g \circ f = 0$.*

[1] Technically, the morphism $T^{-1}(h^{\mathrm{op}})$ goes from $T^{-1}(T(X))$ to $T^{-1}(Z)$, so we have to compose it with the isomorphism $X \xrightarrow{\sim} T^{-1}(T(X))$.

(ii). For every object W of $\mathscr{D}$, the functors $\operatorname{Hom}_{\mathscr{D}}(Z,\cdot) : \mathscr{D} \to \mathbf{Ab}$ *and* $\operatorname{Hom}_{\mathscr{D}}(\cdot,W) : \mathscr{D}^{\mathrm{op}} \to \mathbf{Ab}$ *are cohomological.*

Proof. (i). By (TR1) and (TR4), there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X) \arrow[d, "\mathrm{id}_{T(X)}"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd}$$

This shows that $g \circ f = 0$.

(ii). We show that $\operatorname{Hom}_{\mathscr{D}}(W,\cdot)$ is cohomological. (The case of $\operatorname{Hom}_{\mathscr{D}}(\cdot,W)$ follows by doing the same proof in the opposite category.) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a distinguished triangle. We want to show that the sequence of abelian groups

$$\operatorname{Hom}_{\mathscr{D}}(W,X) \xrightarrow{f_*} \operatorname{Hom}_{\mathscr{D}}(W,Y) \xrightarrow{g_*} \operatorname{Hom}_{\mathscr{D}}(W,Z)$$

is exact. The fact that $g_* \circ f_* = 0$ follows from point (i). Let $u \in \operatorname{Ker}(g_*)$. Then u is a morphism from W to Y such that $g \circ u = 0$. By (TR1), (TR3) and (TR4), there exists a morphism of distinguished triangles

$$\begin{tikzcd} W \arrow[r, "\mathrm{id}_W"] \arrow[d, "v"'] & W \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d] & T(W) \arrow[d, "T(v)"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X) \end{tikzcd}$$

In particular, this gives a morphism $v \in \operatorname{Hom}_{\mathscr{D}}(W,X)$ such that $f_*(v) = u$, so $u \in \operatorname{Im}(f_*)$.

$\square$

Corollary V.1.1.12. *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"'] & T(X) \arrow[d, "T(u)"'] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X') \end{tikzcd}$$

be a morphism of distinguished triangles. If u and v are isomorphisms, then so is w.

Proof. Let W be an object of $\mathscr{D}$. By Proposition V.1.1.11, we have a commutative diagram with exact rows:

$$\begin{tikzcd} \operatorname{Hom}_{\mathscr{D}}(W,X) \arrow[r, "f_*"] \arrow[d, "u_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,Y) \arrow[r, "g_*"] \arrow[d, "v_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,Z) \arrow[r, "h_*"] \arrow[d, "w_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,T(X)) \arrow[r, "T(f)_*"] \arrow[d, "T(u)_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,T(Y)) \arrow[d, "T(v)_*"'] \\ \operatorname{Hom}_{\mathscr{D}}(W,X') \arrow[r, "f'_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,Y') \arrow[r, "g'_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,Z') \arrow[r, "h'_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,T(X')) \arrow[r, "T(f')_*"'] & \operatorname{Hom}_{\mathscr{D}}(W,T(Y')) \end{tikzcd}$$

By the five lemma (Corollary IV.2.2.2), the morphism $w_* = \mathrm{Hom}_{\mathscr{D}}(W, w)$ is an isomorphism. By the Yoneda lemma (Corollary I.3.2.3), this implies that w is an isomorphism.

□

Corollary V.1.1.13. *Let $\mathscr{D}'$ be a full triangulated subcategory of $\mathscr{D}$.*

(i). Let $X \xrightarrow{f} Y \to Z \to T(X)$ be a triangle in $\mathscr{D}'$, and suppose that this triangle is distinguished in $\mathscr{D}$. Then it is also distinguished in $\mathscr{D}'$. (Note that the converse is also true, by the definition of a triangulated subcategory.)

(ii). Let $X \to Y \to Z \to T(X)$ be a distinguished triangle in $\mathscr{D}$. If $X, Y \in \mathrm{Ob}(\mathscr{D}')$, then there exists an object Z' of $\mathscr{D}'$ and an isomorphism $Z \simeq Z'$.

Proof. (i). By (TR2), there exists a distinguished triangle $X \xrightarrow{f} Y \to Z' \to T(X)$ in $\mathscr{D}'$. By (TR4), the identity morphisms of X and Y extend to a morphism between this triangle and $X \xrightarrow{f} Y \to Z \to T(X)$ and, by Corollary V.1.1.12, this morphism is an isomorphism. So the triangle $X \xrightarrow{f} Y \to Z \to T(X)$ is distinguished in $\mathscr{D}'$ by (TR0).

(ii). We proved this during the proof of (i).

□

Corollary V.1.1.14. *Let $f : X \to Y$ be a morphism of $\mathscr{D}$. Then f is an isomorphism if and only if there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$.*

Proof. See Problem A.8.2.

□

V.1.2 The homotopy category

Let $\mathscr{C}$ be a triangulated category, and let $* \in \{+, -, b, \varnothing\}$. We consider the auto-equivalence $T = [1]$ of $K^*(\mathscr{C})$, and we say that a triangle in $K^*(\mathscr{C})$ is distinguished if it is isomorphic to a mapping cone triangle, that is, a triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1],$$

for $f : X \to Y$ a morphism of $K^*(\mathscr{C})$.

Theorem V.1.2.1. *This makes $K^*(\mathscr{C})$ into a triangulated category.*

Lemma V.1.2.2. *Let $f : X \to Y$ be a morphism in $\mathcal{C}(\mathscr{C})$, and consider the mapping cone triangle*

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1].$$

Then there exists a morphism $u : X[1] \to \mathrm{Mc}(\alpha(f))$ such that u is an isomorphism in $K(\mathscr{C})$ and that the following diagram commutes in $K(\mathscr{C})$:

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{id}_Y"'] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"] \arrow[d, "\mathrm{id}_{\mathrm{Mc}(f)}"'] & X[1] \arrow[r, "-f[1]"] \arrow[d, "u"'] & Y[1] \arrow[d, "\mathrm{id}_{Y[1]}"'] \\
Y \arrow[r, "\alpha(f)"'] & \mathrm{Mc}(f) \arrow[r, "\alpha(\alpha(f))"'] & \mathrm{Mc}(\alpha(f)) \arrow[r, "\beta(\alpha(f))"'] & Y[1]
\end{tikzcd}$$

Proof. By definition of the mapping cone, we have, for every $n \in \mathbb{Z}$,

$$\mathrm{Mc}(\alpha(f))^n = Y^{n+1} \oplus \mathrm{Mc}(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

and

$$d^n_{\mathrm{Mc}(\alpha(f))} = \begin{pmatrix} -d_Y^{n+1} & 0 & 0 \\ 0 & -d_X^{n+1} & 0 \\ \mathrm{id}_{Y^{n+1}} & f^{n+1} & d_Y^n \end{pmatrix}.$$

We define $u^n : X[1]^n \to \mathrm{Mc}(\alpha(f))^n$ and $v^n : \mathrm{Mc}(\alpha(f))^n \to X[1]^n$ by

$$u^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix} \quad \text{and} \quad v^n = \begin{pmatrix} 0 & \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}.$$

Straightforward calculations show that the families $(u^n)_{n\in\mathbb{Z}}$ and $(v^n)_{n\in\mathbb{Z}}$ define morphisms of complexes $u : X[1] \to \mathrm{Mc}(\alpha(f))$ and $v : \mathrm{Mc}(\alpha(f)) \to X[1]$, that $v \circ u = \mathrm{id}_{X[1]}$, that $v \circ \alpha(\alpha(f)) = \beta(f)$ and that $\beta(\alpha(f)) \circ u = -f[1]$. To finish the proof, it suffices to show that $u \circ v$ is homotopic to $\mathrm{id}_{\mathrm{Mc}(\alpha(f))}$. Define $s^n : \mathrm{Mc}(\alpha(f))^n \to \mathrm{Mc}(\alpha(f))^{n-1}$ by

$$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have, for every $n \in \mathbb{Z}$,

$$\mathrm{id}_{\mathrm{Mc}(\alpha(f))^n} - u^n \circ v^n = s^{n+1} \circ d^n_{\mathrm{Mc}(\alpha(f))} + d^{n-1}_{\mathrm{Mc}(\alpha(f))} \circ s^n.$$

□

Proof of Theorem V.1.2.1. We check properties (TR0)-(TR5) of Definition V.1.1.4. Properties (TR0) and (TR2) are clear, and (TR3) follows immediately from Lemma V.1.2.2. Let X be an object of $K^*(\mathscr{C})$.The mapping cone triangle corresponding to the unique morphism from 0 to X is $0 \to X \overset{\mathrm{id}_X}{\to} X \to 0[1] = 0$; this triangle is distinguished, and by (TR2), so is the triangle $X \overset{\mathrm{id}_X}{\to} X \to 0 \to X[1]$. So we get (TR1).

We prove (TR4). We may assume that the two triangles are mapping cone triangles, that is, of the form $X \stackrel{f}{\to} Y \stackrel{\alpha(f)}{\to} \mathrm{Mc}(f) \stackrel{\beta(f)}{\to} X[1]$ and $X' \stackrel{f}{\to} Y' \stackrel{\alpha(f')}{\to} \mathrm{Mc}(f') \stackrel{\beta(f')}{\to} X'[1]$. Let $u : X \to X'$ and $v : Y \to Y'$ be morphisms such that $v \circ f = f' \circ u$ in $K^*(\mathscr{C})$. This means that there exist morphisms $s^n : X^n \to Y'^{n-1}$, for $n \in \mathbb{Z}$, such that $v^n \circ f^n - f'^n \circ u^n = s^{n+1} \circ d_X^n + d_{Y'}^{n-1} \circ s^n$. We define $w^n : \mathrm{Mc}(f)^n = X^{n+1} \oplus Y^n \to \mathrm{Mc}(f')^n = X'^{n+1} \oplus Y'^n$ by

$$w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix}.$$

Then it is easy to check that this defines a morphism of complexes $w : \mathrm{Mc}(f) \to \mathrm{Mc}(f')$ and that we have $w \circ \alpha(f) = \alpha(f') \circ v$ and $u[1] \circ \beta(f) = \beta(f') \circ w$. (This even holds in $\mathcal{C}(\mathscr{C})$.)

We finally prove (TR5). We may again assume that the triangles appearing in the statement are mapping cone triangles, so they are of the form $X \stackrel{f}{\to} Y \stackrel{\alpha(f)}{\to} Z' \stackrel{\beta(f)}{\to} X[1]$ with $Z' = \mathrm{Mc}(f)$, $Y \stackrel{g}{\to} Z \stackrel{\alpha(g)}{\to} X' \stackrel{\beta(g)}{\to} Y[1]$ with $X' = \mathrm{Mc}(g)$, and $X \stackrel{g\circ f}{\to} Z \stackrel{\alpha(g\circ f)}{\to} Y' \stackrel{\beta(g\circ f)}{\to} X[1]$ with $Y' = \mathrm{Mc}(g \circ f)$. We define morphisms of complexes $u : Z' \to Y'$ and $v : Y' \to X'$ by taking $u^n : Z'^n = X^{n+1} \oplus Y^n \to Y'^n = X^{n+1} \oplus Z^n$ equal to $\begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & g^n \end{pmatrix}$ and $v^n : Y'^n = X^{n+1} \oplus Z^n \to X'^n = Y^{n+1} \oplus Z^n$ equal to $\begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}_{Z^n} \end{pmatrix}$. We also define $w : X' \to Z'[1]$ as the composition $X' \stackrel{\beta(g)}{\to} Y[1] \stackrel{\alpha(f)[1]}{\to} Z'[1]$. Then it is clear that the diagram appearing in (TR5) is commutative, so we just need to check that the triangle $Z' \stackrel{u}{\to} Y' \stackrel{v}{\to} X' \stackrel{w}{\to} Z'[1]$ is distinguished. This will follow if we can construct two isomorphisms $\varphi : \mathrm{Mc}(u) \to X'$ and $\psi : X' \to \mathrm{Mc}(u)$ that are inverses of each other (in $K^*(\mathscr{C})$) and such that $\varphi \circ \alpha(u) = v$ and $\beta(u) \circ \psi = w$. Remember that

$$\mathrm{Mc}(u)^n = Z'^{n+1} \oplus Y'^n = X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n$$

and

$$X'^n = Y^{n+1} \oplus Z^n.$$

We take

$$\varphi^n = \begin{pmatrix} 0 & \mathrm{id}_{Y^{n+1}} & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z^n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y^{n+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{X^{n+1}} \end{pmatrix}.$$

This defines morphisms of complexes φ and ψ such that $\varphi \circ \alpha(u) = v$, $\beta(u) \circ \psi = w$ and $\varphi \circ \psi = \mathrm{id}_{X'}$. (Even in $\mathcal{C}(\mathscr{C})$.) We show that $\psi \circ \varphi = \mathrm{id}_{\mathrm{Mc}(u)}$ in $K^*(\mathscr{C})$. Define $s^n : \mathrm{Mc}(u)^n \to \mathrm{Mc}(u)^{n-1}$ by

$$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{X^{n+1}} & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{pmatrix}.$$

Then we have $\mathrm{id}_{\mathrm{Mc}(u)^n} = \psi^n \circ \varphi^n = s^{n+1} \circ d^n_{\mathrm{Mc}(u)} + d^{n-1}_{\mathrm{Mc}(u)} \circ s^n$, so we are done.

□

Proposition V.1.2.3. *Let $\mathscr{A}$ be an abelian category.*

(i). Let $f : X \to Y$ be a morphism in $\mathcal{C}(\mathscr{A})$. Then f is a quasi-isomorphism if and only if $\mathrm{Mc}(f)$ is quasi-isomorphic to 0.

(ii). Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathcal{C}(\mathscr{A})$, and define $u : \mathrm{Mc}(f) \to Z$ by $u^n = \begin{pmatrix} 0 & g^n \end{pmatrix}$. Then u is a quasi-isomorphism and $u \circ \alpha(f) = g$.

Proof. (i). This follows immediately from the long exact sequence of Corollary IV.2.2.8.

(ii). The fact that $u \circ \alpha(f) = g$ is obvious. We have an exact sequence in $\mathcal{C}(\mathscr{A})$:

$$0 \to \mathrm{Mc}(\mathrm{id}_X) \xrightarrow{v} \mathrm{Mc}(f) \xrightarrow{u} Z \to 0,$$

where $v^n : \mathrm{Mc}(\mathrm{id}_X)^n = X^{n+1} \oplus X^n \to \mathrm{Mc}(f)^n = X^{n+1} \oplus Y^n$ is define by $v^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$. As $\mathrm{Mc}(\mathrm{id}_X)$ is quasi-isomorphic to 0 by (i), the long exact sequence of cohomology (Corollary IV.2.2.6) shows that u is a quasi-isomorphism.

□

V.2 Localization of categories

V.2.1 Definition and construction

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$.

Definition V.2.1.1. A *localization* of $\mathscr{C}$ by W is a category $\mathscr{C}[W^{-1}]$, together with a functor $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$, such that the following conditions hold:

(a) for every $s \in W$, the morphism $Q(s)$ is an isomorphism;

(b) for any functor $F : \mathscr{C} \to \mathscr{C}'$ such that $F(s)$ is an isomorphism for every $s \in W$, there exists a functor $F_W : \mathscr{C}[W^{-1}] \to \mathscr{C}'$ and an isomorphism $F_W \circ Q \simeq F$;

$$\begin{tikzcd} \mathscr{C} \arrow[r, "Q"] \arrow[d, "F"'] & \mathscr{C}[W^{-1}] \arrow[dl, dashed, "F_W"] \\ \mathscr{C}' & \end{tikzcd}$$

(c) for any category $\mathscr{C}'$ and any functors $G_1, G_2 : \mathscr{C}[W^{-1}] \to \mathscr{C}'$, the map

$$\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}[W^{-1}],\mathscr{C}')}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{C}')}(G_1 \circ Q, G_2 \circ Q)$$

sending $u : G_1 \to G_2$ to $u \circ Q$ is bijective.

Remark V.2.1.2. By condition (c), the functor $(\cdot) \circ Q : \mathrm{Func}(\mathscr{C}[W^{-1}], \mathscr{C}') \to \mathrm{Func}(\mathscr{C}, \mathscr{C}')$ is fully faithful, so the functor F_W of (b) is unique up to unique isomorphism.

The following result follows immediately from the definition (and from Remark V.2.1.2).

Proposition V.2.1.3. *(i). If a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ exists, then it is unique up to an equivalence of categories (and this equivalence is unique up to unique isomorphism).*

(ii). Let W^{op} be the set W, seen as a set of morphisms in $\mathscr{C}^{\mathrm{op}}$. If a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ exists, then $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}[W^{-1}]^{\mathrm{op}}$ is a localization of $\mathscr{C}^{\mathrm{op}}$ by W^{op}.

Theorem V.2.1.4. *There exists a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ of $\mathscr{C}$ by W.*

Proof. For any $X, Y \in \mathrm{Ob}(\mathscr{C})$, we write $W^{\mathrm{op}}(X, Y) = \{f \in \mathrm{Hom}_{\mathscr{C}}(Y, X) \mid f \in W\}$; if $f : Y \to X$ is an element of W, we denote the corresponding element of $W^{\mathrm{op}}(X, Y)$ by $\overline{f}$.

Consider the directed graph $\mathscr{C}'$ defined by:

(1) $\mathrm{Ob}(\mathscr{C}') = \mathrm{Ob}(\mathscr{C})$;

(2) for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y) \sqcup W^{\mathrm{op}}(X, Y)$.

The identity of $\mathscr{C}$ induces a morphism of directed graphs $\mathscr{C} \to \mathscr{C}'$; composing with the obvious morphism $\mathscr{C}' \to \mathscr{P}\mathscr{C}'$, we get a morphism of directed graphs $Q' : \mathscr{C} \to \mathscr{P}\mathscr{C}'$. Consider the smallest equivalence relation $\sim$ on the morphisms of $\mathscr{P}\mathscr{C}'$ satusfying the following conditions:

- for every object X of $\mathscr{C}$, we have $(X; \varnothing; X) \sim (X; \mathrm{id}_X; X)$;
- if $f : X \to Y$ and $g : Y \to Z$ are two morphisms of $\mathscr{C}$, we have $(X; f, g; Z) \sim (X; g \circ f; Z)$;
- if $s : X \to Y$ is a morphism in W, we have $(X; s, \overline{s}; X) \sim (X; \mathrm{id}_X; X)$ and $(Y; \overline{s}, s; Y) \sim (Y; \mathrm{id}_Y; Y)$;
- if $f_1, f_2 : X \to Y$ are morphisms of $\mathscr{P}\mathscr{C}'$ such that $f_1 \sim f_2$, then $h \circ f_1 \circ g \sim h \circ f_2 \circ g$ for all morphisms $g : X' \to X$ and $h : Y \to Y'$ in $\mathscr{P}\mathscr{C}'$.

We can define a category $\mathscr{C}[W^{-1}]$ by taking $\mathrm{Ob}(\mathscr{C}[W^{-1}]) = \mathrm{Ob}(\mathscr{P}\mathscr{C}') = \mathrm{Ob}(\mathscr{C})$ and, for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, setting $\mathrm{Hom}_{\mathscr{C}[W^{-1}]}(X, Y) = \mathrm{Hom}_{\mathscr{P}\mathscr{C}'}(X, Y)/\sim$. Let $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ be the functor obtained by composing $Q' : \mathscr{C} \to \mathscr{P}\mathscr{C}'$ with the obvious functor $\mathscr{P}\mathscr{C}' \to \mathscr{C}[W^{-1}]$.

I claim that $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ is a localization of $\mathscr{C}$ by W. First, let $s : X \to Y$ be an element of W. Then the image in $\mathscr{C}[W^{-1}]$ of the morphism $(Y; \overline{s}; X)$ of $\mathscr{P}\mathscr{C}'$ is an inverse of $Q(s)$, so $Q(s)$ is an isomorphism; this shows condition (a).

We prove (b). Let $F : \mathscr{C} \to \mathscr{D}$ be a functor such that $F(s)$ is an isomorphism for every $s \in W$. We extend F to a morphism of directed graphs $F' : \mathscr{C}' \to \mathscr{D}$ by setting $F'(\overline{s}) = F(s)^{-1}$, for all

$X, Y \in \mathrm{Ob}(\mathscr{C})$ and every $s \in W^{\mathrm{op}}(X,Y)$. By Proposition I.4.12, there exists a unique functor $G : \mathscr{P}\mathscr{C}' \to \mathscr{D}$ extending F'. We check that G descends to a functor $F_W : \mathscr{C}[W^{-1}] \to \mathscr{D}$, which will then satisfy $F_W \circ Q = F$. We must show that, if $f \sim g$ in $\mathscr{P}\mathscr{C}'$, then $G(f) = G(g)$. It suffices to check this for the relation generating $\sim$; there are three cases:

- If X is an object of $\mathscr{C}$, we have $G(X; \varnothing; X) = G(X; \mathrm{id}_X; X) = \mathrm{id}_{F(X)}$.
- If $f : X \to Y$ and $g : Y \to Z$ are two morphisms of $\mathscr{C}$, we have $G(X; f, g; Z) = F(g) \circ F(f) = F(g \circ f) = (X; g \circ f; Z)$.
- If $s : X \to Y$ is a morphism in W, we have $G(X; s, \overline{s}; X) = F(s)^{-1} \circ F(s) = \mathrm{id}_{F(X)} = G(X; \mathrm{id}_X; X)$ and $G(Y; \overline{s}, s; Y) = F(s) \circ F(s)^{-1} = \mathrm{id}_{F(Y)} = G(Y; \mathrm{id}_Y; Y)$.

We finally prove (c). Let $G_1, G_2 : \mathscr{C}[W^{-1}] \to \mathscr{D}$ be two functors, and let $\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{C}[W^{-1}],\mathscr{C}')}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{C}')}(G_1 \circ Q, G_2 \circ Q)$ be the obvious map. By definition of $\mathscr{C}[W^{-1}]$, every morphism of $\mathscr{C}[W^{-1}]$ is equal to a composition $Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1} \circ \ldots \circ Q(f_n) \circ Q(s_n)^{-1}$, where $f_1, \ldots, f_n$ are morphisms of $\mathscr{C}$ and $s_1, \ldots, s_n \in W$. This immediately implies that α is injective. We show that α is surjective. Let $u : G_1 \circ Q \to G_2 \circ Q$ be a morphism of functors. If there exists a morphism of functors $v : G_1 \to G_2$ such that $u = \alpha(v)$, then we have $v(X) = u(X) : G_1(X) \to G_2(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. So, to show that u is in the image of α, we have to show that the family $(u(X))_{X \in \mathrm{Ob}(\mathscr{C})}$ defines a morphism of functors from G_1 to G_2, that is, that the diagram

$$\begin{tikzcd} G_1(X) \arrow[r, "u(X)"] \arrow[d, "G_1(f)"'] & G_2(X) \arrow[d, "G_2(f)"] \\ G_1(Y) \arrow[r, "u(Y)"'] & G_2(Y) \end{tikzcd}$$

is commutative for every morphism $f : X \to Y$ in $\mathscr{C}[W^{-1}]$. Writing $f = Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1} \circ \ldots \circ Q(f_n) \circ Q(s_n)^{-1}$ as before, we see that it suffices to treat the case where f is in the image of Q, where the result follows from the fact that u is a morphism of functors from $G_1 \circ Q$ to $G_2 \circ Q$.

$\square$

Remark V.2.1.5. If $\mathscr{C}$ is a $\mathscr{U}$-category, then $\mathscr{C}[W^{-1}]$ might not be. The only thing that the construction tells us is that $\mathscr{C}[W^{-1}]$ is a $\mathscr{V}$-category for every universe $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$. However, we will be able to control the size of the Hom sets in $\mathscr{C}[W^{-1}]$ if $\mathscr{C}$ has some extra structure, such as a model category structure for which W is the set of weak equivalences. [2]

Proposition V.2.1.6. *Let $G : \mathscr{C} \to \mathscr{D}$ be a full and essentially surjective functor. Suppose that there exists a set of morphisms W_1 of $\mathscr{D}$ such that W is the set of morphisms s of $\mathscr{C}$ such that $G(s) \in W_1$. Let $Q_1 : \mathscr{D} \to \mathscr{D}[W_1^{-1}]$ be a localization of $\mathscr{D}$ by W_1. Then $Q_1 \circ G : \mathscr{C} \to \mathscr{D}[W_1^{-1}]$ is a localization of $\mathscr{C}$ by W.*

[2] Add reference.

Proof. Let $\mathscr{D}'$ be the full subcategory of $\mathscr{D}$ whose objects are the $G(X)$, for $X \in \mathrm{Ob}(\mathscr{C})$. As F is essentially surjective, the inclusion functor $\mathscr{D}' \subset \mathscr{D}$ is fully faithful and essentially surjective, hence an equivalence of categories. So we may assume that $\mathscr{D} = \mathscr{D}'$, i.e. that G is surjective on objects. The fact that $Q_1(G(s))$ is invertible for every $s \in W$ is obvious. Let $F_1, F_2 : \mathscr{D} \to \mathscr{E}$ be functors. We claim that the canonical map $\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{D},\mathscr{E})}(F_1, F_2) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{E})}(F_1 \circ G, F_2 \circ G)$ is bijective. This will finish the proof, because then properties (b) and (c) of the localization $Q_1 : \mathscr{D} \to \mathscr{D}[W_1^{-1}]$ immediately imply the analogous properties for $Q_1 \circ G$ (relatively to W). The injectivity of α follows immediately from the fact that G is surjective on objects. We prove that α is surjective. Let $u \in \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{E})}(F_1 \circ G, F_2 \circ G)$. Let $Y \in \mathrm{Ob}(\mathscr{D})$. If $X, X' \in \mathrm{Ob}(\mathscr{C})$ are such that $G(X) = G(X') = Y$, then, by the fullness of G, there exists a morphism $f : X \to X'$ such that $G(f) = \mathrm{id}_Y$. As u is a morphism of functors, we have $F_1(G(f)) \circ u(X) = u(X') \circ F_1(G(f))$, that is, $u(X) = u(X')$. So we can define $v(Y) : F_1(Y) \to F_2(Y)$ by $v(Y) = u(X)$, for any $X \in \mathrm{Ob}(\mathscr{C})$ such that $Y = G(X)$. If $g : Y \to Y'$ is a morphism of $\mathscr{D}$, then, as G is full and surjective on objects, we can find a morphism $f : X \to X'$ in $\mathscr{C}$ such that $G(f) = g$, and then

$$F_2(g) \circ v(Y) = F_1(G(f)) \circ u(X) = u(X') \circ F_2(G(f)) = v(Y') \circ F_2(g).$$

So the family $(v(Y))_{Y \in \mathrm{Ob}(\mathscr{D})}$ defines a morphism of functors $v : F_1 \to F_2$, and we clearly have $\alpha(v) = u$.

□

Example V.2.1.7. Let $\mathscr{A}$ be an abelian category, and let $* \in \{+, -, b, \varnothing\}$. We take $\mathscr{C} = \mathcal{C}^*(\mathscr{A})$, and W equal to the set of quasi-isomorphisms in $\mathscr{C}$. (See Definition IV.1.5.4.) Then $\mathscr{C}[W^{-1}]$ is denoted by $\mathrm{D}^*(\mathscr{A})$, and called the *derived category* of $\mathscr{A}$. More precisely, if $* = +$ (resp. $* = -$, $* = b$, $* = \varnothing$), we talk about the *bounded below* (resp. *bounded above*, resp. *bounded*, resp. *unbounded*) *derived category*.

The canonical functor $\mathcal{C}^*(\mathscr{A}) \to K^*(\mathscr{A})$ is surjective on objects and full, and W is the inverse image by this functor of the set W_1 of morphisms of $K^*(\mathscr{A})$ that induce isomorphisms on all the cohomology objects, so, by Proposition V.2.1.6, the derived category $\mathrm{D}^*(\mathscr{A})$ is canonically equivalent to $K^*(\mathscr{A})[W_1^{-1}]$. This is very useful because we know that $K^*(\mathscr{A})$ is a triangulated category, and we will show that $\mathrm{D}^*(\mathscr{A})$ inherits that structure.

V.2.2 Multiplicative systems

Let $\mathscr{C}$ and W be as in the preceding subsection.

Definition V.2.2.1. (i). We say that W is a *right multiplicative system* if it satisfies the following conditions:

(S1) For every $X \in \mathscr{C}$, we have $\mathrm{id}_X \in W$.

(S2) For all $s : X \to Y$ and $t : Y \to Z$ in W, we have $t \circ s \in Z$.

(S3) If $f : X \to Y$ and $s : X \to X'$ are two morphisms such that $s \in W$, there exists morphisms $t : Y \to Y'$ and $g : X' \to Y'$ such that $t \in W$ and $g \circ s = t \circ f$.

$$\begin{tikzcd} X' \arrow[r, dashed, "g"] & Y' \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y \arrow[u, dashed, "t"'] \end{tikzcd}$$

(S4) Let $f, g : X \to Y$ be two morphisms. If there exists $s : X' \to X$ in W such that $f \circ s = g \circ s$, then there exists $t : Y \to Y'$ such that $t \circ f = t \circ g$.

(ii). We say that W is a *left multiplicative system* if it satisfies the following conditions:

(S1) For every $X \in \mathscr{C}$, we have $\mathrm{id}_X \in W$.

(S2) For all $s : X \to Y$ and $t : Y \to Z$ in W, we have $t \circ s \in Z$.

(S3') If $f : X \to Y$ and $t : Y' \to Y$ are two morphisms such that $t \in W$, there exists morphisms $s : X' \to X$ and $g : X' \to Y'$ such that $s \in W$ and $t \circ g = f \circ s$.

$$\begin{tikzcd} X' \arrow[r, dashed, "g"] \arrow[d, dashed, "s"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(S4') Let $f, g : X \to Y$ be two morphisms. If there exists $t : Y \to Y'$ in W such that $t \circ f = t \circ g$, then there exists $s : X' \to X$ such that $f \circ s = g \circ s$.

(iii). We say that W is a *multiplicative system* if it is both a left and right multiplicative system.

If W is a right or left multiplicative system, then we can give a simpler construction of the localization $\mathscr{C}[W^{-1}]$.

Let $X \in \mathrm{Ob}(\mathscr{C})$. We denote by $X \backslash W$ (resp. W/X) the category whose objects are morphisms $s : X \to X'$ (resp. $s : X' \to X$) in W and whose morphisms are commutative diagrams $\begin{tikzcd} X \arrow[r, "s_1"] \arrow[dr, "s_2"'] & X_1' \arrow[d, "f"] \\ & X_2' \end{tikzcd}$ (resp. $\begin{tikzcd} X_1' \arrow[r, "s_1"] \arrow[d, "f"'] & X \\ X_2' \arrow[ur, "s_2"'] & \end{tikzcd}$). If W satisfies conditions (S1) and (S2), then we can see it as a subcategory of $\mathscr{C}$, with objects all the objects of $\mathscr{C}$ and morphisms the elements of W. Note that $X \backslash W$ and W/X differ from the slice categories of Definition I.2.2.6 because we are allowing the morphism f to be any morphism of $\mathscr{C}$.

Proposition V.2.2.2. *(i). If W is a right multiplicative system, then the category $X \setminus W$ is filtrant for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(ii). If W is a left multiplicative system, then the category $(W/X)^{\mathrm{op}}$ is filtrant for every $X \in \mathrm{Ob}(\mathscr{C})$.

Proof. It suffices to prove (i). Let $X \in \mathrm{Ob}(\mathscr{C})$. We check conditions (a), (b) and (c) of Definition I.5.6.1. The category $X\backslash W$ is nonempty, because it contains id_X. Let $s : X \to Y$ and $t : X \to Z$ be objects of $X\backslash W$. By condition (S3), there exist morphisms $t' : Y \to T$ and $s' : Z \to T$, with $s' \in W$, such that $t' \circ s = s' \circ t$. In particular, $s' \circ t : X \to T$ is an object of $X\backslash W$, and t' and s' define morphisms from $s : X \to Y$ and $t : X \to Z$ to $s' \circ t$. This shows condition (b). Now let $s : X \to Y$ and $t : X \to Z$ be objects of $X\backslash W$, and let $f, g : Y \to Z$ be morphisms in $X\backslash W$ between these two objects. This means that $f \circ s = t = g \circ s$. By condition (S4), there exists $s' : Z \to T$ in W such that $s' \circ f = s' \circ g$. Then $s' \circ t : X \to T$ is an object of $X\backslash W$, and s' is a morphism from $t : X \to Z$ to $s' \circ t : X \to T$ in $X\backslash W$ such that $s' \circ f = s' \circ g$. This shows condition (c).

$\square$

Definition V.2.2.3. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$. If W is a right multiplicative system, we define

$$\mathrm{Hom}_{\mathscr{C}_r}(X,Y) = \varinjlim_{(Y \to Y') \in \mathrm{Ob}(Y\backslash W)} \mathrm{Hom}_{\mathscr{C}}(X,Y').$$

If W is a left multiplicative system, we define

$$\mathrm{Hom}_{\mathscr{C}_l}(X,Y) = \varinjlim_{(X' \to X) \in \mathrm{Ob}((W/X)^{\mathrm{op}})} \mathrm{Hom}_{\mathscr{C}}(X',Y).$$

Note that we have canonical maps from $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ to $\mathrm{Hom}_{\mathscr{C}_r}(X,Y)$ and $\mathrm{Hom}_{\mathscr{C}_l}(X,Y)$ (whenever these sets are define).

By Propositions V.2.2.2 and I.5.6.2, we have the following descriptions of $\mathrm{Hom}_{\mathscr{C}_r}(X,Y)$ and $\mathrm{Hom}_{\mathscr{C}_l}(X,Y)$:

(1) Elements of $\mathrm{Hom}_{\mathscr{C}_r}(X,Y)$ are diagrams $\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "t"'] \end{tikzcd}$, with $t \in W$, modulo the following equivalence relation: the diagrams $\begin{tikzcd} & Y'_1 & \\ X \arrow[ur, "f_1"] & & Y \arrow[ul, "t_1"'] \end{tikzcd}$ and $\begin{tikzcd} & Y'_2 & \\ X \arrow[ur, "f_2"] & & Y \arrow[ul, "t_2"'] \end{tikzcd}$ are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & Y'_1 \arrow[d] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y'_3 & Y \arrow[ul, "t_1"'] \arrow[l, "t_3"'] \arrow[dl, "t_2"] \\ & Y'_2 \arrow[u] & \end{tikzcd}$$

with $t_3 \in W$.

(2) Elements of $\mathrm{Hom}_{\mathscr{C}_l}(X,Y)$ are diagrams
$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
, with $s \in W$, modulo the following equivalence relation: the diagrams
$$\begin{tikzcd} & X'_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & & Y \end{tikzcd}$$
and
$$\begin{tikzcd} & X'_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$
are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & X'_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X'_3 \arrow[l, "s_3"'] \arrow[r, "f_3"] \arrow[u] \arrow[d] & Y \\ & X'_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

with $s_3 \in W$.

Theorem V.2.2.4. *(i). Suppose that W is a right multiplicative system. Let $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, let $u \in \mathrm{Hom}_{\mathscr{C}_r}(X,Y)$ and $v \in \mathrm{Hom}_{\mathscr{C}_r}(Y,Z)$, and choose representatives $X \xrightarrow{f} Y' \xleftarrow{s} Y$ and $Y \xrightarrow{g} Z' \xleftarrow{t} Z$ of u and v, with $s, t \in W$. By condition (S3), we can find a commutative diagram*

$$\begin{tikzcd} & & T & & \\ & Y' \arrow[ur, "g'"] & & Z' \arrow[ul, "s'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \end{tikzcd}$$

with $s' \in W$, and we define $v \circ u \in \mathrm{Hom}_{\mathscr{C}_r}(X,Z)$ to be the element represented by $X \xrightarrow{g' \circ f} T \xleftarrow{s' \circ t} Z$.

a) This operation is well-defined and associative.

So we can define a category $\mathscr{C}_r$ by setting $\mathrm{Ob}(\mathscr{C}_r) = \mathrm{Ob}(\mathscr{C})$, using the sets $\mathrm{Hom}_{\mathscr{C}_r}(X,Y)$ as Hom *sets, and taking the composition law to be the one just define. We have an obvious functor $Q_r : \mathscr{C} \to \mathscr{C}_r$.*

b) *The functor $Q_r : \mathscr{C} \to \mathscr{C}_r$ is a localization of $\mathscr{C}$ by W.*

c) *Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Then $Q_r(f)$ is an isomorphism if and only if there exist morphisms $g : Y \to Z$ and $h : Z \to T$ in $\mathscr{C}$ such that both $g \circ f$ and $h \circ g$ are in W.*

(ii). Suppose that W is a left multiplicative system. Let $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, let $u \in \mathrm{Hom}_{\mathscr{C}_l}(X, Y)$ and $v \in \mathrm{Hom}_{\mathscr{C}_l}(Y, Z)$, and choose representatives $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $Y \xleftarrow{t} Y' \xrightarrow{g} Z$ of u and v, with $s, t \in W$. By condition (S3'), we can find a commutative diagram

$$\begin{tikzcd}
 & & T \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

with $t' \in W$, and we define $v \circ u \in \mathrm{Hom}_{\mathscr{C}_r}(X, Z)$ to be the element represented by $X \xleftarrow{s \circ t'} T \xrightarrow{g \circ f'} Z$.

a) *This operation is well-defined and associative.*

So we can define a category $\mathscr{C}_l$ by setting $\mathrm{Ob}(\mathscr{C}_l) = \mathrm{Ob}(\mathscr{C})$, using the sets $\mathrm{Hom}_{\mathscr{C}_l}(X, Y)$ as Hom *sets, and taking the composition law to be the one just define. We have an obvious functor $Q_l : \mathscr{C} \to \mathscr{C}_l$.*

b) *The functor $Q_l : \mathscr{C} \to \mathscr{C}_l$ is a localization of $\mathscr{C}$ by W.*

c) *Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Then $Q_l(f)$ is an isomorphism if and only if there exist morphisms $g : Z \to X$ and $h : T \to Z$ in $\mathscr{C}$ such that both $f \circ g$ and $g \circ h$ are in W.*

If W is a right (resp. left) multiplicative system, then we can (and will) identify $Q_r : \mathscr{C} \to \mathscr{C}_r$ (resp. $Q_l : \mathscr{C} \to \mathscr{C}_l$) and $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$. In particular, if W is a multiplicative system, then there is an equivalence of categories $F : \mathscr{C}_r \xrightarrow{\sim} \mathscr{C}_l$ such that $F \circ Q_r \simeq Q_l$.

Proof. It suffices to prove (i).

We first show the following fact, which we will call (*): If $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $s : X \to X'$ is in W, then composition on the right by s induces a bijective map $\alpha : \mathrm{Hom}_{\mathscr{C}_r}(X', Y) \to \mathrm{Hom}_{\mathscr{C}_r}(X, Y)$.

Indeed, let $u = X \xrightarrow{f} Y' \xleftarrow{t} Y$ be an element of $\mathrm{Hom}_{\mathscr{C}_r}(X, Y)$. By condition (S3), we can find a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y'' & \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y' \arrow[u, "t'"'] & Y \arrow[l, "t"'] \end{tikzcd}$$

with $t' \in W$. Then $u' = X' \xrightarrow{f'} Y'' \xleftarrow{t' \circ t} Y$ is an element of $\operatorname{Hom}_{\mathscr{C}_r}(X', Y)$, and $\alpha(u') = u$. This shows that α is surjective. Now let $u_1', u_2' \in \operatorname{Hom}_{\mathscr{C}_r}(X', Y)$ such that $\alpha(u_1') = \alpha(u_2')$. By Proposition V.2.2.2, we can choose representatives of u_1' and u_2' of the form $X' \xrightarrow{f_1} Y' \xleftarrow{t_1} Y$ and $X' \xrightarrow{f_2} Y' \xleftarrow{t_2} Y$; using the same proposition, after composing the morphisms f_1, f_2, t_1, t_2 with some morphism $Y' \to Y''$ in W, we may assume that $f_1 \circ s = f_2 \circ s$. Then condition (S4) says that there exists $t : Y' \to Z$ in W such that $t \circ f_1 = t \circ f_2$, and this implies that $u_1' = u_2'$. So α is injective.

Now, using (*), we see that we can rewrite the definition of the composition law

$$\operatorname{Hom}_{\mathscr{C}_r}(X, Y) \times \operatorname{Hom}_{\mathscr{C}_r}(Y, Z) \to \operatorname{Hom}_{\mathscr{C}_r}(X, Z)$$

as

$$\begin{aligned} \varinjlim_{Y \to Y'} \operatorname{Hom}_{\mathscr{C}_r}(X, Y') \times \varinjlim_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}_r}(Y, Z') &\simeq \varinjlim_{Y \to Y'} \big(\operatorname{Hom}_{\mathscr{C}_r}(X, Y') \times \varinjlim_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}_r}(Y, Z')\big) \\ &\overset{\sim}{\leftarrow} \varinjlim_{Y \to Y'} \big(\operatorname{Hom}_{\mathscr{C}_r}(X, Y') \times \varinjlim_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}_r}(Y', Z')\big) \\ &\to \varinjlim_{Y \to Y'} \varinjlim_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}_r}(X, Z') \\ &\simeq \varinjlim_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}_r}(X, Z'). \end{aligned}$$

(The first isomorphism comes from Proposition I.5.6.4 and the second from (*), the third map comes from the composition law in $\mathscr{C}$ and the fourth isomorphism is obvious.) In particular, the composition law of $\mathscr{C}_r$ is well-defined. To show that it is associative, consider objects $X_1, X_2, X_3, X_4 \in \operatorname{Ob}(\mathscr{C})$ and elements $u_1 \in \operatorname{Hom}_{\mathscr{C}_r}(X_1, X_2)$, $u_2 \in \operatorname{Hom}_{\mathscr{C}_r}(X_2, X_3)$ and $u_3 \in \operatorname{Hom}_{\mathscr{C}_r}(X_3, X_4)$. We choose representatives $X_i \xrightarrow{f_i} X_{i+1}' \xleftarrow{s_i} X_{i+1}$ of u_i for $1 \le i \le 3$, with $s_i \in W$. Using condition (S3), we can construct a commutative diagram

$$\begin{tikzcd} & & & X_4''' & & & \\ & & X_3'' \arrow[ur, "f_3''"] & & X_4'' \arrow[ul, "s_1''"'] & & \\ & X_2' \arrow[ur, "f_2'"] & & X_3' \arrow[ul, "s_1'"'] \arrow[ur, "f_3'"] & & X_4' \arrow[ul, "s_2'"'] & \\ X_1 \arrow[ur, "f_1"] & & X_2 \arrow[ul, "s_1"'] \arrow[ur, "f_2"] & & X_3 \arrow[ul, "s_2"'] \arrow[ur, "f_3"] & & X_4 \arrow[ul, "s_3"'] \end{tikzcd}$$

with $s_1', s_1'', s_2' \in W$. By definition of the composition law, the diagram $X_1 \xrightarrow{f_3''\circ f_2'\circ f_1} X_4''' \xleftarrow{s_1''\circ s_2'\circ s_3} X_4$ represents both $u_3 \circ (u_2 \circ u_1)$ and $(u_3 \circ u_2) \circ u_1$. This finishes the proof of part (a).

We prove part (b). By (*) and the Yoneda lemma (Corollary I.3.2.3), for every $s : X \to X'$ in W, the morphism $Q_r(s)$ is an isomorphism in $\mathscr{C}_r$; this proves property (a) of the localization. As in the proof of Theorem V.2.1.4, property (c) of the localization follows from the fact that every morphism in $\mathscr{C}_r$ is of the form $Q_r(s)^{-1} \circ Q_r(f)$, with $s \in W$ and f a morphism of $\mathscr{C}$. We prove property (b) of the localization. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor such that $F(s)$ is an isomorphism for every $s \in W$. We define a functor $F' : \mathscr{C}_r \to \mathscr{D}$ such that $F' \circ Q_r = F$ in the following way:

(1) On objects, F' is equal to F.

(2) Let $X, Y \in \mathrm{Ob}(\mathscr{C}_r) = \mathrm{Ob}(\mathscr{C})$, and let $u \in \mathrm{Hom}_{\mathscr{C}_r}(X, Y)$. Let $X \xrightarrow{f_1} Y_1' \xleftarrow{t_1} Y$ and $X \xrightarrow{f_2} Y_2' \xleftarrow{t_2} Y$ be two representatives of u. Then there exists a commutative diagram

$$\begin{tikzcd} & Y_1' \arrow[d] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3' & Y \arrow[ul, "t_1"'] \arrow[l, "t_3"'] \arrow[dl, "t_2"] \\ & Y_2' \arrow[u] & \end{tikzcd}$$

with $t_3 \in W$. In particular, we get $F(t_1)^{-1} \circ F(f_1) = F(t_3)^{-1} \circ F(f_3) = F(t_2)^{-1} \circ F(f_2)$, so we can set $F'(u) = F(t_1)^{-1} \circ F(f_1)$. It is easy to check that this does respect composition.

We finally prove part (c). Let $f : X \to Y$ be a morphism of $\mathscr{C}$. If there exist morphisms $g : Y \to Z$ and $h : Z \to T$ in $\mathscr{C}$ such that both $g \circ f$ and $h \circ g$ are in W, then $Q_r(h) \circ Q_r(g)$ and $Q_r(g) \circ Q_r(f)$ are isomorphisms, and this implies that $Q_r(f)$ is an isomorphism (by Corollary I.3.2.9 for example). Conversely, suppose that $Q_r(f)$ is an isomorphism, and let $u \in \mathrm{Hom}_{\mathscr{C}_r}(Y, X)$ be its inverse. We choose a representative $Y \xrightarrow{g'} X' \xleftarrow{t} X$ of u. The fact that $u \circ f = \mathrm{id}_X$ in $\mathscr{C}_r$ means that there exists a morphism $s : X' \to Y$ in W such that $s \circ g \circ f : X \to Y$ is also in W. So, taking $g = s \circ g' : Y \to Z$, we have $g \circ f \in Z$. Also, the morphism $Q_r(g)$ is invertible in $\mathscr{C}_r$, so applying what we just did to g gives a morphism $h : Z \to T$ in $\mathscr{C}$ such that $h \circ g \in W$.

□

V.2.3 Localization of a subcategory

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$. If $\mathscr{I}$ is a subcategory of $\mathscr{C}$ and $W_{\mathscr{I}}$ is the set of morphisms of $\mathscr{I}$ that are in W, then the universal property of the localization $\mathscr{I}[W_{\mathscr{I}}^{-1}]$

gives a canonical functor $\iota : \mathscr{I}[W_{\mathscr{I}}^{-1}] \to \mathscr{C}[W^{-1}]$ extending the inclusion $\mathscr{I} \subset \mathscr{C}$. We want to give conditions for this functor to be fully faithful or an equivalence of categories.

Proposition V.2.3.1. *Suppose that $\mathscr{I}$ is a full subcategory of $\mathscr{C}$, that W is a right multiplicative system and that, for every $s : X \to Y$ in W such that $X \in \mathrm{Ob}(\mathscr{I})$, there exists a morphism $f : Y \to Z$ with $Z \in \mathrm{Ob}(\mathscr{I})$ and $f \circ s \in W$. Then $W_{\mathscr{I}}$ is a right multiplicative system, and the canonical functor $\iota : \mathscr{I}[W_{\mathscr{I}}^{-1}] \to \mathscr{C}[W^{-1}]$ is fully faithful.*

Proof. For the proof that $W_{\mathscr{I}}$ is a right multiplicative system, see problem A.8.1.

Let $X, Y \in \mathrm{Ob}(\mathscr{I})$. By Theorem V.2.2.4(i) and the fullness of $\mathscr{I}$ in $\mathscr{C}$, we have

$$\mathrm{Hom}_{\mathscr{I}[W_{\mathscr{I}}^{-1}]}(X,Y) = \varinjlim_{(Y \to Y') \in \mathrm{Ob}(Y \backslash W_{\mathscr{I}})} \mathrm{Hom}_{\mathscr{C}}(X, Y')$$

and

$$\mathrm{Hom}_{\mathscr{C}[W^{-1}]}(X,Y) = \varinjlim_{(Y \to Y') \in \mathrm{Ob}(Y \backslash W)} \mathrm{Hom}_{\mathscr{C}}(X, Y').$$

By the hypotheses of the proposition, for every object $s : Y \to Y'$ of $Y \backslash W$, there exists a morphism $f : Y' \to Y''$ from s to the object $f \circ s : Y \to Y''$ of $Y \backslash W$ such that $f \circ s$ is in the full subcategory $Y \backslash W_{\mathscr{I}}$. [3] Using Proposition I.5.6.2, we see easily that this implies that the canonical morphism

$$\varinjlim_{(Y \to Y') \in \mathrm{Ob}(Y \backslash W_{\mathscr{I}})} \mathrm{Hom}_{\mathscr{C}}(X, Y') \to \varinjlim_{(Y \to Y') \in \mathrm{Ob}(Y \backslash W)} \mathrm{Hom}_{\mathscr{C}}(X, Y')$$

is bijective, which is what we wanted to prove.

□

Corollary V.2.3.2. *Suppose that $\mathscr{I}$ is a full subcategory of $\mathscr{C}$, that W is a right multiplicative system and that, for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $s : X \to Y$ in W such that $Y \in \mathrm{Ob}(\mathscr{I})$. Then the canonical functor $\iota : \mathscr{I}[W_{\mathscr{I}}^{-1}] \to \mathscr{C}[W^{-1}]$ is an equivalence of categories.*

Proof. We already know that ι is fully faithful by Proposition V.2.3.1 (whose hypotheses follow from those of the corollary). As morphisms of W become isomorphisms in $\mathscr{C}[W^{-1}]$, the last hypothesis of the corollary implies that ι is essentially surjective.

□

[3] We say that the full subcategory $Y \backslash W_{\mathscr{I}}$ is *cofinal* in $Y \backslash W$.

V.2.4 Localization of functors

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$. We fix a localization $W : \mathscr{C} \to \mathscr{C}[W^{-1}]$ of $\mathscr{C}$ by W.

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. In general, the functor F does not factor through $\mathscr{C}[W^{-1}]$, so we introduce the following notions:

Definition V.2.4.1. (i). A *right localization* of F by W is a functor $F_W : \mathscr{C}[W^{-1}] \to \mathscr{D}$ and a morphism of functors $\tau : F \to F_W \circ Q$ such that, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, the map

$$\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}[W^{-1}],\mathscr{D})}(F_W, G) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q)$$

sending $u : F_W \to G$ to $\tau(u \circ Q)$ is bijective. If F has a right localization, we say that F is *right localizable*.

(ii). A *left localization* of F by W is a functor $F_W : \mathscr{C}[W^{-1}] \to \mathscr{D}$ and a morphism of functors $\tau : F_W \circ Q \to F$ such that, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, the map

$$\mathrm{Hom}_{\mathrm{Func}(\mathscr{C}[W^{-1}],\mathscr{D})}(G, F_W) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(G \circ Q, F)$$

sending $u : G \to F_W$ to $\tau(u \circ Q)$ is bijective. If F has a left localization, we say that F is *left localizable*.

In other words, a right (resp. left) localization (F_W, τ) of F is a couple representing the functor $\mathrm{Func}(\mathscr{C}_W, \mathscr{D}) \to \mathbf{Set}$, $G \longmapsto \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q)$ (resp. the functor $\mathrm{Func}(\mathscr{C}_W, \mathscr{D})^{\mathrm{op}} \to \mathbf{Set}$, $G \longmapsto \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(G \circ Q, F)$). In particular, by Corollaries I.3.2.5 and I.3.2.8, right and left localizations are unique up to unique isomorphism if they exist.

Remark V.2.4.2. If F sends every element of W to an isomorphism in $\mathscr{D}$, then the functor $F_W : \mathscr{C}[W^{-1}] \to \mathscr{D}$ given by the universal property of the localization is both a right and left localization of F.

Remark V.2.4.3. A right (resp. left) localization of F by W is also called a *left Kan extension* (resp. *right Kan extension*) of F along the functor $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$. Note that right and left are exchanged, and this is not a mistake.

Proposition V.2.4.4. *Let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$ and $W_{\mathscr{I}}$ be the set of morphisms of $\mathscr{I}$ that are in W. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose that:*

(a) W is a right multiplicative system;

(b) for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $s : X \to Y$ in W such that $Y \in \mathrm{Ob}(\mathscr{I})$;

(c) for every $s \in W_{\mathscr{I}}$, the morphism $F(s)$ is an isomorphism.

Then F is right localizable. In fact, if $\iota : \mathscr{I}[W_{\mathscr{I}}^{-1}] \to \mathscr{C}[W^{-1}]$ is the equivalence of categories of Corollary V.2.3.2, if ι^{-1} is a quasi-inverse of ι and if $F_{W_{\mathscr{I}}} : \mathscr{I}[W_{\mathscr{I}}^{-1}] \to \mathscr{D}$ is the functor extending $F_{|\mathscr{I}}$ whose existence is given by condition (c) and by the universal property of the localization, then the functor $F_{W_{\mathscr{I}}} \circ \iota^{1} : \mathscr{C}[W^{-1}] \to \mathscr{D}$ is a right localization of F.

Proof. We write $F_W = F_{W_\mathscr{I}} \circ \iota^{-1}$. We also denote by ι the inclusion functor $\mathscr{I} \subset \mathscr{C}$ and by $Q_\mathscr{I} : \mathscr{I} \to \mathscr{I}[W_\mathscr{I}^{-1}]$ the localization functor. We have a (non-commutative) diagram of categories and functors:

$$\begin{tikzcd}
\mathscr{I} \arrow[r, "\iota"] \arrow[d, "Q_\mathscr{I}"'] & \mathscr{C} \arrow[d, "Q"'] \arrow[ddr, "F"] & \\
\mathscr{I}[W_\mathscr{I}^{-1}] \arrow[r, "\iota", "\sim"'] \arrow[drr, "F_{W_\mathscr{I}}"'] & \mathscr{C}[W^{-1}] \arrow[dr, "F_W"] & \\
 & & \mathscr{D}
\end{tikzcd}$$

Let $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$ be a functor. By problem A.8.4, the map

$$\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F \circ \iota, G \circ Q \circ \iota)$$

induced by composition on the right by ι is bijective.

Using this fact and the universal property of the functor $F_{W_\mathscr{I}}$, we get, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, a chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q) &\overset{\sim}{\to} \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F \circ \iota, G \circ Q \circ \iota) \\
&\simeq \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F_{W_\mathscr{I}} \circ Q_\mathscr{I}, G \circ \iota \circ Q_\mathscr{I}) \\
&\simeq \mathrm{Hom}_{\mathrm{Func}(\mathscr{I}[W_\mathscr{I}^{-1}],\mathscr{D})}(F_{W_\mathscr{I}}, G \circ \iota) \\
&\simeq \mathrm{Hom}_{\mathrm{Func}(\mathscr{C}[W^{-1}],\mathscr{D})}(F_W, G).
\end{aligned}$$

This shows that F_W is a right localization of F by W.

□

V.3 Localization of triangulated categories

In this section, we fix a triangulated category $(\mathscr{D}, T)$.

V.3.1 Null systems

Instead of fixing a set of morphisms to invert, we can fix a set of objects to send to 0. If this set satisfies some natural conditions, we can do this in a controlled way, and the result will still be a triangulated category. [4]

Definition V.3.1.1. A *null system* in $\mathscr{D}$ is a set $\mathscr{N}$ of objects of $\mathscr{D}$ such that:

[4] There is a similar theory for abelian categories, called the Serre quotient by a thick subcategory (although it is also a localization and not a quotient in the sense of Definition IV.1.3.1).

(N1) $0 \in \mathscr{N}$;

(N2) for every $X \in \mathrm{Ob}(\mathscr{C})$, we have $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$;

(N3) if $X \to Y \to Z \to T(X)$ is a distinguished triangle and if $X, Y \in \mathscr{N}$, then $Z \in \mathscr{N}$.

Remark V.3.1.2. If $\mathscr{N}$ is a null system, then it is stable by isomorphism. See problem A.8.3(a).

Definition V.3.1.3. Let $\mathscr{N}$ be a null system. We denote by $W_{\mathscr{N}}$ the set of morphisms $f : X \to Y$ in $\mathscr{D}$ such that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z \in \mathscr{N}$.

Theorem V.3.1.4. *Let $\mathscr{N}$ be a null system in $\mathscr{D}$, and let $Q : \mathscr{D} \to \mathscr{D}[W_{\mathscr{N}}^{-1}]$ be a localization of $\mathscr{D}$ by $W_{\mathscr{N}}$; we also write $\mathscr{D}/\mathscr{N} = \mathscr{D}[W_{\mathscr{N}}^{-1}]$.*

(i). The set $W_{\mathscr{N}}$ is a multiplicative system.

(ii). Define $T_{\mathscr{N}} : \mathscr{D}/\mathscr{N} \to \mathscr{D}/\mathscr{N}$ by $T_{\mathscr{N}}(X) = T(X)$ for every $x \in \mathrm{Ob}(\mathscr{D}/\mathscr{N}) = \mathrm{Ob}(\mathscr{D})$ and $T_{\mathscr{N}}(Q(f) \circ Q(s)^{-1}) = Q(T(f)) \circ Q(T(s))^{-1}$ for f a morphism of $\mathscr{D}$ and $s \in W_{\mathscr{N}}$. Then $T_{\mathscr{N}}$ is well-defined and an auto-equivalence of $\mathscr{D}/\mathscr{N}$, and we have $T_{\mathscr{N}} \circ Q = Q \circ T$.

(iii). We say that a triangle of $(\mathscr{D}/\mathscr{N}, T_{\mathscr{N}})$ is distinguished if it is isomorphic to the image by Q of a distinguished triangle of $\mathscr{D}$. Then $\mathscr{D}/\mathscr{N}$ is a triangulated category, and the functor $Q : \mathscr{D} \to \mathscr{D}/\mathscr{N}$ is triangulated.

(iv). If $X \in \mathscr{N}$, then $Q(X) = 0$.

(v). Let $F : \mathscr{D} \to \mathscr{D}'$ be a triangulated functor such that $F(X) = 0$ for every $X \in \mathscr{N}$. Then there exists a functor $F_{\mathscr{N}} : \mathscr{D}/\mathscr{N} \to \mathscr{D}'$ such that $F_{\mathscr{N}} \circ Q \simeq F$, and $F_{\mathscr{N}}$ is unique up to a unique isomorphism of functors.

Proof. Point (i) is proved in problem A.8.3 and points (ii) and (iii) are proved in problem A.8.5. If $X \in \mathscr{N}$, then the morphism $0 \to X$ is in $W_{\mathscr{N}}$ (because the triangle $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0$ is diatinguished), so $0 = Q(0) \to Q(X)$ is an isomorphism in $\mathscr{D}/\mathscr{N}$; this proves (iv). We show (v). It suffices to prove that the functor F sends every element of $W_{\mathscr{N}}$ to an isomorphism. Let $f \in W_{\mathscr{N}}$, and consider a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$, with $Z \in \mathscr{N}$. Then the triangle $F(X) \xrightarrow{F(f)} F(Y) \to F(Z) = 0 \to F(T(X))$ is distinguished, so $F(f)$ is an isomorphism by problem A.8.2.

□

We also have a triangulated version of Proposition V.2.3.1 about the localization of a subcategory.

Proposition V.3.1.5. *Let $\mathscr{N}$ be a null system in $\mathscr{D}$ and $\mathscr{I}$ be a full triangulated subcategory of $\mathscr{D}$. Then $\mathscr{N}_{\mathscr{I}} = \mathscr{N} \cap \mathrm{Ob}(\mathscr{I})$ is a null system in $\mathscr{I}$, and $W_{\mathscr{N}_{\mathscr{I}}}$ is the set of morphisms of $\mathscr{I}$ that are in $W_{\mathscr{N}}$. Suppose that one of the following conditions is satisfied:*

(a) *For any morphism* $Y \to Z$ *with* $Y \in \mathrm{Ob}(\mathscr{I})$ *and* $Z \in \mathscr{N}$*, there exists a factorization* $Y \to Z' \to Z$ *with* $Z' \in \mathscr{N}_{\mathscr{I}}$.

(b) *For any morphism* $Z \to Y$ *with* $Y \in \mathrm{Ob}(\mathscr{I})$ *and* $Z \in \mathscr{N}$*, there exists a factorization* $Z \to Z' \to Y$ *with* $Z' \in \mathscr{N}_{\mathscr{I}}$.

Then the canonical functor $\iota : \mathscr{I}/\mathscr{N}_{\mathscr{I}} \to \mathscr{D}/\mathscr{N}$ *is fully faithful.*

Proof. The first statement follows immediately from the definition and from Corollary V.1.1.13.

To prove the second statement, we want to use Proposition V.2.3.1, so we have to check the hypothesis of this proposition. We treat the case where (b) holds (the other case follows by considering the opposite categories). Let $s : X \to Y$ be a morphism of $W_{\mathscr{N}}$ such that $X \in \mathrm{Ob}(\mathscr{I})$, and complete s to a distinguished triangle $X \xrightarrow{s} Y \to Z \xrightarrow{f} T(X)$ with $Z \in \mathscr{Z}$. The object $T(X)$ is isomorphic to an object of $\mathscr{I}$, so, by (b), the morphism $f : Z \to T(X)$ factors as $Z \xrightarrow{u} Z' \xrightarrow{f'} T(X)$ with $Z' \in \mathscr{N}_{\mathscr{I}}$. We embed $f' : Z' \to T(X)$ into a distinguished triangle $X \to Y' \to Z' \to T(X)$ with $Y' \in \mathrm{Ob}(\mathscr{I})$; by (TR4), there exists a morphism of distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "u"'] & T(X) \arrow[d, "\mathrm{id}_{T(X)}"'] \\
X \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X)
\end{tikzcd}$$

As $Z' \in \mathscr{N}$, this means that $v \circ s \in W_{\mathscr{N}}$.

□

Corollary V.3.1.6. *With the notation of Proposition V.3.1.5, assume that one of the following two conditions holds:*

(a') *For every* $X \in \mathrm{Ob}(\mathscr{D})$*, there exists a distinguished triangle* $X \to Y \to Z \to T(X)$ *with* $Y \in \mathrm{Ob}(\mathscr{I})$ *and* $Z \in \mathscr{N}$.

(b') *For every* $X \in \mathrm{Ob}(\mathscr{D})$*, there exists a distinguished triangle* $Y \to X \to Z \to T(X)$ *with* $Y \in \mathrm{Ob}(\mathscr{I})$ *and* $Z \in \mathscr{N}$.

Then the canonical functor $\iota : \mathscr{I}/\mathscr{N}_{\mathscr{I}} \to \mathscr{D}/\mathscr{N}$ *is an equivalence of categories.*

Proof. Condition (a') immediately implies the hypothesis of Corollary V.2.3.2, so we get the result. If condition (b') holds, then condition (a') holds in the opposite category.

□

V.3.2 Localization of triangulated functors

Proposition V.3.2.1. *Let $\mathscr{N}$ be a null system in $\mathscr{D}$ and let $F : \mathscr{D} \to \mathscr{D}'$ be a triangulated functor. Assume that there exists a full triangulated subcategory $\mathscr{I}$ of $\mathscr{D}$ such that:*

(a) For every $X \in \mathrm{Ob}(\mathscr{D})$, there exists a distinguished triangle $X \to Y \to Z \to T(X)$ with $Y \in \mathrm{Ob}(\mathscr{I})$ and $Z \in \mathscr{N}$.

(b) For any $Y \in \mathscr{N}_{\mathscr{I}}$, we have $F(Y) = 0$.

Then F is right localizable, and its right localization is also a triangulated functor.

Proof. Conditions (a) and (b) imply the hypotheses of Proposition V.2.4.4, so F is right localizable, and we get a right localization of F by taking $F_{\mathscr{N}_{\mathscr{I}}} \circ \iota^{-1}$, where ι^{-1} is a quasi-inverse of the equivalence of categories $\iota : \mathscr{I}/N_{\mathscr{I}} \to \mathscr{D}/\mathscr{N}$ and $F_{N_{\mathscr{I}}} : \mathscr{I}/\mathscr{N}_{\mathscr{I}} \to \mathscr{D}'$ comes from condition (b) and the universal property of $\mathscr{I}/\mathscr{N}_{\mathscr{I}}$. As both ι and $F_{\mathscr{N}_{\mathscr{I}}}$ are triangulated, so is $F_{\mathscr{N}_{\mathscr{I}}} \circ \iota^{-1}$.

□

The case of bifunctors

Suppose that we have two triangulated categories $\mathscr{D}$ and $\mathscr{D}'$ and null systems $\mathscr{N}$ and $\mathscr{N}'$ in $\mathscr{D}$ and $\mathscr{D}'$ respectively. Then $(\mathscr{D}/\mathscr{N}) \times (\mathscr{D}'/\mathscr{N}')$ is a localization of $\mathscr{D} \times \mathscr{D}'$ by the set of morphisms $W_{\mathscr{N}} \times W_{\mathscr{N}'}$.

The folowing proposition is easy.

Proposition V.3.2.2. *Let $\mathscr{I} \subset \mathscr{D}$ and $\mathscr{I}' \subset \mathscr{D}'$ be full triangulated subcategories and $F : \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ be a bifunctor that is triangulated in each variable. Suppose that the following conditions hold:*

(a) For any $X \in \mathrm{Ob}(\mathscr{D})$, there exists a distinguished triangle $X \to Y \to Z \to T(X)$ with $Y \in \mathrm{Ob}(\mathscr{I})$ and $Z \in \mathscr{N}$.

(b) For any $X' \in \mathrm{Ob}(\mathscr{D}')$, there exists a distinguished triangle $X' \to Y' \to Z' \to T(X')$ with $Y' \in \mathrm{Ob}(\mathscr{I}')$ and $Z' \in \mathscr{N}'$.

(c) If $Y \in \mathrm{Ob}(\mathscr{I})$ and $Y' \in \mathscr{N}' \cap \mathrm{Ob}(\mathscr{I}')$, then $F(Y, Y') = 0$.

(d) If $Y \in \mathscr{N} \cap \mathrm{Ob}(\mathscr{I})$ and $Y' \in \mathrm{Ob}(\mathscr{I}')$, then $F(Y, Y') = 0$.

Then F is right localizable and its right localization is triangulated in each variable. Moreover, if we denote by $F_{\mathscr{N},\mathscr{N}'}$ a right localization of F, we can calculate it in the following way: If $X \in \mathrm{Ob}(\mathscr{D})$ and $X' \in \mathrm{Ob}(\mathscr{D}')$, choose a morphism $X \to Y$ in $W_{\mathscr{N}}$ and a morphism $X' \to Y'$ in $W_{\mathscr{N}'}$ such that $Y \in \mathrm{Ob}(\mathscr{I})$ and $Y' \in \mathrm{Ob}(\mathscr{I}')$. Then

$$F_{\mathscr{N},\mathscr{N}'}(X, Y) = F(Y, Y').$$

V.4 Derived category of an abelian category

In this section, we fix an abelian category $\mathscr{A}$.

V.4.1 Definition

We already defined the derived category of $\mathscr{A}$ in Example V.2.1.7, but we did not have many tools to study it then.

Definition V.4.1.1. Let $* \in \{+, -, b, \varnothing\}$. We denote by $\mathscr{N}^*(\mathscr{A})$ the set of objects X of $K^*(\mathscr{A})$ such that $\mathrm{H}^n(X) = 0$ for every $n \in \mathbb{Z}$.

Proposition V.4.1.2. *The set $W_{\mathscr{N}^*(\mathscr{A})}$ is the set of quasi-isomorphisms in $K^*(\mathscr{A})$ (that is, morphisms that induce isomorphisms on all cohomology objects), so the derived category $D^*(\mathscr{A})$ is canonically equivalent to $K^*(\mathscr{A})/\mathscr{N}^*(\mathscr{A})$. In particular, we have a natural structure of triangulated category on $D^*(\mathscr{A})$ that makes the localization functor $Q : K^*(\mathscr{A}) \to D^*(\mathscr{A})$ a triangulated functor.*

Proof. The first statement follows from the long exact sequence of cohomology, and the rest of the proposition is an immediate consequence of this.

□

We usually omit the notation "Q" and use the same notation for complexes and morphisms between them and for their images in the derived category.

We collect some useful facts about derived categories.

Proposition V.4.1.3. *(i). For every $n \in \mathbb{Z}$, the functor $\mathrm{H}^n : \mathcal{C}^*(\mathscr{A}) \to \mathscr{A}$ induces a cohomological functor $\mathrm{H}^n : \mathrm{D}^*(\mathscr{A}) \to \mathscr{A}$.*

(ii). Let f be a morphism in $\mathcal{C}^*(\mathscr{A})$ (resp. in $K^*(\mathscr{A})$). Then f becomes an isomorphism in $D^*(\mathscr{A})$ if and only if it is a quasi-isomorphism.

(iii). A morphism $f : X \to Y$ in $\mathrm{D}^*(\mathscr{A})$ is an isomorphism if and only if $\mathrm{H}^n(f) : \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ is an isomorphism for every $n \in \mathbb{Z}$.

(iv). Let $0 \to X \xrightarrow{f} Y \to Z \to 0$ be an exact sequence in $\mathcal{C}^*(\mathscr{A})$. Then there is a morphism $Z \to X[1]$ in $D^*(\mathscr{A})$ such that $X \to Y \to Z \to X[1]$ is a distinguished triangle isomorphic to the image in $D^*(\mathscr{A})$ of the mapping cone triangle $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1]$.

Proof. (i). The functor H^n sends quasi-isomorphisms to isomorphisms, so it factors through a functor $\mathrm{D}^*(\mathscr{A}) \to \mathscr{A}$, that we still denote by H^n. The fact that this functor is cohomological follows from the definition of distinguished triangles in $\mathrm{D}^*(\mathscr{A})$ and from the long exact sequence of cohomology of a mapping cone triangle (Corollary IV.2.2.8).

(ii). If f is a morphism of complexes of objects of $\mathscr{A}$, whether or not it is a quasi-isomorphism depends only on its image in $K^*(\mathscr{A})$, so it suffices to consider morphisms in $K^*(\mathscr{A})$.

It is clear that quasi-isomorphisms in $K^*(\mathscr{A})$ become isomorphisms in $\mathrm{D}^*(\mathscr{A})$ (by definition of a localization). Conversely, let $f : X \to Y$ be a morphism of $K^*(\mathscr{A})$ that becomes an isomorphism in $\mathrm{D}^*(\mathscr{A})$. By Theorem V.2.2.4(i)(c), there exists morphisms $g : Y \to Z$ and $h : Z \to T$ in $K^*(\mathscr{A})$ such that $h \circ g$ and $g \circ f$ are quasi-isomorphisms. Let $n \in \mathbb{Z}$. Then $\mathrm{H}^n(h) \circ \mathrm{H}^n(g)$ and $\mathrm{H}^n(g) \circ \mathrm{H}^n(f)$ are isomorphisms in $\mathscr{A}$, so $\mathrm{H}^n(g)$ is nijective and surjective, hence it is an isomorphism, and this implies that $\mathrm{H}^n(f)$ is also an isomorphism.

(iii). It is obvious that an isomorphism of $\mathrm{D}^*(\mathscr{A})$ induces an isomorphism in cohomology. Conversely, let $f : X \to Y$ be a morphism of $\mathrm{D}^*(\mathscr{A})$ and suppose that $\mathrm{H}^n(f)$ is an isomorphism for every $n \in \mathbb{Z}$. We can write $f = g \circ s^{-1}$, where $g : X \to Y'$ and $s : Y \to Y'$ are morphisms of $K^*(\mathscr{A})$ and s is a quasi-isomorphism. Then $\mathrm{H}^n(g) = \mathrm{H}^(f) \circ \mathrm{H}^n(s)$ is an isomorphism for every $n \in \mathbb{Z}$, so g is a quasi-isomorphism, so it becomes an isomorphism in $\mathrm{D}^*(\mathscr{A})$, and f is an isomorphism in $\mathrm{D}^*(\mathscr{A})$.

(iv). By Proposition V.1.2.3(ii), there exists a morphism of complexes $u : \mathrm{Mc}(f) \to Z$ such that $u \circ \alpha(f) = g$ and u is a quasi-isomorphism. Remember that we denote by Q the localization functor $\mathcal{C}^*(\mathscr{A}) \to \mathrm{D}^*(\mathscr{A})$. The morphism $Q(u)$ is an isomorphism, so we get a triangle $Q(X) \overset{Q(f)}{\to} Q(Y) \overset{Q(g)}{\to} Q(Z) \overset{Q(u)^{-1} \circ Q(\beta(f))}{\to} Q(X)[1]$ in $\mathrm{D}^*(\mathscr{A})$, and this triangle is distinguished because it is isomorphic to the image by Q of the mapping cone triangle $X \overset{f}{\to} Y \overset{\alpha(f)}{\to} \mathrm{Mc}(f) \overset{\beta(f)}{\to} X[1]$.

□

V.4.2 Truncation functors

Definition V.4.2.1. Let $n \in \mathbb{Z}$. We define the truncation functors $\tau^{\leq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^{\leq n}(\mathscr{A})$ and $\tau^{\geq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^{\geq n}(\mathscr{A})$ by

$$\tau^{\leq n}(X) = (\ldots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}(d_X^n) \to 0 \to 0 \to \ldots)$$

and

$$\tau^{\geq n}(X) = (\ldots \to 0 \to 0 \to \mathrm{Coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \ldots).$$

For every complex X, the identity morphism of X induces morphisms $\tau^{\leq n}(X) \to X$ and $X \to \tau^{\geq n}(X)$, that are actually morphisms of functors. The reason we truncated the way we did (and not by brutally making the X^i equal to 0 for $i < n$ or $i > n$) is the following result, whose proof follows immediately from the definition of the cohomology objects.

Proposition V.4.2.2. *The morphism $\tau^{\leq n}(X) \to X$ induces an isomorphism on H^k for $k \leq n$ (and is zero on H^k for $k > n$), while the morphism $X \to \tau^{\geq n}(X)$ induces an isomorphism on H^k for $k \geq n$ (and is zero on H^k for $k < n$).*

In particular, if $f : X \to Y$ is a quasi-isomorphism in $\mathcal{C}(\mathscr{A})$, then $\tau^{\leq n}(f)$ and $\tau^{\geq n}(f)$ are also quasi-isomorphisms.

Using the universal property of the localization, we get the following corollary.

Corollary V.4.2.3. *The truncation functors $\tau^{\leq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^-(\mathscr{A})$ and $\tau^{\geq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^+(\mathscr{A})$ induce functors $\tau^{\leq n} : \mathrm{D}(\mathscr{A}) \to \mathrm{D}^-(\mathscr{A})$ and $\tau^{\geq n} : \mathrm{D}(\mathscr{A}) \to \mathrm{D}^+(\mathscr{A})$. Similarly, the truncation functors $\tau^{\leq n} : \mathcal{C}^+(\mathscr{A}) \to \mathcal{C}^b(\mathscr{A})$ and $\tau^{\geq n} : \mathcal{C}^-(\mathscr{A}) \to \mathcal{C}^b(\mathscr{A})$ induce functors $\tau^{\leq n} : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^b(\mathscr{A})$ and $\tau^{\geq n} : \mathrm{D}^-(\mathscr{A}) \to \mathrm{D}^b(\mathscr{A})$.*

Corollary V.4.2.4. *The inclusions $\mathcal{C}^b(\mathscr{A}) \subset \mathcal{C}^\pm(\mathscr{A}) \subset \mathcal{C}(\mathscr{A})$ induce fully faithful functors $\mathrm{D}^b(\mathscr{A}) \to \mathrm{D}^\pm(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$, and the essential image of $\mathrm{D}^b(\mathscr{A})$ (resp. $\mathrm{D}^+(\mathscr{A})$, resp. $\mathrm{D}^-(\mathscr{A})$) is the full subcategory with objects the $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ such that $\mathrm{H}^n(X) = 0$ for $|n| >> 0$ (res. $n << 0$, resp. $n >> 0$).*

Similary, if $[a, b]$ is an interval with $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ and if $W_{[a,b]}$ is the set of morphisms of $\mathcal{C}^{[a,b]}(\mathscr{A})$ that are quasi-isomorphisms, then the inclusion $\mathcal{C}^{[a,b]}(\mathscr{A}) \subset \mathcal{C}(\mathscr{A})$ induces a fully faithful functor $\mathcal{C}^{[a,b]}(\mathscr{A})[W_{[a,b]}^{-1}] \to \mathrm{D}(\mathscr{A})$ whose essential image is the full subcategory with objects the $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ such that $\mathrm{H}^n(X) = 0$ for $n \not\in [a, b]$.

Proof. We apply Proposition V.2.3.1. All the cases are similar, let us treat for example that of the inclusion $\mathcal{C}^+(\mathscr{A}) \subset \mathcal{C}(\mathscr{A})$. Let $f : X \to Y$ be a quasi-isomorphism in $\mathcal{C}(\mathscr{A})$ with $X \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{A}))$. We want to find $g : Y \to Z$ such that $Z \in \mathcal{C}^+(\mathscr{A})$ and $g \circ f$ is a quasi-isomorphism. Let $n \in \mathbb{Z}$ be such that $\mathrm{H}^m(X) = 0$ for $m < n$, let $Z = \tau^{\geq n}(Y)$ and let $g : Y \to Z$ be the morphism given by the morphism of functors $\mathrm{id}_{\mathcal{C}(\mathscr{A})} \to \tau^{\geq m}$. We claim that $g \circ f$ is a quasi-isomorphism. Indeed, if $m < n$, then $\mathrm{H}^m(X) = 0$ and $\mathrm{H}^m(Z) = 0$, so $\mathrm{H}^m(g \circ f)$ is an isomorphism. If $m \geq n$, then $\mathrm{H}^m(f)$ is an isomorphism by hypothesis, and $\mathrm{H}^m(g)$ is an isomorphism by Proposition V.4.2.2, so $\mathrm{H}^m(g \circ f)$ is an isomorphism.

□

From now on, we will identify $\mathrm{D}^b(\mathscr{A})$, $\mathrm{D}^+(\mathscr{A})$ and $\mathrm{D}^-(\mathscr{A})$ to their essential images in $\mathrm{D}(\mathscr{A})$.

Remark V.4.2.5. If $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$, the following conditions are equivalent:

(a) $X \in \mathrm{Ob}(\mathrm{D}^+(\mathscr{A}))$ (resp. $X \in \mathrm{Ob}(\mathrm{D}^-(\mathscr{A}))$, $X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))$);

(b) $\mathrm{H}^n(X) = 0$ for $n << 0$ (resp. $n >> 0$, resp. $|n| >> 0$);

(c) the morphism $X \to \tau^{\geq n}(X)$ (resp. the morphism $\tau^{\leq n}(X) \to X$, resp. both of them) is an isomorphism for $n << 0$ (resp. $n >> 0$, resp. $|n| >> 0$).

If $a \leq b$ are elements of $\mathbb{Z} \cup \{\pm\infty\}$, we also write $\mathrm{D}^{[a,b]}(\mathscr{A})$ for the full subcategory of $\mathrm{D}(\mathscr{A})$ with objects the $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ such that $\mathrm{H}^n(X)$ for $n \not\in [a, b]$. If $b = +\infty$ (resp. $a = -\infty$), we also write $\mathrm{D}^{\geq a}$ (resp. $\mathrm{D}^{\leq b}$) instead of $\mathrm{D}^{[a,b]}$).

By the proposition, the category $\mathrm{D}^{[a,b]}(\mathscr{A})$ is the localization of $\mathcal{C}^{[a,b]}(\mathscr{A})$ by the quasi-isomorphisms in this category. In particular, if $a = b = 0$, we see that $\mathrm{D}^{[0,0]}(\mathscr{A})$ is the localization of $\mathcal{C}^{[0,0]}(\mathscr{A}) = \mathscr{A}$ by the set of isomorphisms of $\mathscr{A}$, so that the localization functor $\mathscr{A} \xrightarrow{\sim} \mathcal{C}^{[0,0]}(\mathscr{A}) \to \mathrm{D}^{[0,0]}(\mathscr{A})$ is an equivalence. We use this functor to identify $\mathscr{A}$ to the full subcategory of $\mathrm{D}(\mathscr{A})$ whose objects are the complexes with cohomology concentrated in degree 0.

Remark V.4.2.6. Consider a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathscr{A}$. By Proposition V.4.1.3(iv), this sequence defines a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\mathrm{D}(\mathscr{A})$. We have $\mathrm{H}^n(h) = 0$ for every $n \in \mathbb{Z}$, because either $n \neq 0$ and then $\mathrm{H}^n(C) = 0$, or $n = 0$ and then $\mathrm{H}^n(A[1]) = \mathrm{H}^{n+1}(A) = 0$. However, the morphism h is not always zero. In fact, we claim that $h = 0$ if and only if the original short exact sequence is split.

Indeed, let X be an object of $\mathrm{D}(\mathscr{A})$, and apply the cohomological functor $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, \cdot)$ to the distinguished triangle $a \to B \to C \xrightarrow{+1}$. We get an exact sequence

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, C) \xrightarrow{h_*} \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, A[1]).$$

If g has a section, then $s_* : \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, C) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, B)$ is a section of g_*, so g_* is surjective, so $h_* = 0$. As this holds for every X, the Yoneda lemma (see for example Corollary I.3.2.3) implies that $h = 0$. Conversely, suppose that $h = 0$. Then g_* is surjective for every $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$. Taking $X = C$ and using the fact that $\mathscr{A}$ is a full subcategory of $\mathrm{D}(\mathscr{A})$, we get that the morphism $g_* : \mathrm{Hom}_{\mathscr{A}}(C, B) \to \mathrm{Hom}_{\mathscr{A}}(C, C)$ is surjective; then any preimage of id_C is a section of g.

Proposition V.4.2.7. *Let $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ and $n \in \mathbb{Z}$.*

(i). We have distinguished triangles, functorial in X:

$$\tau^{\leq n} X \to X \to \tau^{\geq n} X \xrightarrow{+1},$$

$$\tau^{\leq n-1} X \to \tau^{\leq n} X \to \mathrm{H}^n(X)[-n] \xrightarrow{+1}$$

and

$$\mathrm{H}^n(X)[-n] \to \tau^{\geq n} X \to \tau^{\geq n+1} X \xrightarrow{+1} .$$

(ii). We have isomorphisms, functorial in X:

$$\mathrm{H}^n(X)[-n] \simeq \tau^{\leq n}\tau^{\geq n} X \simeq \tau^{\geq n}\tau^{\leq n} X.$$

Proof. We want to use Proposition V.4.1.3(iv). Unfortunately, if $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{A}))$, the sequence of complexes $0 \to \tau^{\leq n} X \to X \to \tau^{\geq n} X \to 0$ is not exact in general, so we introduce modifications of the functors $\tau^{\leq n}$ and $\tau^{\geq n}$. Define $\widetilde{\tau}^{\leq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^{\leq n}(\mathscr{A})$ and $\widetilde{\tau}^{\geq n} : \mathcal{C}(\mathscr{A}) \to \mathcal{C}^{\geq n}(\mathscr{A})$ by

$$\widetilde{\tau}^{\leq n}(X) = (\ldots \to X^{n-2} \to X^n \to \mathrm{Im}(d_X^n) \to 0 \to 0 \to \ldots)$$

and
$$\widetilde{\tau}^{\geq n}(X) = (\ldots \to 0 \to 0 \to \mathrm{Im}(d_X^{n-1}) \to X^n \to X^{n+1} \to X^{n+2} \to \ldots).$$
For every $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{A}))$, we have obvious morphisms of complexes
$$\tau^{\leq n}X \to \widetilde{\tau}^{\leq n}X \to X \to \widetilde{\tau}^{\geq n}X \to \tau^{\geq n}X,$$
and the morphisms $\tau^{\leq n}X \to \widetilde{\tau}^{\leq n}X$ and $\widetilde{\tau}^{\geq n}X \to \tau^{\geq n}X$ are clearly quasi-isomorphisms. So the functors $\widetilde{\tau}^{\leq n}$ and $\widetilde{\tau}^{\geq n}$ also induce endofunctors of the derived category, that we will denote by the same symbols, and we have isomorphisms $\tau^{\leq n} \overset{\sim}{\to} \widetilde{\tau}^{\leq n}$ and $\widetilde{\tau}^{\geq n} \overset{\sim}{\to} \tau^{\geq n}$ of endofunctors of $\mathrm{D}(\mathscr{A})$.

Hence, to prove (i), we can use the truncation functors τ or $\widetilde{\tau}$. Let X be an object of $\mathcal{C}(\mathscr{A})$. Then we have exact sequences in $\mathcal{C}(\mathscr{A})$:
$$0 \to \widetilde{\tau}^{\leq n}X \to X \to \tau^{\geq n}X \to 0,$$
$$0 \to \widetilde{\tau}^{\leq n-1}X \to \tau^{\leq n}X \to \mathrm{H}^n(X)[-n] \to 0$$
and
$$0 \to \mathrm{H}^n(X)[-n] \to \tau^{\geq n}X \to \widetilde{\tau}^{\geq n}X \to 0.$$
Proposition V.4.1.3(iv) gives the three distinguished triangles of point (i).

We prove (ii). Applying the functor $\tau^{\leq n}$ the first morphism of the third triangle of (i), we get a morphism $\tau^{\leq n}(\mathrm{H}^n(X)[-n]) \to \tau^{\leq n}\tau^{\geq n}X$. As $\mathrm{H}^n(X)[-n]$ only has cohomology in degree n, the canonical morphism $\tau^{\leq n}(\mathrm{H}^n(X)[-n]) \to \mathrm{H}^n(X)[-n]$ is an isomorphism in $\mathrm{D}(\mathscr{A})$, so we get a morphism $\mathrm{H}^n(X)[-n] \to \tau^{\leq n}\tau^{\geq n}X$ in $\mathrm{D}(\mathscr{A})$, which induces an isomorphism on cohomology objects, hence is an isomorphism. We prove the second isomorphism of (ii) in the same way, using the second triangle of (i).

□

The following result is proved in problem A.9.3(a).

Corollary V.4.2.8. *Let $n \in \mathbb{Z}$. If $X \in \mathrm{Ob}(\mathrm{D}^{\leq n}(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^{\geq n+1}(\mathscr{A}))$, then* $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) = 0$.

Corollary V.4.2.9. *Let $n \in \mathbb{Z}$. Then $\tau^{\leq n}$ is right adjoint to the inclusion $\mathrm{D}^{\leq n}(\mathscr{A}) \subset \mathrm{D}(\mathscr{A})$, and $\tau^{\geq n}$ if left adjoint to the inclusion $\mathrm{D}^{\geq n}(\mathscr{A}) \subset \mathrm{D}(\mathscr{A})$.*

Proof. We prove the first statement (the proof of the second statement is similar). Let $X \in \mathrm{Ob}(\mathrm{D}^{\leq n}(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$. By Proposition V.4.2.7, we have a distinguished triangle (functorial in Y)
$$\tau^{\leq n}Y \to Y \to \tau^{\geq n+1}Y \overset{+1}{\to}.$$
As $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, \cdot)$ is a cohomological functor, we get an exact equence of abelian groups
$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, (\tau^{\geq n+1}Y)[-1]) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, \tau^{\leq n}Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, \tau^{\geq n+1}Y).$$

But $\tau^{\geq n+1}Y$ and $(\tau^{\geq n+1}Y)[-1]$ are in $\mathrm{D}^{\geq n+1}(\mathscr{A})$, so, by Corollary V.4.2.8, we have $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X,(\tau^{\geq n+1}Y)[-1]) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X,\tau^{\geq n+1}Y) = 0$, so the morphism

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X,\tau^{\leq n}Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X,Y)$$

induced by $\tau^{\leq n}Y \to Y$ is an isomorphism.

□

V.4.3 Resolutions

Proposition V.4.3.1. *Let $\mathscr{A}$ be an abelian category with enough injective objects, and let $\mathscr{I}$ be the full subcategory of injective objects of $\mathscr{A}$. Then the composition $K^+(\mathscr{I}) \to K^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{A})$ is an equivalence of categories.*

If $\mathscr{A}$ has enough projective objects and $\mathscr{P}$ is the full subcategory of projective objects of $\mathscr{A}$, then the composition $K^-(\mathscr{P}) \to K^-(\mathscr{A}) \to \mathrm{D}^-(\mathscr{A})$ is an equivalence of categories.

Proof. It suffices to prove the first statement. By Corollary IV.4.1.11, for every object X of $K^+(\mathscr{A})$, there exists a quasi-isomorphism $X \to I$ with $I \in \mathrm{Ob}(K^+(\mathscr{I}))$. So, by Corollary V.2.3.2, the inclusion $K^+(\mathscr{I}) \subset K^+(\mathscr{A})$ induces an equivalence of categories $K^+(\mathscr{I})[W_{\mathscr{I}}^{-1}] \to \mathrm{D}^+(\mathscr{A})$, where $W_{\mathscr{I}}^{-1}$ is the set of morphisms of $K^+(\mathscr{I})$ that induce isomorphisms on the cohomology objects. But, by Theorem IV.3.2.1(iii), any element of $W_{\mathscr{I}}$ is already an isomorphism of $K^+(\mathscr{I})$, so $K^+(\mathscr{I})[W_{\mathscr{I}}^{-1}] = K^+(\mathscr{I})$.

□

Remark V.4.3.2. The proposition implies that, if $\mathscr{A}$ is an abelian $\mathscr{U}$-category wih enough injective objects, then $\mathrm{D}^+(\mathscr{A})$ is a $\mathscr{U}$-category; this does not necessarily hold if $\mathscr{A}$ does not have enough injective objects.

Here is a more general version of Proposition V.4.3.1.

Proposition V.4.3.3. *Let $\mathscr{C}$ be a full additive subcategory of $\mathscr{A}$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, there exists a monomorphism $A \to A'$ with $A' \in \mathrm{Ob}(\mathscr{C})$. Let $\mathscr{N}_{\mathscr{C}}$ be the set of objects X of $K^+(\mathscr{C})$ such that $\mathrm{H}^n(X) = 0$ for every $n \in \mathbb{Z}$. Then the inclusion $K^+(\mathscr{C}) \to K^+(\mathscr{A})$ induces an equivalence of categories $K^+(\mathscr{C})/\mathscr{N}_{\mathscr{C}} \to \mathrm{D}^+(\mathscr{A})$, and $K^+(\mathscr{C})/\mathscr{N}_{\mathscr{C}}$ is the localization of $K^+(\mathscr{C})$ by the set of quasi-isomorphisms between objects of $K^+(\mathscr{C})$.*

Proof. The last statement follows from Proposition V.3.1.5. To prove the equivalence, we check the hypothesis of Corollary V.2.3.2. This means that, for every $X \in \mathrm{Ob}(K^+(\mathscr{A}))$, we have to find a quasi-isomorphism $X \to Y$ with $Y \in \mathrm{Ob}(K^+(\mathscr{C}))$. This is done in Corollary IV.4.1.12.

□

Corollary V.4.3.4. *In the situation of Proposition V.4.3.3, suppose moreover that there exists an integer $d \geq 0$ such that, for every exact sequence $X^0 \to X^1 \to \ldots \to X^d \to 0$ in $\mathscr{A}$ with $X^0, X^1, \ldots, X^{d-1} \in \mathrm{Ob}(\mathscr{C})$, we have $X^d \in \mathrm{Ob}(\mathscr{C})$.*

Then the inclusion $K^b(\mathscr{C}) \to K^b(\mathscr{A})$ induces an equivalence of categories $K^b(\mathscr{C})/(\mathscr{N}_{\mathscr{C}} \cap \mathrm{Ob}(K^b(\mathscr{C}))) \to \mathrm{D}^b(\mathscr{A})$.

Proof. We need to show that, for every $X \in \mathrm{Ob}(K^b(\mathscr{A}))$, there exists a quasi-isomorphism $f : X \to Y$ with $Y \in \mathrm{Ob}(K^+(\mathscr{A}))$. By the proposition, we can find a quasi-isomorphism $X \to Y$ with $Y \in \mathrm{Ob}(K^+(\mathscr{A}))$. Fix $N \in \mathbb{Z}$ such that $X^n = 0$ for $n \geq N$. We have $X = \tau^{\leq N+d}X$, so we get a morphism $f' : X \to \tau^{\leq N+d}Y$; if $n \leq N+d$, then $\mathrm{H}^n(f') = \mathrm{H}^n(f)$ is an isomorphism, and if $n > N+d$, then $\mathrm{H}^n(f') : \mathrm{H}^n(X) = 0 \to \mathrm{H}^n(\tau^{\leq N+d}Y) = 0$ is also an isomorphism; so f' is a quasi-isomorphism. To finish the proof, it suffices to show that $Y' = \tau^{\leq N+d}Y$ is in $K(\mathscr{C})$ (it is obviously bounded). The only Y'^n that might not be in $\mathscr{C}$ is $Y'^{N+d} = \mathrm{Ker}(d_Y^{N+d})$. But, as $\mathrm{H}^i(Y') = \mathrm{H}^i(X) = 0$ for $i \geq N$, the complex

$$Y'^N = Y^N \to Y'^{N+1} = Y^{N+1} \to \ldots \to Y'^{N+d-1} = Y^{N+d-1} \to Y'^{N+d} \to 0 = Y'^{N+d+1}$$

is exact, so the hypothesis on $\mathscr{C}$ implies that Y'^{N+d} is an object of $\mathscr{C}$.

□

V.4.4 Derived functors

Definition V.4.4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $* \in \{b, +, -, \varnothing\}$. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. By Remark IV.1.4.5, it defines an additive functor $K(F) : K^*(\mathscr{A}) \to K^*(\mathscr{B})$.

(i). Suppose that F is left exact. If the composition $K^*(\mathscr{A}) \overset{K(F)}{\to} K^*(\mathscr{B}) \to \mathrm{D}^*(\mathscr{B})$ has a right localization $\mathrm{D}^*(\mathscr{A}) \to \mathrm{D}^*(\mathscr{B})$, we call this right localization a *(total) right derived functor* of F and denote it by R^*F or RF.

(ii). Suppose that F is right exact. If the composition $K^*(\mathscr{A}) \overset{K(F)}{\to} K^*(\mathscr{B}) \to \mathrm{D}^*(\mathscr{B})$ has a left localization $\mathrm{D}^*(\mathscr{A}) \to \mathrm{D}^*(\mathscr{B})$, we call this left localization a *(total) left derived functor* of F and denote it by L^*F or LF.

Proposition V.4.4.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact additive functor between abelian categories, and suppose that $\mathscr{A}$ has enough injective objects. Then F has a right derived functor $RF : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{B})$, and, for every $n \in \mathbb{N}$, the composition of the functor $\mathrm{H}^n \circ R^+F : K^+(\mathscr{A}) \to \mathscr{B}$ and of the embedding $\mathscr{A} \to K^+(\mathscr{A})$ of Remark IV.1.1.5 (sending an object of $\mathscr{A}$ to a complex concentrated in degree 0) is equal to the nth right derived functor R^nF.*

Proof. Let $\mathscr{I}$ be the full subcatgeory of injective objects of $\mathscr{A}$. We have seen in the proof of Proposition V.4.3.1 that the full subcategory $K^+(\mathscr{I})$ of $K^+(\mathscr{A})$ satifies the hypothesis (b) of Proposition V.2.4.4

□

More generally:

Definition V.4.4.3. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact additive functor, and let $\mathscr{C}$ be a full additive subcategory of $\mathscr{A}$. We say that $\mathscr{C}$ is F*-injective* if it satisfies the following properties:

(a) For every $A \in \mathrm{Ob}(\mathscr{A})$, there exists a monomorphism $A \to A'$ with $A' \in \mathrm{Ob}(\mathscr{C})$.

(b) For every exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ with $A, B \in \mathrm{Ob}(\mathscr{C})$, we have $C \in \mathrm{Ob}(\mathscr{C})$.

(c) For every exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$ with $A, B, C \in \mathrm{Ob}(\mathscr{C})$, the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.

There is a similar definition of a F-projective subcategory if F is right exact.

Example V.4.4.4. (1) If $\mathscr{A}$ has enough injective objects, then the additive subcategory of injective objects of $\mathscr{A}$ is F-injective for every left exact functor $F : \mathscr{A} \to \mathscr{B}$.

(2) The category of flat right R-modules is $(\cdot) \otimes_R M$-projective for every right R-module M.

(3) If $\mathscr{A}$ has enough injective objects and $F : \mathscr{A} \to \mathscr{B}$ is left exact, then the category of F-acyclic objects (see Definition IV.3.3.4) is F-injective.

Proposition V.4.4.5. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact additive functor, and suppose that there exists a F-injective subcategory $\mathscr{C}$ of $\mathscr{A}$. Then F admits a right derived functor $RF : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{B})$, and we can calculate RF in the following way: If $X \in \mathrm{Ob}(K^+(\mathscr{A}))$, take a quasi-isomorphism $X \to Y$ with $Y \in \mathrm{Ob}(K^+(\mathscr{C}))$ (this exists by Proposition V.4.3.3); then $RF(X)$ is the image of $KF(Y)$ in $\mathrm{D}^+(\mathscr{B})$.*

Proof. We want to apply Proposition V.3.2.1, so we must check the hyppotheses of that proposition. Condition (a) follows from Proposition V.4.3.3. Condition (b) says that, if $X \in \mathrm{Ob}(\mathcal{C}^+(\mathscr{C}))$ is acyclic, then so is $F(X)$. Let's prove this. We claim that $\mathrm{Ker}(d_X^n) = \mathrm{Im}(d_X^{n-1})$ is an object of $\mathscr{C}$ for every $n \in \mathbb{Z}$. If n is small enough, this holds because these objects are 0. Suppose that we know the claim for n, and let's prove it $n+1$. As X is acyclic, we have an exact sequence

$$0 \to \mathrm{Im}(d_X^{n-1}) \to X^n \to \mathrm{Coker}(d_X^{n-1}) \xrightarrow{\sim} \mathrm{Im}(d_X^n) = \mathrm{Ker}(d_X^{n+1}) \to 0.$$

By condition (b) in Definition V.4.4.3, this implies that $\mathrm{Ker}(d_X^{n+1})$ is an object of $\mathscr{C}$. Now condition (c) in Definition V.4.4.3 implies that the sequence

$$0 \to F(\mathrm{Im}(d_X^{n-1})) = F(\mathrm{Ker}(d_X^n)) \to F(X^n) \to F(\mathrm{Im}(d_X^n)) = F(\mathrm{Ker}(d_X^{n+1})) \to 0$$

is exact for every $n \in \mathbb{Z}$, and so the complex $F(X)$ is acyclic.

□

The formula for RF immediately gives the following corollary.

Corollary V.4.4.6. *In the situation of Proposition V.4.4.5, suppose that the F-injective subcategory $\mathscr{C}$ satisfies the hypothesis of Corollary V.4.3.4. Then the right derived functor $RF : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{B})$ sends $\mathrm{D}^b(\mathscr{A})$ to $\mathrm{D}^b(\mathscr{B})$.*

Composition of derived functors

Let $F : \mathscr{A} \to \mathscr{A}'$ and $G : \mathscr{A}' \to \mathscr{A}''$ be left exact functors between abelian categories. If the functors F, G and $G \circ F$ admit right derived functors $RF : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{B})$, $RG : \mathrm{D}^+(\mathscr{B}) \to \mathrm{D}^+(\mathscr{C})$ and $R(G \circ F) : \mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{C})$, then, by the universal property of the right localization of a functor, we have a canonical morphism of functors $R(G \circ F) \to (RG) \circ (RF)$, induced by the equality $K(G \circ F) = (KG) \circ (KF)$.

Proposition V.4.4.7. *Suppose that there exists a F-injective subcategory $\mathscr{C}$ of $\mathscr{A}$ and an G-injective subcategory $\mathscr{C}'$ of $\mathscr{A}'$. Suppose also that $F(\mathscr{C}) \subset \mathscr{C}'$. Then the category $\mathscr{C}$ is $(G \circ F)$-injective, and the canonical morphism $R(G \circ F) \to (RG) \circ (RF)$ is an isomorphism of functors.*

Proof. The fact that $\mathscr{C}$ is $(G \circ F)$-injective follows immediately from Definition V.4.4.3, and the isomorphism $R(G \circ F) \xrightarrow{\sim} (RG) \circ (RF)$ follows immediately from the formula for right derived functors in Proposition V.4.4.5.

□

Remark V.4.4.8. Here is a partial converse. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects, and that the morphism $R(G \circ F) \to (RG) \circ (RF)$ is an isomorphism of functors. Then the functor F sends injective objects to G-acyclic objects.

Indeed, let I be an injective object of $\mathscr{A}$. Then $(RG)(RF(I)) \simeq R(G \circ F)(I)$ is concentrated in degree 0, so $RF(I)$ is G-acyclic.

Derived bifunctors

Let $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{A}''$ be three abelian categories, and $F : \mathscr{A} \times \mathscr{A}' \to \mathscr{A}''$ be a bi-additive functor that is left exact in each variable.

Definition V.4.4.9. If $\mathscr{C}$ if a full additive subcategory of $\mathscr{A}$ and $\mathscr{C}'$ if a full additive subcategory of $\mathscr{A}'$, we say that the pair $(\mathscr{C}, \mathscr{C}')$ is F-*injective* if it satisfies the following conditions:

(a) For every $X \in \mathrm{Ob}(\mathscr{C})$, the category $\mathscr{C}'$ is $F(X, \cdot)$-injective.

(b) For every $X' \in \mathrm{Ob}(\mathscr{C}')$, the category $\mathscr{C}$ is $F(\cdot, X')$-injective.

Lemma V.4.4.10. *Suppose that $(\mathscr{C}, \mathscr{C}')$ is F-injective. Let $X \in \mathrm{Ob}(K^+(\mathscr{C}))$ and $X' \in \mathrm{Ob}(K^+(\mathscr{C}'))$, and suppose that X or X' is acyclic. Then $\mathrm{Tot}(F(X, X'))$ is acyclic.*

Applying Proposition V.3.2.2, we get the following corollary.

Corollary V.4.4.11. *If there exists a F-injective pair $(\mathscr{C},\mathscr{C}')$, then the functor $K^+(\mathscr{A}) \times K^+(\mathscr{A}') \overset{KF}{\to} K^+(\mathscr{A}'') \to \mathrm{D}^+(\mathscr{A}'')$ admits a right localization $RF : \mathrm{D}^+(\mathscr{A}) \times \mathrm{D}^+(\mathscr{A}') \to \mathrm{D}^+(\mathscr{A}'')$, which is triangulated in each variable.*

For example, if $\mathscr{A}$ has enough injective objects and if the functor $F(I,\cdot) : \mathscr{A}' \to \mathscr{A}''$ is exact for every injective object I of $\mathscr{A}$, then the pair $(\mathscr{I},\mathscr{A}')$ (where $\mathscr{I}$ is the category of injective objects of $\mathscr{A}$) is F-injective, so the corollary applies. We can obviously switch the roles of $\mathscr{A}$ and $\mathscr{A}'$, and we have similar results for right exact functors and left localizations.

Example V.4.4.12. (1) Let R be a ring. Then the functor $(\cdot)\otimes_R(\cdot) : \mathbf{Mod}_R \times {}_R\mathbf{Mod} \to \mathbf{Ab}$ admits a left localization $(\cdot)\otimes^L_R(\cdots) : \mathrm{D}^-(\mathbf{Mod}_R) \times \mathrm{D}^-({}_R\mathbf{Mod}) \to \mathrm{D}^-(\mathbf{Ab})$, called the *derived tensor product* functor.

Let $M \in \mathrm{Ob}(\mathbf{Mod}_R)$ and $N \in \mathrm{Ob}({}_R\mathbf{Mod})$. If $P^\bullet \to M$ and $Q^\bullet \to N$ are flat resolutions, then we have

$$M \otimes^L_R N \simeq P^\bullet \otimes_R N \simeq M \otimes_R Q^\bullet \simeq \mathrm{Tor}(P^\bullet \otimes_R Q^\bullet).$$

In particular, for every $n \in \mathbb{Z}$, we get

$$\mathrm{Tor}^R_n(M,N) = \mathrm{H}^{-n}(M \otimes^L_R N).$$

If R is commutative, we can similarly construct a derived tensor product functor $(\cdot)\otimes^L_R(\cdot) : \mathrm{D}^-({}_R\mathbf{Mod}) \times \mathrm{D}^-({}_R\mathbf{Mod}) \to \mathrm{D}^-({}_R\mathbf{Mod})$.

(2) More generally, let $(\mathscr{C},\mathscr{T})$ be a site satisfying the condition before Definition III.2.2.6 (so that the sheafification functor is defined), let R be a commutative ring (or a sheaf of commutative rings on $\mathscr{C}_{\mathscr{T}}$) and let $\mathscr{A} = \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, R)$ be the category of sheaves of R-modules on $\mathscr{C}_{\mathscr{T}}$. We are interested in the bifunctor $(\cdot)\otimes_R(\cdot) : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ sending a pair of sheaves of R-modules $(\mathscr{F},\mathscr{G})$ to the sheafification of the presheaf $X \longmapsto \mathscr{F}(X)\otimes_R\mathscr{G}(X)$. Even though $\mathscr{A}$ does not have enough projective objects in general, its flat objects form a $\mathscr{F}\otimes_R(\cdot)$-exact subcategory for every $\mathscr{F} \in \mathrm{Ob}(\mathscr{A})$, and we can use this to define the derived bifunctor $(\cdot)\otimes^L_R(\cdot) : \mathrm{D}^-(\mathscr{A}) \times \mathrm{D}^-(\mathscr{A}) \to \mathrm{D}^-(\mathscr{A})$.

(3) Suppose that the abelian category $\mathscr{A}$ has enough injective objects or enough projective objects. Then the functor $\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{A})} : \mathcal{C}^+(\mathscr{A}) \times \mathcal{C}^-(\mathscr{A}) \to \mathcal{C}^+(\mathscr{A})$ of Definition IV.1.6.4 has a right localization $R\,\mathrm{Hom}_{\mathscr{A}} : \mathrm{D}^+(\mathscr{A}) \times \mathrm{D}^-(\mathscr{A}) \to \mathrm{D}^+(\mathscr{A})$. [5]

Suppose for example that $\mathscr{A}$ has enough injective objects. Let $A, B \in \mathrm{Ob}(\mathscr{A})$, and let $B \to I^\bullet$ be an injective resolution. Then

$$R\,\mathrm{Hom}_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(A, I^\bullet),$$

so, for every $n \in \mathbb{Z}$, we have

$$\mathrm{Ext}^n_{\mathscr{A}}(A,B) = \mathrm{H}^n(R\,\mathrm{Hom}_{\mathscr{A}}(A,B)).$$

[5] Note that $\underline{\mathrm{Hom}}_{\mathcal{C}(\mathscr{A})}$ induces a functor on the homotopy categories, so we can apply our results.

Theorem V.4.4.13. *Suppose that $\mathscr{A}$ has enough injective objects or enough projective objects. Then, for every $X \in \mathrm{Ob}(\mathrm{D}^-(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^+(\mathscr{A}))$, we have*

$$\mathrm{H}^n(R\operatorname{Hom}_{\mathscr{A}}(X,Y)) \simeq \operatorname{Hom}_{\mathrm{D}(\mathscr{A})}(X,Y[n]).$$

We will need the following lemma, which generalizes Theorem IV.3.2.1(iv).

Lemma V.4.4.14. *Let $X,Y \in \mathrm{Ob}(K(\mathscr{A}))$, let $f : X \to Y$ be a quasi-isomorphism, and let $I \in \mathrm{Ob}(K^+(\mathscr{A}))$ be a bounded below complex of injective objects. Then the map $(\cdot)\circ f : \operatorname{Hom}_{K(\mathscr{A})}(Y,I) \to \operatorname{Hom}_{K(\mathscr{A})}(X,I)$ is bijective.*

Proof. We have a distinguished triangle in $K(\mathscr{A})$:

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1],$$

and $\mathrm{Mc}(f)$ is acyclic by Corollary IV.2.2.8. Applying the cohomological functor $\operatorname{Hom}_{K(\mathscr{A})}(\cdot, I)$ gives an exact sequence

$$\operatorname{Hom}_{K(\mathscr{A})}(\mathrm{Mc}(f),I) \to \operatorname{Hom}_{K(\mathscr{A})}(Y,I) \xrightarrow{(\cdot)\circ f} \operatorname{Hom}_{K(\mathscr{A})}(X,I) \to \operatorname{Hom}_{K(\mathscr{A})}(\mathrm{Mc}(f)[-1],I).$$

As $\mathrm{Mc}(f)$ is acyclic, we have $\operatorname{Hom}_{K(\mathscr{A})}(\mathrm{Mc}(f),I) = \operatorname{Hom}_{K(\mathscr{A})}(\mathrm{Mc}(f)[-1],I) = 0$ by Theorem IV.3.2.1(i). This implies the result.

□

Proof of Theorem V.4.4.13. The functor $R\operatorname{Hom}_{\mathscr{A}}$ commutes with shifts (i.e. $R\operatorname{Hom}_{\mathscr{A}}(X[n],Y[m]) \simeq R\operatorname{Hom}_{\mathscr{A}}(X,Y)[m-n]$) because it is triangulated in each variable, so it suffices to treat the case $n=0$. It also suffices to treat the case where $\mathscr{A}$ has enough injective objects.

Let $Y \to I$ be a quasi-isomorphism with I a bounded below complex of injective objects of $\mathscr{A}$. Then $R\operatorname{Hom}_{\mathscr{A}}(X,Y) \simeq \underline{\operatorname{Hom}}_{C(\mathscr{A})}(X,I)$, so, by Proposition IV.1.6.5, we have

$$\mathrm{H}^0(R\operatorname{Hom}_{\mathscr{A}}(X,Y)) \simeq \operatorname{Hom}_{K(\mathscr{A})}(X,I) \xrightarrow{A} \operatorname{Hom}_{\mathrm{D}(\mathscr{A})}(X,I),$$

where the map A is induced by the localization functor $K(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. So it suffices to prove that this map A is bijective. We use the fact that quasi-isomorphisms form a left multiplicative system and the description of the Hom sets in the localization as the sets $\operatorname{Hom}_{\mathscr{C}_l}$ of Definition V.2.2.3.

Let $g \in \operatorname{Hom}_{\mathrm{D}(\mathscr{A})}(X,I)$. Then we can write $g = f \circ s^{-1}$, where $s : X' \to X$ is a quasi-isomorphism in $K(\mathscr{A})$ and $f : X' \to I$ is a morphism in $K(\mathscr{A})$. By Lemma V.4.4.14, the map $(\cdot)\circ s : \operatorname{Hom}_{K(\mathscr{A})}(X',I) \to \operatorname{Hom}_{K(\mathscr{A})}(X,I)$ is bijective, so there exists $h : X \to I$ in $K(\mathscr{A})$ such that $f = h \circ s$, and then $A(h) = g$. So Q is surjective. To show that Q is injective, consider a morphism $f : X \to I$ in $K(\mathscr{A})$ such that $Q(f) = 0$. Then there exists a quasi-isomorphism $s : X' \to X$ such that $f \circ s = 0$. Using again Lemma V.4.4.14, we see that this implies that $f = 0$.

□

V.4.5 Ext groups

Fix an abelian category $\mathscr{A}$.

Definition V.4.5.1. Let $X, Y \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$. For every $n \in \mathbb{Z}$, we write

$$\mathrm{Hom}^n_{\mathscr{A}}(X,Y) = \mathrm{Ext}^n_{\mathscr{A}}(X,Y) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y[n]).$$

By the calculation in Example V.4.4.12(2), this is compatible with Definition IV.3.4.3 if X and Y are objects of $\mathscr{A}$, and by Theorem V.4.4.13, we have

$$\mathrm{Ext}^n_{\mathscr{A}}(X,Y) = \mathrm{H}^n R\,\mathrm{Hom}_{\mathscr{A}}(X,Y)$$

if $\mathscr{A}$ has enough injective or enough projective objects and if $X \in \mathrm{Ob}(\mathrm{D}^-(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^+(\mathscr{A}))$.

We now give a description of the groups $\mathrm{Ext}^n_{\mathscr{A}}(A,B)$ for $A, B \in \mathrm{Ob}(\mathscr{A})$ that holds even when $\mathscr{A}$ does not have enough injective or projective objects. We already know that $\mathrm{Ext}^n_{\mathscr{A}}(A,B) = 0$ if $n < 0$ (by Corollary V.4.2.8) and that $\mathrm{Ext}^0_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(A,B)$ (by Remark V.4.2.5).

Definition V.4.5.2. Let $A, B \in \mathrm{Ob}(\mathscr{A})$ and let $i \geq 1$ be an integer. A degree i *Yoneda extension* of A by B is an exact sequence in $\mathscr{A}$:

$$0 \to B \to E_{i-1} \to \ldots \to E_1 \to E_0 \to A \to 0.$$

We say that two such exact sequences $0 \to B \to E_{i-1} \to \ldots \to E_0 \to A \to 0$ and $0 \to B \to E'_{i-1} \to \ldots \to E'_0 \to A \to 0$ are equivalent if there exists a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & E_{i-1} \arrow[r] & \ldots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] \arrow[d, equal] & E''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \ldots \arrow[r] & E''_1 \arrow[r] \arrow[u] \arrow[d] & E''_0 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E'_{i-1} \arrow[r] & \ldots \arrow[r] & E'_1 \arrow[r] & E'_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

It is not clear that “being equivalent” is an equivalence relation on Yoneda extensions, but it follows from the next result.

Proposition V.4.5.3. *Let $A, B \in \mathrm{Ob}(\mathscr{A})$. Let $c = (0 \to B \to E_{i-1} \to \ldots \to E_0 \to A \to 0)$ be a Yoneda extension of A by B. We write $\alpha(c) \in \mathrm{Ext}^i_{\mathscr{A}}(A,B)$ for the morphism $f \circ s^{-1}$, where f is the obvious morphism of complexes*

$$(\ldots \to 0 \to B \to E_{i-1} \to \ldots E_0 \to 0 \to \ldots) \to B[i]$$

and s is the quasi-isomorphism

$$(\ldots \to 0 \to B \to E_{i-1} \to \ldots E_0 \to 0 \to \ldots) \to A$$

(with E_0 in degree 0 on the left hand side of both morphisms).

Then every element of $\mathrm{Ext}^i_{\mathscr{A}}(A,B)$ is of the form $\alpha(c)$ for some Yoneda extension, and two Yoneda extensions c and c' are equivalent if and only if $\alpha(c) = \alpha(c')$.

Proof. We start with a construction. Suppose that $X \in \mathrm{Ob}(K(\mathscr{A}))$ and that we have morphisms $f : X \to B[i]$, $s : X \to A$ such that s is a quasi-isomorphism. We construct a Yoneda extension $c(f,s)$ in the following way: Let $Y = \tau^{\geq 0}X$, and let $g : Y \to B[i]$ and $t : Y \to A$ be the compositions of f and s with the canonical morphism $\tau^{\geq 0}X \to X$. As A is concentrated in degree 0, the morphism t is still a quasi-isomorphis, so we have an exact sequence

$$\ldots X^{-2} \to X^{-1} \to X^0 \to A \to 0,$$

and the morphism $g : Y \to B[i]$ is zero except in every degree, except for $g^{-i} = f^{-i} : X^{-i} \to B$. So we get a diagram with an exact first row:

$$\begin{tikzcd}
\ldots \arrow[r] & X^{-i-1} \arrow[r] & X^{-i} \arrow[r] \arrow[d, "f^{-i}"'] & X^{-i+1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X^0 \arrow[r] & A \arrow[r] & 0 \\
 & & B \arrow[r, "u"'] & X^{-i+1} \oplus_{X^{-i}} B & & & &
\end{tikzcd}$$

We take $E_{i-1} = X^{-i+1} \oplus_{X^{-i}} B$ and $E_j = X^{-j}$ for $0 \leq j \leq i-2$. By Corollary II.2.1.16, the morphism $u : B \to E_{i-1}$ is injective. As $d_X^{-i+1} \circ d_X^{-i}$, the morphism $\begin{pmatrix} d_X^{-i+1} & 0 \end{pmatrix} : X^{-i+1} \oplus B \to E_{i-2}$ factors through a morphism $d : E_{i-1} \to E_{i-2}$, and we have $\mathrm{Ker}\, d = \mathrm{Im}\, u$. So we get an exact sequence

$$0 \to B \stackrel{u}{\to} E_{i-1} \stackrel{d}{\to} E_{i-2} \stackrel{d_X^{-i+2}}{\to} E_{i-3} \to \ldots \to E_1 \stackrel{d_X^{-1}}{\to} E_0 \to A \to 0,$$

which is the desired Yoneda extension $c(f,s)$. Also, the commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & X^{-i-1} \arrow[r] & X^{-i} \arrow[r] \arrow[d, "f^{-i}"'] & X^{-i+1} \arrow[r] \arrow[d] & \ldots \arrow[r] & X^0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & B \arrow[r, "u"'] & E_{i-1} \arrow[r] & \ldots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

and the construction of α imply that $\alpha(c(f,s)) = f \circ s^{-1} \in \mathrm{Hom}_{D(\mathscr{A})}(A, B[i])$.

Now we prove the proposition. Let $\xi \in \mathrm{Ext}^i_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(A, B[i])$. Then we can write $\xi = f \circ s^{-1}$, where $f : X \to B[i]$ and $s : X \to A$ are morphisms in $K(\mathscr{A})$ and s is a quasi-isomorphism, and then $c(f,s)$ is a Yoneda extension such that $\alpha(c(f,s)) = \xi$.

Let $c = (0 \to B \to E_{i-1} \to \ldots \to E_0 \to A \to 0)$ and $c' = (0 \to B \to E'_{i-1} \to \ldots \to E'_0 \to A \to 0)$ be two degree n Yoneda extensions. If c and c'

are equivalent, then the description of $\mathrm{Hom}_{\mathscr{C}_l}$ after Definition V.2.2.3 implies that $\alpha(c) = \alpha(c')$. Conversely, suppose that $\alpha(c) = \alpha(c')$. Then there exists a commutative diagram

$$(*)\qquad \begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A & X'' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u] \arrow[d] & B[i] \\ & X' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

where $f : X \to B[n]$, $s : X \to A$, $f' : X' \to B[n]$ and $s' : X \to A'$ are constructed from c and c' as in the statement, and where s'' is a quasi-isomorphism. Let $c(f'', s'') = (0 \to B \to E''_{i-1} \to \ldots \to E''_0 \to A \to 0)$. Then the commutative diagram (*) gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & E_{i-1} \arrow[r] & \ldots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[u, equal] \arrow[d, equal] & E''_{i-1} \arrow[r] \arrow[u] \arrow[d] & \ldots \arrow[r] & E''_1 \arrow[r] \arrow[u] \arrow[d] & E''_0 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E'_{i-1} \arrow[r] & \ldots \arrow[r] & E'_1 \arrow[r] & E'_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

so c and c' are equivalent.

$\square$

VI Model categories

VI.1 Model categories

Model categories will give us a convenient framework in which to localize categories in a controlled way, similar to the equivalence $K^+(\mathscr{I}) \xrightarrow{\sim} \mathrm{D}^+(\mathscr{A})$ of Proposition V.4.3.1.

VI.1.1 Definition

Definition VI.1.1.1. Let $\mathscr{C}$ be a $\mathscr{U}$-category. A *model structure* on $\mathscr{C}$ is the data of three sets of morphisms containing all identity morphisms and stable by composition:

(1) the set W of *weak equivalences*;

(2) the set Fib of *fibrations*;

(3) the set Cof of *cofibrations*.

An element of $W \cap \mathrm{Fib}$ is called an *acyclic fibration* or a *trivial fibration*, and an element of $W \cap \mathrm{Cof}$ is called an *acyclic cofibration* or a *trivial cofibration*.

The category $\mathscr{C}$ and the sets W, Fib and Cof are assumed to satisfy the following axioms:

(MC1) The category $\mathscr{C}$ has all $\mathscr{U}$-small limits and colimits.

(MC2) The set of weak equivalences W satisfy the two out of three property, that is, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of $\mathscr{C}$ such that two out of the three morphisms f, g and $g \circ f$ are in W, then the third of these morphisms is also in W.

(MC3) The sets W, Fib and Cof are *stable by retracts*, that is, if we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r] & B \end{tikzcd}$$

such that g is in W (resp. Fib, resp. Cof), then so is f. [1]

[1]If we have such a commutative diagram, we say that f is a *retract* of g.

(MC4) If we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

then there exists a diagonal morphism $h : B \to X$ making the diagram commute if either of the following holds:

- i is a cofibration and p is an acyclic fibration;
- i is an acyclic cofibration and p is a fibration.

(MC5) For every morphisms f of $\mathscr{C}$, we can write $f = p_1 \circ i_1 = p_2 \circ i_2$, where:

- i_1 is a cofibration and p_1 is an acyclic fibration;
- i_2 is an acyclic cofibration and p_2 is a fibration.

A category with a model structure is called a *model catgeory*.

Remark VI.1.1.2. The axioms of model categories are self-dual: $(\mathscr{C}, W, \mathrm{Fib}, \mathrm{Cof})$ is a model category if and only if $(\mathscr{C}^{\mathrm{op}}, W^{\mathrm{op}}, \mathrm{Cof}^{\mathrm{op}}, \mathrm{Fib}^{\mathrm{op}})$ is.

Let $\mathscr{C}$ be a model category. By axiom MC1, the category $\mathscr{C}$ has an initial object $\varnothing$ and a final object $*$.

Definition VI.1.1.3. Let X be an object of $\mathscr{C}$. We say that X is *fibrant* if the unique morphism $X \to *$ is a fibration, and *cofibrant* if the unique morphism $\varnothing \to X$ is a cofibration.

Definition VI.1.1.4. The *homotopy category* of $\mathscr{C}$ is the localization $\mathscr{C}[W^{-1}]$.

VI.1.2 Some consequences of the axioms

Definition VI.1.2.1. Let $\mathscr{C}$ be a category and let $i : A \to B$ and $p : X \to Y$ be two morphisms of $\mathscr{C}$. We say that i has the *left lifting property* (or LLP) relatively to p, or thet p has the *right lifting property* (or RLP) relatively to i if, for every commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists a diagonal morphism $h : B \to X$ making the diagram commute.

Proposition VI.1.2.2. *Let $\mathscr{C}$ be a model category.*

(i). A morphism of $\mathscr{C}$ is a cofibration (resp. an acyclic cofibration) if and only if it has the left lifting property relatively to all acyclic fibrations (resp. to all fibrations).

(ii). A morphism of $\mathscr{C}$ is a fibration (resp. an acyclic fibration) if and only if it has the right lifting property relatively to all acyclic cofibrations (resp. to all cofibrations).

Proof. It suffices to prove point (i) (point (ii) is just point (i) in the opposite category). The "only if" direction follows from axiom MC4. We prove the other direction. Let $i : A \to B$ be a morphism that has the right lifting property relatively to acyclic fibrations. By axiom MC5, we can write $i = p_1 \circ i_1$, where $i_1 : A \to A'$ is a cofibration and $p_1 : A' \to B$ is an acyclic fibration. By assumption, there exists a morphism $h : B \to A'$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i"'] & A' \arrow[d, "p_1"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "h"] & B \end{tikzcd}$$

So we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i"'] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i_1"'] & A \arrow[d, "i"'] \\ B \arrow[r, "h"'] & A' \arrow[r, "p_1"'] & B \end{tikzcd}$$

and $h \circ p_1 = \mathrm{id}_{A'}$, which shows that i is a retract of i_1, hence a cofibration by axiom MC3. If i has the right lifting property relatively to all fibrations, then we write $i = p_2 \circ i_2$ with p_2 a fibration and i_2 an acyclic cofibration, and the same proof shows that i is a retract of i_2, hence an acyclic cofibration by MC3.

□

Corollary VI.1.2.3. *Let $\mathscr{C}$ be a model category. Then any two of the sets W,* Fib*,* Cof *determine the third.*

Proof. It follows immediately from the proposition that W and Fib (resp. W and Cof) determine Cof (resp. Fib). Suppose that we know Fib and Cof. Then we also know the sets of acyclic fibratins and of acyclic cofibrations, by the proposition. But a morphism of $\mathscr{C}$ is in W if and only it is of the form $p \circ i$, with p an acyclic fibration and i an acyclic cofibration. Indeed, the condition is obviously sufficient. Conversely, let $s \in W$; by MC5, we can write $s = p \circ i$, with p an acyclic fibration and i a cofibration, and by MC2, the morphism i is a weak equivalence, hence an acyclic cofibration.

□

Corollary VI.1.2.4. *Let $\mathscr{C}$ be a model category, and let*

$$(*) \qquad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Y' \end{tikzcd}$$

be a commutative square in $\mathscr{C}$.

(i). If the square (*) is cartesian and g is a fibration (resp, an acyclic fibration), then f is also a fibration (resp. an acyclic fibration).

(ii). If the square (*) is cocartesian and f is a cofibration (resp, an acyclic cofibration), then g is also a cofibration (resp. an acyclic cofibration).

In particular, a finite product of fibrant objects is fibrant, and a finite coproduct of cofibrant objects is cofibrant.

Proof. All the statements have similar proofs. Let us prove the first statement of (i). We use Proposition VI.1.2.2 to check that f is a fibration. So let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "k"'] & Y \end{tikzcd}$$

be a commutative diagram, with i an acyclic cofibration. By MC4, there exists $h' : B \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "g"] \\ B \arrow[r] \arrow[urr, dashed, "h'"] & Y \arrow[r] & Y' \end{tikzcd}$$

As the square (*) is cartesian, there exists a unique morphism $h : B \to X$ such that $f \circ h' = k$ and $u \circ h = h'$, and then it is easy to check that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "k"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

is commutative.

$\square$

Lemma VI.1.2.5 (Ken Brown's lemma)**.** *Let $\mathscr{C}$ be a model category, let $\mathscr{D}$ be a category with a set of morphisms $W_{\mathscr{D}}$ that contains all identity morphisms and satisfies the two out of three axiom MC2, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor.*

(i). If F sends acyclic cofibrations between cofibrant objects to morphisms of $W_{\mathscr{D}}$, then it sends any weak equivalence between cofibrant objects to a morphism of $W_{\mathscr{D}}$.

(ii). If F sends acyclic fibrations between fibrant objects to morphisms of $W_{\mathscr{D}}$, then it sends any weak equivalence between fibrant objects to a morphism of $W_{\mathscr{D}}$.

Proof. It suffices to prove (ii). Let $s : A \to B$ be a weak equivalence, with A and B fibrant objects of $\mathscr{C}$. Consider the cartesian square

$$\begin{tikzcd} A \times B \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

By Corollary VI.1.2.4, both p_1 and p_2 are fibrations. Using axiom MC5, we factor the morphism $(\mathrm{id}_A, f) : A \to A \times B$ into an acyclic cofibration $i : A \to C$ followed by a fibration $p : C \to A \times B$; as $A \times B$ is fibrant, so is C. We have $p_1 \circ p \circ i = \mathrm{id}_A$ and $p_2 \circ p \circ i = f$, so, by axiom MC2, the morphisms $p_1 \circ p$ and $p_2 \circ p$ are weak equivalences, hence acyclic fibrations. By the hypothesis on F, the morphisms $F(p_2 \circ p)$ and $F(p_1 \circ p)$ are in $W_{\mathscr{D}}$. As $F(p_1 \circ p) \circ F(i) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \in W_{\mathscr{D}}$, the two out of three axiom for $W_{\mathscr{D}}$ implies that $F(i) \in W_{\mathscr{D}}$. Finally, the fact that $F(f) = F(p_2 \circ p) \circ F(i)$ (and the two out of three property for $W_{\mathscr{D}}$) implies that $F(f) \in W_{\mathscr{D}}$.

□

VI.1.3 Examples

The stable category of modules

Definition VI.1.3.1. Let R be a ring. We say that R is a *(left) Frobenius ring* if the projective and injective objects of ${}_R\mathbf{Mod}$ coincide.

Example VI.1.3.2. If G is a finite group and k is a field, then the group algebra $k[G]$ is a Frobenius ring.

We fix a Frobenius ring R.

Definition VI.1.3.3. We say that two morphisms $f, g : M \to N$ in ${}_R\mathbf{Mod}$ are *stably equivalent* if $f - g$ factors through a projective R-module.

It is easy to see that morpshism that are stably equivalent to 0 form an ideal in ${}_R\mathbf{Mod}$. (See Definition IV.1.3.1.)

Definition VI.1.3.4. The *stable category of (left) R-modules* is the quotient of ${}_R\mathbf{Mod}$ by the ideal of morphisms that are stably equivalent to 0.

We say that a morphism of ${}_R\mathbf{Mod}$ is a *stable equivalence* if it becomes an isomorphism in the stable category of R-modules.

Theorem VI.1.3.5. *There exists a model structure on ${}_R\mathbf{Mod}$ for which the cofibrations are the injections, the fibrations are the surjections and the weak equivalences are the stable equivalences. The homotopy category of ${}_R\mathbf{Mod}$ is the stable category of R-modules.*

Model structures on complexes

Let R be a ring, and let $\mathscr{C} = \mathcal{C}({}_R\mathbf{Mod})$. Then there are two commonly used model structures on $\mathscr{C}$ (see Section 2.3 of [5], and also problem A.10.2 for the first one):

(i). The *projective model structure*, for which W is the set of quasi-isomorphisms and Fib is the set of surjections.

(ii). The *injective model structure*, for which W is the set of quasi-isomorphisms and Cof is the set of injections.

The projective model structure restricts to a model structure on $\mathcal{C}^-({}_R\mathbf{Mod})$ and on $\mathcal{C}^{\leq n}({}_R\mathbf{Mod})$, for every $n \in \mathbb{Z}$; on these categories, a cofibration is an injective morphism whose cokernel is a complex of projective objects. Similary, the injective model structure restricts to a model structure on $\mathcal{C}^+({}_R\mathbf{Mod})$ and on $\mathcal{C}^{\geq n}({}_R\mathbf{Mod})$, for every $n \in \mathbb{Z}$; on these categories, a fibration is a surjective morphism whose kernel is a complex of injective objects.

Both model structures have the same homotopy category, which is the derived category of ${}_R\mathbf{Mod}$. The projective model structure is useful when we want to derived right exact functors, and the injective model structure is useful when we want to derived left exact functors.

Model structures on abelian categories

There is more general theory of abelian model structures on abelian categories generalizing the first two examples (remember that, if $\mathscr{A}$ is an abelian category, then so is $\mathcal{C}(\mathscr{A})$). They are related to another structure called a cotorsion pair. See Hovey's article [6] for a survey.

For example, if $\mathscr{A}$ is a Grothendieck abelian category, then there is an injective model structure on $\mathcal{C}(\mathscr{A})$. Also, if $\mathscr{A}$ is a category of sheaves of $\mathscr{O}$-modules on a site (where $\mathscr{O}$ is a sheaf of commutative rings), then, even though $\mathscr{A}$ usually does not have enough projective objects, we can use flat objects instead to define a "flat model structure" that allows us to derived the tensor product functor.

Model structures on the category of topological spaces

Definition VI.1.3.6. Let $f : X \to Y$ be a continuous map. We say that f is a *weak homotopy equivalence* if, for every $x \in X$ and every $n \in \mathbb{N}$, the map $\pi_n(f,x) : \pi_n(X,x) \to \pi_n(Y, f(x))$ is bijective.

There are two standard model structures on $\mathbf{Top}$ having weak homotopy equivalences as their set of weak equivalences:

(i). The *classical Quillen model structure* (see Section 2.4 of [5]), for which:

a) the fibrations are the *Serre fibrations*, i.e. the morphisms having the right lifting property relatively to all inclusions $D^n \to D^n \times [0,1]$, $x \longmapsto (x,0)$, where D^n is closed unit disk in $\mathbb{R}^n$;

b) the cofibrations are the retracts of relative cell complexes (see Section 2.1.2 of [5] and the discussion under Definition 2.4.3 of that book).

(ii). The *mixed model structure* (see Sections 17.3 and 17.4 of [11]), for which:

a) the fibrations are the *Hurewicz fibrations*, i.e. the maps that have the right lifting property with respect to the inclusion $X \to X \times [0,1]$, $x \longmapsto (x,0)$, for every topological space X;

b) the cofibrant objects are the spaces that are homotopy equivalent to CW complexes.

Both model structures have variants for $\mathbf{Top}_*$, the category of pointed topological spaces, and the classical Quillen model structure has variants for the subcategories of compactly generated spaces and Kelley spaces. See Section 2.4 of [5].

VI.2 Homotopy in model categories

In this section, we fix a model category $\mathscr{C}$.

VI.2.1 Left and right homotopies

Definition VI.2.1.1. Let $A \in \mathrm{Ob}(\mathscr{C})$.

(i). A *cylinder object* for A is a factorization $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ of the morphism $A \sqcup A \to A$ induced by $(\mathrm{id}_A, \mathrm{id}_A)$.

$$\begin{tikzcd} A \sqcup A \arrow[r, "i"] \arrow[dr] & A \wedge I \arrow[d, "s"] \\ & A \end{tikzcd}$$

where i is a cofibration and s is a weak equivalence.

(ii). A *path object* for A is a factorization $A \stackrel{t}{\to} A^I \stackrel{p}{\to} A \times A$ of the diagonal morphism $A \to A \times A$ (induced by $(\mathrm{id}_A, \mathrm{id}_A)$).

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[dr] & A^I \arrow[d, "p"] \\ & A \times A \end{tikzcd}$$

where p is a fibration and t is a weak equivalence.

Remark VI.2.1.2. By axiom MC5, every object of $\mathscr{C}$ has a cylinder object and a path object. Note that these cylinder and path objects are not unique.

Example VI.2.1.3. Let X be a topological space. We write $I = [0, 1]$.

(1) The morphisms $i_0, i_1 : X \to X \times I$ defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ are acyclic cofibrations for the mixed model structure on $\mathbf{Top}$, and the first projection $p : X \times I \to X$ is a weak homotopy equivalence, so $X \sqcup X \stackrel{i_0 \sqcup i_1}{\to} X \times I \stackrel{p}{\to} X$ is a cylinder object for X in the mixed model structure. If X is a CW complex, it is also a cylinder object in the classical Quillen model structure.

If we work in $\mathbf{Top}_*$ and x_0 is the base point of X, then we have to use the *smash product* $X \wedge I$ instead, where $X \wedge I$ is the quotient of $X \wedge I$ by the equivalence relations that identifies (x_0, t) and $(x, 0)$ with $(x_0, 0)$, for every $x \in X$ and every $t \in I$.

Note that $p : X \times I \to X$ is a Serre fibration, but it is not a Hurewicz fibration in general. (It is if X is paracompact.)

(2) The Let X^I be the space of continuous maps from I to X (with the compact-open topology), let $p : X^I \to X \times X$ be the map $\gamma \longmapsto (\gamma(0), \gamma(1))$ and $i : X \to X^I$ be the map sending $x \in X$ to the constant function $t \longmapsto x$ on I. Then i is a weak homotopy equivalence and p is a Serre fibration, so we get a path object for X in the classical Quillen model structure on $\mathbf{Top}$.

Lemma VI.2.1.4. *Let $A \in \mathrm{Ob}(\mathscr{C})$.*

(i). Let $A \sqcup A \stackrel{i}{\to} A \wedge I \stackrel{s}{\to} A$ be a cylinder object for A, let $j_0, j_1 : A \to A \sqcup A$ be the two canonical morphisms, and let $i_0 = i \circ j_1$, $i_1 = i \circ j_1$. Then i_0 and i_1 are weak equivalences, and they are acyclic cofibrations if A is cofibrant.

(ii). Let $A \stackrel{t}{\to} A^I \stackrel{p}{\to} A \times A$ be a path object for A, let $q_0, q_1 : A \times A \to A$ be the two canonical projections, and let $p_0 = q_0 \circ p$, $p_1 = q_1 \circ p$. Then p_0 and p_1 are weak equivalences, and they are acyclic fibrations if A is fibrant.

Proof. It suffices to prove (i). The compositions of i_0 and i_1 with the morphism $A \sqcup A \to A$ induced by $(\mathrm{id}_A, \mathrm{id}_A)$ are both equal to id_A, so we have $s \circ i_0 = s \circ i_1 = \mathrm{id}_A$. As s is a weak equivalence, axiom MC2 implies that i_0 and i_1 are weak equivalences.

Suppose that A is cofibrant. We have a cocartesian square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & A \arrow[d, "j_0"] \\ A \arrow[r, "j_1"'] & A \sqcup A \end{tikzcd}$$

so, by Corollary VI.1.2.4, the morphisms j_0 and j_1 are cofibrations. As i is a cofibration, we deduce that i_0 and i_1 are also cofibrations.

□

Definition VI.2.1.5. Let $f, g : A \to X$ be two morphisms of $\mathscr{C}$.

(i). A *left homotopy* from f to g is a morphism $H : A \wedge I \to X$ making the following diagram commute:

$$\begin{tikzcd} A \sqcup A \arrow[r, "i"] \arrow[dr, "f \sqcup g"'] & A \wedge I \arrow[d, "H"] \\ & X \end{tikzcd}$$

where $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ is a cylinder object for A. If a left homotopy from f to g exists, we say that f and g are *left homotopic* and we write $f \overset{l}{\sim} g$.

(ii). A *right homotopy* from f to g is a morphism $K : A \to X^I$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f \times g"] \arrow[d, "K"'] & X \times X \\ X^I \arrow[ur, "p"'] & \end{tikzcd}$$

where $X \xrightarrow{t} X^I \xrightarrow{p} X \times X$ is a path object for X. If a right homotopy from f to g exists, we say that f and g are *right homotopic* and we write $f \overset{r}{\sim} g$.

Remark VI.2.1.6. In general, the relations $\overset{l}{\sim}$ and $\overset{r}{\sim}$ are *not* equivalence relations.

Remark VI.2.1.7. We use the notation of Definition VI.2.1.5. If $f \overset{l}{\sim} g$ (resp. $f \overset{r}{\sim} g$), then f is a weak equivalence if and only g is a weak equivalence.

Let us prove the statement about left homotopies. Suppose that $f \overset{l}{\sim} g$. Let $H : A \wedge I \to X$ be a left homotopy from f to g. Let $i_0, i_1 : A \to A \wedge I$ be the two morphisms defined in Lemma VI.2.1.4. By this lemma, the morphisms i_0 and i_1 are weak equivalences. We also have $f = H \circ i_0$ and $g = H \circ i_1$. Hence, if f (resp. g) is a weak equivalence, so is H by axiom MC2, and then $g = H \circ i_1$ (resp. $f = H \circ i_0$) is also a weak equivalence.

Proposition VI.2.1.8. *Let $f, g : A \to X$, $h : B \to A$ and $k : X \to Y$ be morphisms of $\mathscr{C}$.*

(i). If $f \overset{l}{\sim} g$, then $k \circ f \overset{l}{\sim} k \circ g$.

(ii). If $f \overset{r}{\sim} g$*, then* $f \circ h \overset{r}{\sim} g \circ h$*.*

(iii). If X *is fibrant and* $f \overset{l}{\sim} g$*, then* $f \circ h \overset{l}{\sim} g \circ h$*.*

(iv). If A *is cofibrant and* $f \overset{r}{\sim} g$*, then* $k \circ f \overset{r}{\sim} k \circ g$*.*

Proof. It suffices to prove the statements about left homotopies, so suppose that we have a left homotopy $H : A \wedge I \to X$ from f to g, with $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ a cylinder object for A. Then $k \circ H$ is a left homotopy from $k \circ f$ to $k \circ g$, so we get (i).

We assume that X is fibrant, and we want to prove that $f \circ h \overset{l}{\sim} g \circ h$. Write $s = s'' \circ s'$, where $s' : A \wedge I \to C$ is an acyclic cofibration and $s'' : C \to A$ is an acyclic fibration. Then $A \sqcup A \xrightarrow{s' \circ i} C \xrightarrow{s''} A$ is also a cylinder object for A. As X is fibrant and s' is an acyclic cofibration, there exists a morphism $H' : C \to X$ making the following diagram commute

$$\begin{tikzcd} A \wedge I \arrow[r, "H"] \arrow[d, "s'"'] & X \arrow[d] \\ C \arrow[r] \arrow[ur, "H'"] & * \end{tikzcd}$$

that is, such that $H' \circ s' = H$. Then $H' : C \to X$ is also a homotopy from f to g. So, after replacing H by H', we may and will assume that s is an acyclic fibration (and not just a weak equivalence). Now let $B \sqcup B \xrightarrow{j} B \wedge I \xrightarrow{t} B$ be a cylinder object for B. We have a commutative diagram

$$\begin{tikzcd} B \sqcup B \arrow[r, "i \circ (h \sqcup h)"] \arrow[d, "j"'] & A \wedge I \arrow[d, "s"] \\ B \wedge I \arrow[r, "h \circ t"'] \arrow[ur, dashed, "l"] & A \end{tikzcd}$$

As s is an acyclic fibration and j is a cofibration, there exists a morphism $l : B \wedge I \to A \wedge I$ making the diagram commute. Then $H \circ l : B \wedge I \to X$ is a left homotopy from $f \circ h$ to $g \circ h$.

$\square$

Proposition VI.2.1.9. *Let* $A, X \in \mathrm{Ob}(\mathscr{C})$*. Then the relations* $\overset{l}{\sim}$ *and* $\overset{r}{\sim}$ *are reflexive and symmetric on* $\mathrm{Hom}_{\mathscr{C}}(A, X)$*. Moreover:*

(i). If A *is cofibrant, then left homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathscr{C}}(A, X)$*.*

(ii). If X *is fibrant, then right homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathscr{C}}(A, X)$*.*

Proof. It suffices to prove the statements about $\overset{l}{\sim}$.

If $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ is a cylinder object for A, then, for every $f : A \to X$, the morphism $f \circ s$ is a left homotopy from f to f. So $\overset{l}{\sim}$ is reflexive.

Let $f, g \in \mathrm{Hom}_{\mathscr{C}}(A, X)$, and let $H : A \wedge I \to X$ be a left homotopy from f to g, where $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ is a cylinder object for A. Composing i with the morphism $\delta : A \sqcup A \to A \sqcup A$ that switches the two copies of A, we get a new cylinder object $A \sqcup A \xrightarrow{i \circ \delta} A \wedge I \xrightarrow{s} A$ for A, and we have $H \circ (i \circ \delta) = g \sqcup f$, so, using this new cylinder object, the morphism H defines a homotopy from g to f. So $\stackrel{l}{\sim}$ is symmetric.

Assume that A is cofibrant, and let $f, g, h : A \to X$ be morphisms such that $f \stackrel{l}{\sim} g$ and $g \stackrel{l}{\sim} h$. We choose a left homotopy $H : A \wedge I \to X$ from f to g and $H' : (A \wedge I)' \to X$ from g to h, where $A \sqcup A \xrightarrow{i} A \wedge I \xrightarrow{s} A$ and $A \sqcup A \xrightarrow{i'} (A \wedge I)' \xrightarrow{s'} A$ are cylinder objects for A. Let $i_0, i_1 : A \to A \wedge I$ and $i'_0, i'_1 : A \to (A \wedge I)'$ be the morphisms defined in Lemma VI.2.1.4, and define $(A \wedge I)''$ as the pushout of i'_0 and i_1; in other words, the following square is cocartesian:

$$\begin{tikzcd}
A \arrow[r, "i'_0"] \arrow[d, "i_1"'] & (A \wedge I)' \arrow[d] \\
A \wedge I \arrow[r] & (A \wedge I)''
\end{tikzcd}$$

By Lemma VI.2.1.4, the morphisms i'_0 and i_1 are acyclic cofibrations, hence, by Proposition VI.1.2.4, so are the two canonical morphisms $A \wedge I \to (A \wedge I)''$ and $(A \wedge I)' \to (A \wedge I)''$. So the canonical morphism $j : A \to (A \wedge I)''$ is an acyclic cofibration. Also, by the universal property of the pushout, the morphisms $s : A \wedge I \to A$ and $s' : (A \wedge I)' \to A$ induce a morphism $s'' : (A \wedge I)'' \to A$ such that $s'' \circ j = \mathrm{id}_A$. By axiom MC2, the morphism s'' is acyclic. Moreover, the pair $(i_0 : A \to A \wedge I, i'_1 : A \to (A \wedge I)')$ defines a morphism $i'' : A \sqcup A \to (A \wedge I)''$ such that $s'' \circ i'' : A \sqcup A \to A$ is the morphism induced by $(\mathrm{id}_A, \mathrm{id}_A)$. Note that we do not claim that i'' is a cofibration.

Now consider the morphism $H'' : (A \wedge I)'' \to X$ induced by H and H' (using the universal property of the pushout). We have $H'' \circ i'' = (H \circ i_0) \sqcup (H' \circ i'_1) = f \sqcup h$.

$$\begin{tikzcd}
A \arrow[r, "i'_0"] \arrow[d, "i_1"'] & (A \wedge I)' \arrow[d] \arrow[ddr, "H'"] & \\
A \wedge I \arrow[r] \arrow[drr, "H"'] & (A \wedge I)'' \arrow[dr, "H''"] & \\
& & X
\end{tikzcd}$$

Using axiom MC5, write $i'' = q \circ j$, where $j : A \sqcup A \to C$ is a cofibration and $q : C \to (A \wedge I)''$ is an acyclic fibration. Then $A \sqcup A \xrightarrow{j} C \xrightarrow{s'' \circ q} A$ is a cylinder object for A, and $H'' \circ q : C \to X$ is a left homotopy from f to h. So we have proved that $f \stackrel{l}{\sim} h$.

□

Definition VI.2.1.10. Let $A, X \in \mathrm{Ob}(\mathscr{C})$. We denote by $\pi^l(A, X)$ (resp. $\pi^r(A, X)$) the set of equivalence classes in $\mathrm{Hom}_{\mathscr{C}}(A, X)$ for the equivalence relation generated by $\stackrel{l}{\sim}$ (resp. $\stackrel{r}{\sim}$).

Corollary VI.2.1.11. *Let* $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$.

(i). If Z *is fibrant, then the composition law* $\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$, $(f,g) \longmapsto g \circ f$ *induces a map* $\pi^l(X,Y) \times \pi^l(Y,Z) \to \pi^l(X,Z)$.

(ii). If X *is cofibrant, then the composition law* $\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$, $(f,g) \longmapsto g \circ f$ *induces a map* $\pi^r(X,Y) \times \pi^r(Y,Z) \to \pi^r(X,Z)$.

Proof. As before, it suffices to treat the case of left homotopies. Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ such that $f \overset{l}{\sim} f'$ and $g \overset{l}{\sim} g'$. We need to show that $g \circ f$ and $g' \circ f'$ define the same element of $\pi^l(X,Z)$. By Proposition VI.2.1.8(i), we have $g \circ f \overset{l}{\sim} g \circ f'$; by (iii) of the same proposition and the assumption that Z is fibrant, we have $g \circ f' \overset{l}{\sim} g' \circ f'$. This implies the desired conclusion.

□

Proposition VI.2.1.12. *(i). Let* A *be a cofibrant object of* $\mathscr{C}$ *and* $h : X \to Y$ *be an acyclic fibration or a weak equivalence between fibrant objects. Then composition (on the left) by* h *induces a bijection* $\pi^l(A,X) \xrightarrow{\sim} \pi^l(A,Y)$.

(ii). Let X *be a fibrant object of* $\mathscr{C}$ *and* $h : A \to B$ *be an acyclic cofibration or a weak equivalence between cofibrant objects. Then composition (on the right) by* h *induces a bijection* $\pi^r(B,X) \xrightarrow{\sim} \pi^r(A,X)$.

Proof. It suffices to prove (i).

The fact that composition on the left by h gives a well-defined map $\alpha : \pi^l(A,X) \xrightarrow{\sim} \pi^l(A,Y)$ follows from Proposition VI.2.1.8(i). Note also that by Proposition VI.2.1.9, the relation $\overset{l}{\sim}$ is an equivalence relation on $\mathrm{Hom}_{\mathscr{C}}(A,X)$ and $\mathrm{Hom}_{\mathscr{C}}(A,Y)$.

Suppose first that h is an acyclic fibration. We show that α is surjective. Let $g \in \mathrm{Hom}_{\mathscr{C}}(A,Y)$. Consider the commutative square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ A \arrow[r, "g"'] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

As A is cofibrant and h is an acyclic fibration, there exists a morphism $f : A \to X$ such that $h \circ f = g$, and then α sends the class of f to the class of g. We show that α is injective. Let $f, f' \in \mathrm{Hom}_{\mathscr{C}}(A,X)$ such that $h \circ f \overset{l}{\sim} h \circ f'$. Choose a left homotopy $H : A \wedge I \to Y$ from $h \circ f$ to $h \circ f'$, where $A \sqcup A \xrightarrow{i} A \wedge A \to A$ is a cylinder object for A. As h is an acyclic fibration and i is a cofibration, we can find $K : A \wedge I \to X$ making the following diagram commute:

$$\begin{tikzcd} A \sqcup A \arrow[r, "f \sqcup g"] \arrow[d, "i"'] & X \arrow[d, "h"] \\ A \wedge I \arrow[r, "H"'] \arrow[ur, "K"] & Y \end{tikzcd}$$

This morphism K is a left homotopy from f to g, so f and g have the same class in $\pi^l(A, X)$. This finishes the proof in the case where h is an acyclic fibration.

Now suppose that X and Y are fibrant and that h is a weak equivalence. Let $\mathscr{D} = \mathbf{Set}$ and $W_{\mathscr{D}}$ be the set of isomorphisms in $\mathscr{D}$; this set $W_{\mathscr{D}}$ clearly contains the identity morphisms and satisfies the two out of three axiom. Consider the functor $F : \mathscr{C} \to \mathscr{D}$, $Z \longmapsto \pi^l(A, Z)$. By the first case, the functor F sends acyclic fibrations to morphisms in $W_{\mathscr{D}}$, so, by Ken Brown's lemma (Lemma VI.1.2.5), it sends any weak equivalence between fibrant objects to a bijection. Applying this to h gives the result.

□

VI.2.2 Comparing left and right homotopies

Proposition VI.2.2.1. *Let $f, g : A \to X$ be morphisms of $\mathscr{C}$.*

(i). Suppose that $f \overset{l}{\sim} g$ and that A is cofibrant. If $X \overset{t}{\to} X \times I \overset{p}{\to} X \times X$ is a path object for X, then there exists a right homotopy $H : A \to X^I$ from f to g.

$$\begin{tikzcd} & X^I \arrow[d, "p"] \\ A \arrow[r, "f \times g"'] \arrow[ur, "H"] & X \times X \end{tikzcd}$$

(ii). Suppose that $f \overset{r}{\sim} g$ and that X is fibrant. If $A \sqcup A \overset{i}{\to} A \wedge I \overset{s}{\to} A$ is a cylinder object for A, then there exists a left homotopy $H : A \wedge I \to X$ from f to g.

$$\begin{tikzcd} A \sqcup A \arrow[r, "f \sqcup g"] \arrow[d, "i"'] & X \\ A \wedge I \arrow[ur, "H"'] & \end{tikzcd}$$

Proof. If suffices to prove (i). Let $K : A \wedge I \to X$ be a left homotopy from f to g, where $A \sqcup A \overset{i}{\to} A \wedge I \overset{s}{\to} A$ is a cylinder object for A. As A is cofibrant, the morphisms $i_0, i_1 : A \to A \wedge I$ defined in Lemma VI.2.1.4 are acyclic cofibrations. Also, as $K \circ i = f \sqcup g$ and $s \circ i = \mathrm{id}_A \sqcup \mathrm{id}_A$, we have $(s \times K) \circ i_0 = \mathrm{id}_A \times f$ and $(s \times K) \circ i_1 = \mathrm{id}_A \times g$. In particular, as $p \circ t : X \to X \times X$ is the diagonal morphism, we have a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[r, "t"] & X^I \arrow[d, "p"] \\
A \wedge I \arrow[r, "s\times K"'] \arrow[urr, dashed, "H"] & A \times X \arrow[r, "f\times \mathrm{id}_X"'] & X \times X
\end{tikzcd}$$

As p is a fibration and i_0 is an acyclic cofibration, we have a morphism $H : A \wedge I \to X^I$ making the diagram commutative, and then $H \circ i_1 : A \to X^I$ satisfies $p \circ (H \circ i_1) = (f \times \mathrm{id}_X) \circ (s \times K) \circ i_1 = f \times g$, hence is a right homotopy from f to g.

□

Corollary VI.2.2.2. *Let $f, g : A \to X$ be two morphisms of $\mathscr{C}$, with A cofibrant and X fibrant. Fix a cylinder object $A \sqcup A \to A \wedge I \to A$ for A and a path object $X \to X^I \to X \times X$ for X. Then the following conditions are equivalent:*

(i). $f \overset{l}{\sim} g$;

(ii). there exists a left homotopy $H : A \wedge I \to X$ from f to g;

(iii). $f \overset{r}{\sim} g$;

(iv). there exists a right homotopy $K : A \to X^I$ from f to g.

In particular, the relations $\overset{l}{\sim}$ and $\overset{r}{\sim}$ are equal equivalence relations on $\mathrm{Hom}_{\mathscr{C}}(A, X)$. We denote the quotient by $\pi(A, X)$.

VI.2.3 The Whitehead theorem

Definition VI.2.3.1. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$. We say that two morphisms $f, g : X \to Y$ are *homotopic* if they are both left and right homotopic; in that case, we write $f \sim g$. We say that $f : X \to Y$ is a *homotopy equivalence* if there exists $f' : Y \to X$ such that $f \circ f' \sim \mathrm{id}_Y$ and $f' \circ f \sim \mathrm{id}_X$.

In algebraic topology, the Whitehead theorem says that a continuous map between CW complexes that induces isomorphisms on all homotopy groups is a homotopy equivalence. The following theorem is a formal version of this.

Theorem VI.2.3.2. *Let A and B be two objects of $\mathscr{C}$ that are fibrant and cofibrant, and let $f : A \to B$ be a morphism. Then f is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. First suppose that f is a weak equivalence. Let X be a cofibrant object of $\mathscr{C}$; by Proposition VI.2.1.12 and Corollary VI.2.2.2, the map $f \circ (\cdot) : \pi(X, A) \to \pi(X, B)$ is bijective. Taking $X = B$, we get a morphism $g : B \to A$ such that $f \circ g \sim \mathrm{id}_B$. By Proposition VI.2.1.8, this implies in particular that $f \circ g \circ f \sim f$, so, taking $X = A$ in the first sentence, we deduce that $g \circ f \sim \mathrm{id}_A$. This shows that f is a homotopy equivalence.

Now suppose that f is a homotopy equivalence. We want to show that f is weak equivalence. By axiom MC5, we can write $f = p \circ i$, where $i : A \to C$ is an acyclic cofibration and $p : C \to B$ is a fibration, and it suffices to prove that p is a weak equivalence. Let $g : B \to A$ be a morphism such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$, and let $H : B \wedge I \to B$ be a left homotopy from $f \circ g$ to id_B, where $B \sqcup B \to B \wedge I \to B$ is a cylinder object for B. Let $i_0, i_1 : B \to B \wedge I$ be the morphisms of Lemma VI.2.1.4; by that lemma and the fact that B is cofibrant, they are acyclic cofibrations. In particular, by axiom MC4, there exists a morphism $K : B \wedge I \to C$ making the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "i \circ g"] \arrow[d, "i_0"'] & C \arrow[d, "p"] \\ B \wedge I \arrow[r, "H"'] \arrow[ur, "K"] & B \end{tikzcd}$$

If $q = K \circ i_1$, then K is a left homotopy from $K \circ i_0 = i \circ g$ to q. Moreover, we have $p \circ q = p \circ K \circ i_1 = H \circ i_1 = \mathrm{id}_B$. Note that C is fibrant and cofibrant, so, by the first part of the proof, the weak equivalence i is a homotopy equivalence. Choose a morphism $j : C \to A$ such that $j \circ i \sim \mathrm{id}_A$ and $i \circ j \sim \mathrm{id}_C$. Then we have $p \sim p \circ i \circ j = f \circ j$, so $q \circ p \sim (i \circ g) \circ (f \circ j) \sim i \circ \mathrm{id}_A \circ j \sim \mathrm{id}_C$. Let $H' : C \wedge I \to C$ be a left homotopy from id_C to $q \circ p$, where $C \sqcup C \to C \wedge I \to C$ is a cylinder object for C, and $i'_0, i'_1 : C \to C \wedge I$ are the morphisms of Lemma VI.2.1.4; by that lemma again, these morphisms are acyclic cofibrations. As $H' \circ i'_0 = \mathrm{id}_C$, the two out of three property for weak equivalences (axiom MC2) implies that H' is a weak equivalence, hence so is $q \circ p = H' \circ i'_1$. (We are reproving part of Remark VI.2.1.7.) But we have a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "p"'] & C \arrow[r, "\mathrm{id}_C"] \arrow[d, "q \circ p"'] & C \arrow[d, "p"] \\ B \arrow[r, "q"'] & C \arrow[r, "p"'] & C \end{tikzcd}$$

(remember that $p \circ q = \mathrm{id}_B$), so p is a retract of $q \circ p$, hence it is a weak equivalence by axiom MC3.

$\square$

VI.3 The homotopy category

As before, we fix a model category $\mathscr{C}$.

VI.3.1 Fibrant and cofibrant replacements

Definition VI.3.1.1. Let X be an object of $\mathscr{C}$. A *cofibrant replacement* of X is an acyclic fibration $Q(X) \to X$ with $Q(X)$ cofibrant, and a *fibrant replacement* of X is an acyclic cofibration $X \to R(X)$ with $R(X)$ fibrant.

Example VI.3.1.2. If $\mathscr{C} = \mathcal{C}^-(\mathscr{A})$ with the model structure of problem A.10.2 (where $\mathscr{A}$ is an abelian category that has enough projective objects), then a projective resolution of an object of $\mathscr{A}$ is a cofibrant replacement.

By axiom MC5 applied to the morphisms $\varnothing \to X$ and $X \to *$, cofibrant and fibrant replacements of X always exist. Note that, despite the notation, we are not claiming that we can make these functorial in X. The best we can do is the following proposition.

For every X of $\mathscr{C}$, we choose a cofibrant replacement $p_X : Q(X) \to X$ of X and a fibrant replacement $i_X : Q(X) \to RQ(X)$ of $Q(X)$, such that $p_X = \mathrm{id}_X$ if X is cofibrant and $i_X = \mathrm{id}_{Q(X)}$ if $Q(X)$ is fibrant. As i_X is a cofibration and $Q(X)$ is cofibrant, the object $RQ(X)$ is also cofibrant, so it is fibrant and cofibrant.

Proposition VI.3.1.3. *Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Then there exists a commutative diagram:*

$$\begin{tikzcd} X \arrow[d, "f"'] & Q(X) \arrow[l, "p_X"'] \arrow[r, "i_X"] \arrow[d, "f_1"'] & RQ(X) \arrow[d, "f_2"] \\ Y & Q(Y) \arrow[l, "p_Y"] \arrow[r, "i_Y"'] & RQ(Y) \end{tikzcd}$$

Then the morphisms f_1 and f_2 are uniquely determined up to homotopy.

Moreover, if f is a weak equivalence, then so are f_1 and f_2.

Proof. The last sentence follows immediately from the two out of three property for weak equivalences (axiom MC2).

We get the morphism $f_1 : Q(X) \to Q(Y)$ by applying axiom MC4 to the following commutative square:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Q(Y) \arrow[d, "p_Y"] \\ Q(X) \arrow[r, "f \circ p_X"'] \arrow[ur, dashed, "f_1"] & Y \end{tikzcd}$$

Suppose that $f_1, f_1' : Q(X) \to Q(Y)$ are two morphisms such that $p_Y \circ f_1 = f \circ p_X = p_Y \circ f_1'$, and let $Q(X) \sqcup Q(X) \to Q(X) \wedge I \to Q(X)$ be a cylinder object for $Q(X)$. As p_Y is an acyclic fibration, axiom MC4 implies that there exists $H : Q(X) \wedge I \to Q(Y)$ making the following diagram commute:

$$\begin{tikzcd}
Q(X) \sqcup Q(X) \arrow[rr, "f_1 \sqcup f_1'"] \arrow[d] & & Q(Y) \arrow[d, "p_Y"] \\
Q(X) \wedge I \arrow[r] \arrow[urr, "H"] & Q(X) \arrow[r, "f \circ p_X"'] & Y
\end{tikzcd}$$

This morphisms H is a left homotopy from f_1 to f_1', so f_1 is uniquely determined up to left homotopy. By Proposition VI.2.2.1 (and the fact that $Q(X)$ is cofibrant), it is also uniquely determined up to right homotopy.

We get the morphism $f_2 : RQ(X) \to RQ(Y)$ by applying axiom MC4 to the following commutative square:

$$\begin{tikzcd}
Q(X) \arrow[r, "i_Y \circ f_1"] \arrow[d, "i_X"'] & RQ(Y) \arrow[d] \\
RQ(X) \arrow[r] \arrow[ur, dashed, "f_2"] & *
\end{tikzcd}$$

Suppose that $f_1, f_1' : Q(X) \to Q(Y)$ are two morphisms such that $p_Y \circ f_1 = f \circ p_X = p_Y \circ f_1'$, and that $f_2, f_2' : RQ(X) \to RQ(Y)$ are two morphisms such that $f_2 \circ i_X = i_Y \circ f_1$ and $f_2' \circ i_X = i_Y \circ f_1'$. We have already seen that $f_1 \overset{l}{\sim} f_1'$, so, by Proposition VI.2.1.8, we have $i_Y \circ f_1 \overset{l}{\sim} i_Y \circ f_1'$; as $Q(X)$ is cofibrant and $RQ(Y)$ is fibrant, this implies that $i_Y \circ f_1 \overset{r}{\sim} i_Y \circ f_1'$ thanks to Corollary VI.2.2.2. Let $K : Q(X) \to RQ(Y)^I$ be right homotopy from $i_Y \circ f_1$ to $i_Y \circ f_1'$, where $RQ(Y) \to RQ(Y)^I \xrightarrow{p} RQ(Y) \times RQ(Y)$ is a path object for $RQ(Y)$. Then $p \circ K = (i_Y \circ f_1, i_Y \circ f_1') : Q(X) \to RQ(Y) \times RQ(Y)$, so the following square commutes:

$$\begin{tikzcd}
Q(X) \arrow[r, "K"] \arrow[d, "i_X"'] & RQ(Y)^I \arrow[d, "p"] \\
RQ(X) \arrow[r, "(f_2, f_2')"'] \arrow[ur, dashed, "H'"] & RQ(Y) \times RQ(Y)
\end{tikzcd}$$

By axiom MC4, there exists a morphism $H' : RQ(X) \to RQ(Y)^I$ such that $p \circ H' = (f_2, f_2')$, and this morphisms is a right homotopy from f_2 to f_2'. As $RQ(X)$ and $RQ(Y)$ are fibrant and cofibrant, Corollary VI.2.2.2 implies that f_2 and f_2' are homotopic.

$\square$

Definition VI.3.1.4. Let $\mathscr{C}_c$ (resp. $\mathscr{C}_f$, resp $\mathscr{C}_{cf}$) be the full subcategory of $\mathscr{C}$ whose objects are the cofibrant (resp. fibrant, resp. fibrant and cofibrant) objects. We consider the following four categories:

(i). The category $\pi(\mathscr{C}_c)$ having the same objects as $\mathscr{C}_c$ and such that $\mathrm{Hom}_{\pi(\mathscr{C}_c)}(X, Y) = \pi^r(X, Y)$.

(ii). The category $\pi(\mathscr{C}_f)$ having the same objects as $\mathscr{C}_f$ and such that $\mathrm{Hom}_{\pi(\mathscr{C}_f)}(X, Y) = \pi^l(X, Y)$.

(iii). The category $\pi(\mathscr{C}_{cf})$ having the same objects as $\mathscr{C}_{cf}$ and such that $\mathrm{Hom}_{\pi(\mathscr{C}_{cf})}(X,Y) = \pi(X,Y)$.

(iv). The category $\mathrm{Ho}(\mathscr{C})$ having the same objects as $\mathscr{C}$ and such that, for all $A, B \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(A,B) = \pi(RQ(A), RQ(B))$.

The Hom sets of $\pi(\mathscr{C}_{cf})$ and $\mathrm{Ho}(\mathscr{C})$ are well-defined by Corollary VI.2.2.2, and Corollary VI.2.1.11 implies that composition is well-defined on all four of the categories.

Corollary VI.3.1.5. *(i). The assignement $X \longmapsto Q(X)$ extends to a functor $Q : \mathscr{C} \to \pi(\mathscr{C}_c)$. More precisely, if $f : X \to Y$ is a morphism of $\mathscr{C}$ and $f_1 : Q(X) \to Q(Y)$ is as in Proposition VI.3.1.3, then $Q(f)$ is the class of f_1 in $\pi^r(X,Y)$.*

This functor Q sends weak equivalences to weak equivalences, where a weak equivalence in $\pi(\mathscr{C}_c)$ is the class of a weak equivalence in $\mathscr{C}_c$.

(ii). Similary, if we choose a fibrant replacement $X \to R(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$, then the assignment $X \longmapsto R(X)$ extends to a functor $R : \mathscr{C} \to \pi(\mathscr{C}_f)$, and this functor sends weak equivalences to weak equivalences.

(iii). The assignement $X \longmapsto RQ(X)$ extends to a functor $RQ : \mathscr{C} \to \pi(\mathscr{C}_{cf})$. More precisely, if $f : X \to Y$ is a morphism in $\mathscr{C}$ and $f_2 : RQ(X) \to RQ(Y)$ is as in Proposition VI.3.1.3, then we take $RQ(f)$ equal to the class of f_2 in $\pi(RQ(X), RQ(Y))$.

This functor sends weak equivalences to isomorphisms.

(iv). The functor $\pi(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C})$ induced by the inclusion $\mathscr{C}_{cf} \subset \mathscr{C}$ is an equivalence of categories, and the composition $\mathscr{C} \overset{RQ}{\to} \pi(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C})$ is equal to the functor $p : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ that sends every $X \in \mathrm{Ob}(\mathscr{C})$ to X and every $f : X \to Y$ to the class of f_2 in $\pi(RQ(X), RQ(Y))$, where $f_2 : RQ(X) \to RQ(Y)$ is as in Proposition VI.3.1.3.

Proof. We prove (i) (note that (ii) is just (i) in the opposite category). Let $f : X \to Y$ be a morphism of $\mathscr{C}$ and let $f_1 : Q(X) \to Q(Y)$ is as in Proposition VI.3.1.3. By that proposition, the morphism f_1 is uniquely determined up to homotopy. So its class in $\pi^r(X,Y)$ only depends on f, and it makes sense to take $Q(f)$ equal to this class. It is easy to see that Q is a functor. The last statement of (i) follows immediately from the last statement of Proposition VI.3.1.3.

Proposition VI.3.1.3 immediately gives the first part of (iii), and the second part of (iii) follows from Theorem VI.2.3.2. As for (iv), it is an easy consequence of (iii).

□

VI.3.2 Calculation of the homotopy category

Theorem VI.3.2.1. *The functor $p : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ is a localization of the category $\mathscr{C}$ by W. It sends left or right homotopic morphisms of $\mathscr{C}$ to the same morphism of $\mathrm{Ho}(\mathscr{C})$. Moreover, a morphism f of $\mathscr{C}$ is a weak isomorphism if and only if $p(f)$ is an isomorphism.*

In particular, we get equivalences of categories

$$\mathscr{C}[W^{-1}] \simeq \mathrm{Ho}(\mathscr{C}) \simeq \pi(\mathscr{C}_{cf}).$$

Note that we could have exchanged the roles of Q and R in the theorem (and the results preceding it).

Proof. We check the conditions of Definition V.2.1.1. We already know that p sends weak equivalences to isomorphisms, so we get condition (a).

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor such that $G(s)$ is an isomorphism for every $s \in W$. We construct a functor $G' : \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ such that $G' \circ p = G$. As p is the identity on objects, we take $G'(X) = G(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Let $\varphi \in \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(X, Y)$, and let $f_2 : RQ(X) \to RQ(Y)$ be a morphism of $\mathscr{C}$ representing φ. By Lemma VI.3.2.2, the morphism $G(f_2)$ only depends on φ. We have a diagram:

$$\begin{tikzcd} X & Q(X) \arrow[l, "p_X"'] \arrow[r, "i_X"] & RQ(X) \arrow[d, "f_2"] \\ Y & Q(Y) \arrow[l, "p_Y"'] \arrow[r, "i_Y"] & RQ(Y) \end{tikzcd}$$

where every horizontal morphism is a weak equivalence, so we can define $G'(\varphi)$ by

$$G'(\varphi) = G(p_Y) \circ G(i_Y)^{-1} \circ G(f_2) \circ G(i_X) \circ G(p_X)^{-1};$$

this only depends on φ and not on the choice of f_2. It is easy to see that G' is a functor (for example, if $f_2 : RQ(X) \to RQ(Y)$ and $g_2 : RQ(Y) \to RQ(Z)$ represent $\varphi \in \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(X, Y)$ and $\psi \in \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(Y, Z)$, then $g_2 \circ f_2$ represents $\psi \circ \varphi$), and we have $G' \circ p = G$ by definition of p. This shows condition (b).

We check condition (c). Let $G_1, G_2 : \mathrm{Ho}(\mathscr{C}) \to \mathscr{C}'$ be two functors. We want to show that composing by p on the right induces a bijection

$$\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathrm{Ho}(\mathscr{C}),\mathscr{D})}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(G_1 \circ p, G_2 \circ p).$$

Let $\varphi : X \to Y$ be a morphism in $\mathrm{Ho}(\mathscr{C})$, and let $f_2 : RQ(X) \to RQ(Y)$ be a morphism in $\mathscr{C}$ representing φ. Then we have $\varphi = p(p_Y) \circ p(i_Y)^{-1} \circ p(f_2) \circ p(i_X) \circ p(p_X)^{-1}$ in $\mathrm{Ho}(\mathscr{C})$. As in the proof of Theorem V.2.1.4, this implies that α in injective (that is, a morphism of functors $G_1 \to G_2$ is determined by its restriction to the image of p). Conversely, let $u : G \circ p \to G_2 \circ p$ be a morphism of functors. If there exists a morphism of functors $v : G_1 \to G_2$ such that $u = \alpha(v)$, then we have $v(X) = u(X) : G_1(X) \to G_2(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. So, to show that u is in the image of α, we have to show that the family $(u(X))_{X\in\mathrm{Ob}(\mathscr{C})}$ defines a morphism of functors from G_1 to G_2, that is, that the diagram

$$\begin{tikzcd} G_1(X) \arrow[r, "u(X)"] \arrow[d, "G_1(f)"'] & G_2(X) \arrow[d, "G_2(f)"] \\ G_1(Y) \arrow[r, "u(Y)"'] & G_2(Y) \end{tikzcd}$$

is commutative for every morphism $\varphi : X \to Y$ in $\mathrm{Ho}(\mathscr{C})$. As φ is a composition of morphisms $p(f)$ and $p(s)^{-1}$, for f a morphism of $\mathscr{C}$ and $s \in W$, it suffices to treat the case where $\varphi = p(f)$ or $\varphi = p(s)^{-1}$, but then the conclusion is clear.

The second statement follows immediately from Lemma VI.3.2.2(iii).

We prove the last statement. Let f be a morphism of $\mathscr{C}$. If f is a weak equivalence, we already know that $p(f)$ is an isomorphism. Suppose that $p(f)$ is an isomorphism. Then the morphism $f_2 : RQ(X) \to RQ(Y)$ of Proposition VI.3.1.3 is a homotopy equivalence, hence a weak equivalence by Theorem VI.2.3.2. As we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & Q(X) \arrow[l, "p_X"'] \arrow[r, "i_X"] \arrow[d, "f_1"'] & RQ(X) \arrow[d, "f_2"] \\ Y & Q(Y) \arrow[l, "p_Y"'] \arrow[r, "i_Y"] & RQ(Y) \end{tikzcd}$$

where all the horizontal morphisms are weak equivalences, this implies that f_1 is a weak equivalence, hence that f is a weak equivalence.

$\square$

Lemma VI.3.2.2. *Let $G : \mathscr{C} \to \mathscr{D}$ be a functor.*

(i). Suppose that G sends weak equivalences between cofibrant objects to isomorphisms, and let $f, g : X \to Y$ be two morphisms of $\mathscr{C}$, with X and Y cofibrant. If $f \overset{r}{\sim} g$, then $G(f) = G(g)$.

(ii). Suppose that G sends weak equivalences between fibrant objects to isomorphisms, and let $f, g : X \to Y$ be two morphisms of $\mathscr{C}$, with X and Y fibrant. If $f \sim g$, then $G(f) = G(g)$.

(iii). Suppose that G sends weak equivalences to isomorphisms, and let $f, g : X \to Y$ be two morphisms of $\mathscr{C}$. If $f \overset{l}{\sim} g$ or $f \overset{r}{\sim} g$, then $G(f) = G(g)$.

Proof. Note that (ii) is just (i) in the opposite categories. We prove (i). Let $H : X \to Y^I$ be a right homotopy from f to g, where $Y \xrightarrow{t} Y^I \xrightarrow{p} Y \times Y$ is a path object for Y. By axiom MC5, we can write $t = p' \circ t'$, where $t' : Y \to (Y^I)'$ is an acyclic cofibration and p' is a fibration;

by the two out of three property for weak equivalences, the morphism p' is actually an acyclic fibration. As X is cofibrant, there exists a morphism $H' : X \to (Y^I)'$ making the following diagram commute:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & (Y^I)' \arrow[d, "p'"] \\ X \arrow[r, "H"'] \arrow[ur, "H'"] & Y^I \end{tikzcd}$$

Then $Y \xrightarrow{t'} (Y^I)' \xrightarrow{p \circ p'} Y \times Y$ is also a path object for Y, and $H' : X \to (Y^I)'$ is a right homotopy from f to g. In other words, we may (and will) assume that $t : Y \to Y^I$ is an acyclic cofibration. As Y is cofibrant, this implies that Y^I is also cofibrant. Also, as t is a weak equivalence, the hypothesis on G implies that $G(t)$ is an isomorphism. Write $p = (p_0, p_1)$, where p_0, p_1 are morphisms from Y^I to Y. Then $p_0 \circ t = p_1 \circ t = \mathrm{id}_Y$, so $G(p_0) = G(p_1) = G(t)^{-1}$. On the other hand, we have $p_0 \circ H = f$ and $p_1 \circ H = g$, so this implies that $G(f) = G(g)$.

We prove (iii) in the case where $f \overset{l}{\sim} g$. (The other case follows by considering the opposite categories.) Let $H : X \wedge I \to X$ be a left homotopy from f to g, where $X \sqcup X \to X \wedge I \xrightarrow{s} X$ is a cylinder object for X. Let $i_0, i_1 : X \to X \wedge I$ be the morphisms of Lemma VI.2.1.4. As $s \circ i_0 = s \circ i_1 = \mathrm{id}_X$, we get $G(s) \circ G(i_0) = G(s) \circ G(i_1) = \mathrm{id}_{G(X)}$. As s is a weak equivalence, the morphism $G(s)$ is an isomorphism, and so $G(i_0) = G(i_1)$. As $f = H \circ i_0$ and $g = H \circ i_1$, this implies that $G(f) = G(g)$.

□

VI.4 Localization of functors

We fix a model category $\mathscr{C}$.

VI.4.1 Existence of the localization

Theorem VI.4.1.1. *Let $F : \mathscr{C} \to \mathscr{D}$.*

(i). Suppose that F sends weak equivalences between cofibrant objects to isomorphisms. Then F is left localizable.

More precisely, we can construct a left localization of F in the following way: By Lemma VI.3.2.2(i), the restriction of the functor F to $\mathscr{C}_c$ induces a functor $F' : \pi(\mathscr{C}_c) \to \mathscr{D}$. For every $X \in \mathrm{Ob}(\mathscr{C})$, applying F to the morphism $p_X : Q(X) \to X$ gives a morphism $\tau'(X) : F' \circ Q(X) \to F(X)$; by Proposition VI.3.1.3 and Lemma VI.3.2.2(i), this defines a morphism of functors $\tau' : F' \circ Q \to F$. The hypothesis on F and the last statement of Corollary VI.3.1.5 imply that $F' \circ Q : \mathscr{C} \to \mathscr{D}$ sends weak equivalences to isomorphisms, and the universal property of the localization gives a functor $LF : \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ and an *isomorphism of functors $LF \circ p \simeq F' \circ Q$. We denote by $\tau : LF \circ p \to F$ the composition of the isomorphism $LF \circ p \simeq F' \circ Q$ and of the morphism $\tau' : F' \circ Q \to F$. Then (LF, τ) is a left localization of F.*

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "p"'] \arrow[d, "Q"] \arrow[dd, bend left=60, "F"] & \\ \mathrm{Ho}(\mathscr{C}) \arrow[dr, "LF"'] & \pi(\mathscr{C}_c) \arrow[d, "F'"] \arrow[r, phantom, "\overset{\tau'}{\Longrightarrow}"] & {} \\ & \mathscr{D} & \end{tikzcd}$$

(ii). Suppose that F sends weak equivalences between fibrant objects to isomorphisms. Then F is right localizable, and we can construct a right localization of F by applying the construction of (i) in the opposite categories.

Proof. We prove (i). Let $G : \mathrm{Ho}(\mathscr{C}) \to \mathscr{D}$ be a functor and $s : G \circ p \to F$ be a morphism of functors. We want to show that there exists a unique morphism of functors $u : G \to LF$ such that $s = \tau(u \circ p)$.

Suppose that such a u exists. Then, for every $X \in \mathrm{Ob}(\mathscr{C})$, we have a commutative diagram:

$$\begin{tikzcd} (G\circ p)(Q(X)) \arrow[r, "u(Q(X))"] \arrow[d, "(G\circ p)(p_X)"', "\wr"] \arrow[rr, bend left, "s(Q(X))"] & (LF\circ p)(Q(X)) \arrow[r, "\tau(Q(X))", "\sim"'] \arrow[d, "\wr"', "(LF\circ p)(p_X)"] & F(Q(X)) \arrow[d, "F(p_X)"] \\ (G\circ p)(X) \arrow[r, "u(X)"'] \arrow[rr, bend right, "s(X)"'] & (LF\circ p)(X) \arrow[r, "\tau(X)"'] & F(X) \end{tikzcd}$$

where $\tau(Q(X))$ is an isomorphism because $Q(X)$ is cofibrant (hence $\tau'(Q(X)) = F(p_{Q(X)}) = \mathrm{id}_{Q(X)}$).

This implies that:

(*) $$u(X) = (LF \circ p)(p_X) \circ \tau(Q(X))^{-1} \circ s(Q(X)) \circ (G \circ p)(p_X)^{-1}.$$

In particular, the morphism of functors u is uniquely determined by the property that $\tau(u \circ p) = s$.

To show the existence of u, we need to prove that formula (*) defines a morphism of functors from G to LF. So we need to check that, for every morphism $f : X \to Y$ in $\mathrm{Ho}(\mathscr{C})$, the diagram

$$\begin{tikzcd} G(X) \arrow[r, "u(X)"] \arrow[d, "G(f)"'] & LF(X) \arrow[d, "LF(f)"] \\ G(Y) \arrow[r, "u(Y)"'] & LF(Y) \end{tikzcd}$$

commutes. If f is of the form $p(g)$ or $p(g)^{-1}$ where g is a morphism of $\mathscr{C}$, this follows easily from formulas (*) and the fact that s is a morphism of functors. But we have seen in the proof of Theorem VI.3.2.1 that every morphism of $\mathrm{Ho}(\mathscr{C})$ is a composition of morphisms of that type, so we are done.

□

VI.4.2 Quillen adjunctions and Quillen equivalences

In this subsection, we fix two model categories $(\mathscr{C}, W_{\mathscr{C}}, \mathrm{Fib}_{\mathscr{C}}, \mathrm{Cof}_{\mathscr{C}})$ and $(\mathscr{D}, W_{\mathscr{D}}, \mathrm{Fib}_{\mathscr{D}}, \mathrm{Cof}_{\mathscr{D}})$, and we denote by $p_{\mathscr{C}} : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ and $p_{\mathscr{D}} : \mathscr{D} \to \mathrm{Ho}(\mathscr{D})$ the localization functors.

Definition VI.4.2.1. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. A *(total) left derived functor* (resp. *(total) right derived functor*) is a left (resp. right) localization of the functor $p_{\mathscr{D}} \circ F : \mathscr{C} \to \mathrm{Ho}(\mathscr{D})$.

Proposition VI.4.2.2. *Let $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors; we don't assume anything about teh categories $\mathscr{C}$ and $\mathscr{D}$. Let S (resp. T) be a set of morphisms in $\mathscr{C}$ (resp. $\mathscr{D}$), and let f (resp. g) be a morphism in $\mathscr{C}$ (resp. $\mathscr{D}$). Then:*

(i). f has the left lifting property relatively to $G(T)$ if and only $F(f)$ has the left lifting property relatively to T;

(ii). g has the right lifting property relatively to $F(S)$ if and only if $G(g)$ has the right lifting property relatively to S.

Proof. The proofs of (i) and (ii) are dual, so we only prove (i).

Write $f : A \to B$, consider two morphisms $u : A \to G(X)$ and $v : B \to G(Y)$ in $\mathscr{C}$, and let $p : X \to Y$ be an element of T. By Lemma I.4.2 (that is, by Problem A.1.7), the square

$$\begin{tikzcd} F(A) \arrow[r, "u^\sharp"] \arrow[d, "F(f)"'] & X \arrow[d, "p"] \\ F(B) \arrow[r, "v^\sharp"'] \arrow[ur, dashed, "h^\sharp"] & Y \end{tikzcd}$$

is commutative if and only if the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & G(X) \arrow[d, "G(p)"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "h"] & G(Y) \end{tikzcd}$$

is commutative. If f has the left lifting property relatively to $G(T)$, then there exists $h : B \to G(X)$ making the second diagram commute, and then $h^\sharp : F(B) \to X$ makes the first diagram commute (by Lemma I.4.2 again), so $F(f)$ has the left lifting property relatively to T.

Conversely, if $F(f)$ has the left lifting property relatively to T, then there exists $h^\sharp : B \to G(X)$ making the first diagram commute, and then $h : F(B) \to G(Y)$ makes the second diagram commute, so f has the left lifting property relatively to $G(T)$.

□

Corollary VI.4.2.3. *Assume again that $\mathscr{C}$ and $\mathscr{D}$ are model categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors, and suppose that (F, G) is a pair of adjoint functors. We denote the functorial isomorphism $\mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(F(\cdot), \cdot)$ by $f \longmapsto f^\sharp$.*

Then the following conditions are equivalent:

(a) $F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}}$ *and* $G(\mathrm{Fib}_{\mathscr{D}}) \subset \mathrm{Fib}_{\mathscr{C}}$;

(b) $F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}}$ *and* $F(\mathrm{Cof}_{\mathscr{C}} \cap W_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}} \cap W_{\mathscr{D}}$;

(c) $G(\mathrm{Fib}_{\mathscr{D}}) \subset \mathrm{Fib}_{\mathscr{C}}$ *and* $G(\mathrm{Fib}_{\mathscr{D}} \cap W_{\mathscr{D}}) \subset \mathrm{Fib}_{\mathscr{C}} \cap W_{\mathscr{C}}$.

Proof. Suppose that G preserves fibrations (i.e. that $G(\mathrm{Fib}_{\mathscr{D}}) \subset \mathrm{Fib}_{\mathscr{C}}$). We show that F preserves cofibrations if and only if G preserves acyclic fibrations, which shows the equivalence of (a) and (c). (The equivalence of (a) and (b) then follows by considering the opposite categories.) If G preserves acyclic fibrations, then any cofibration i in $\mathscr{C}$ has the left lifting property relatively to $G(W_{\mathscr{D}} \cap \mathrm{Fib}_{\mathscr{D}})$, so, by Proposition VI.4.2.2, the morphism $F(i)$ has the left lifting property relatively to $W_{\mathscr{D}} \cap \mathrm{Fib}_{\mathscr{D}}$, hence is a cofibration by Proposition VI.1.2.2. Conversely, if F preserves cofibrations, then any acyclic fibration p in $\mathscr{D}$ has the right lifting property relatively to $F(\mathrm{Cof}_{\mathscr{C}})$, so, by Proposition VI.4.2.2, the morphism $G(p)$ has the right lifting property relatively to $\mathrm{Cof}_{\mathscr{C}}$, hence is an acyclic fibration by Proposition VI.1.2.2.

□

Definition VI.4.2.4. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors, and suppose that (F, G) is a pair of adjoint functors. We denote the functorial isomorphism $\mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(F(\cdot), \cdot)$ by $f \longmapsto f^\sharp$.

(i). We say that (F, G) is a *Quillen adjunction* between $\mathscr{C}$ and $\mathscr{D}$ (or that F is a *left Quillen functor*, or that G is a *right Quillen functor*) if the equivalent conditions of Corollary VI.4.2.3 are satisfied.

(ii). We say that (F, G) is a *Quillen equivalence* between $\mathscr{C}$ and $\mathscr{D}$ if it is a Quillen adjunction and if, for every cofibrant $A \in \mathrm{Ob}(\mathscr{C})$, every fibrant $X \in \mathrm{Ob}(\mathscr{D})$ and every morphism $f : A \to G(X)$ in $\mathscr{C}$, we have:

$$f \in W_{\mathscr{C}} \Leftrightarrow f^\sharp \in W_{\mathscr{D}}.$$

Theorem VI.4.2.5. *Let (F, G) be a Quillen adjunction between $\mathscr{C}$ and $\mathscr{D}$. Then the total left derived functor $LF : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ of F and the total right derived functor $RG : \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$ of G exist, and the functors (LF, RG) form a pair of adjoint functors.*

If moreover (F,G) is a Quillen equivalence, then the functors LF and RG are mutually quasi-inverse equivalences of categories.

Proof. The functor F sends acyclic cofibrations to weak equivalences, so, by Ken Brown's lemma (Lemma VI.1.2.5), it sends any weak equivalence between cofibrant objects to a weak equivalence. In particular, the functor $p_{\mathscr{D}} \circ F$ sends weak equivalences between cofibrant objects to isomorphisms. By Theorem VI.4.1.1, this functor has a left localization $LF : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$. A similar proof shows that $p_{\mathscr{C}} \circ G : \mathscr{D} \to \mathrm{Ho}(\mathscr{C})$ has a right localization $RG : \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$.

As F is a left adjoint, it preserves all colimits by Proposition I.5.4.3, so it sends initial objects of $\mathscr{C}$ to initial objects of $\mathscr{D}$; as F also preserves cofibrations, it preserves cofibrant objects. Similarly, the functor G preserves fibrant objects. So, to show that LF and RG form an adjoint pair, it suffices by the construction of these functors in Theorem VI.4.1.1 to show that, for every cofibrant $A \in \mathrm{Ob}(\mathscr{C})$ and every fibrant $X \in \mathrm{Ob}(\mathscr{D})$, the adjunction isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(A, G(X)) \simeq \mathrm{Hom}_{\mathscr{D}}(F(A), X)$$

preserves the homotopy relation (which is equal to the left and right homotopy relations by Corollary VI.2.2.2 and is an equivalence relation by Proposition VI.2.1.9).

Let $f, g : A \to G(X)$ be two morphisms. Suppose that $f \overset{l}{\sim} g$. Let $H : A \wedge I \to G(X)$ be a left homotopy from f to g, where $A \sqcup A \overset{i}{\to} A \wedge I \overset{s}{\to} A$ is a cylinder object for A. As A is cofibrant and i is a cofibration, the object $A \wedge I$ is also cofibrant. In particular, the objects $F(A \wedge I)$ and $F(A)$ are cofibrant, so, by the start of the proof, the morphism $F(s) : F(A \wedge I) \to F(A)$ is a weak equivalence; as moreover F preserves cofibrations and commutes with colimits, the diagram $F(A) \sqcup F(A)A \overset{F(i)}{\to} F(A \wedge I) \overset{F(s)}{\to} F(A)$ is a cylinder object for $F(A)$. By Lemma I.4.2, the morphism $H^\sharp : F(A \wedge I) \to X$ makes the diagram

$$\begin{tikzcd} F(A) \sqcup F(A) \arrow[r, "F(i)"] \arrow[dr, "f^\sharp \sqcup g^\sharp"'] & F(A \wedge I) \arrow[d, "H^\sharp"] \\ & X \end{tikzcd}$$

commute, so it is a left homotopy from $f^\sharp$ to $g^\sharp$. A similar proof shows that, if $f^\sharp \overset{r}{\sim} g^\sharp$, then we have $f \overset{r}{\sim} g$.

We prove the last statement. Let $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ be the unit and counit of the adjunction (F,G), and let $\eta' : \mathrm{id}_{\mathrm{Ho}(\mathscr{C})} \to RG \circ LF$ and $\varepsilon' : LF \circ RG \to \mathrm{id}_{\mathrm{Ho}(\mathscr{D})}$ be the unit and counit of the adjunction (LF, RG). It suffices to show that η' and ε' are isomorphisms of functors. Let A be a cofibrant object of $\mathscr{C}$, and let $i : F(A) \to R(F(A))$ be a fibrant replacement of $F(A)$ in $\mathscr{D}$. We have $LF(A) \simeq F(A)$, hence $LG(LF(A)) \simeq G(R(F(A)))$, and, by definition of the adjunction between LF and RG, the morphism $\eta'(A) : A \to RG(LF(A))$ is the composition

$$A \overset{\eta(A)}{\to} G(F(A)) \overset{G(i)}{\to} G(R(F(A))) \simeq LG(LF(A)).$$

By Proposition I.4.6, we have $i = (G(i) \circ \eta(A))^\sharp$. As A is cofibrant and $R(F(A))$ is fibrant, and as i is a weak equivalence, the definition of a Quillen equivalence implies that $G(i) \circ \eta(A)$ is a weak equivalence. This implies that $\eta'(A)$ is an isomorphism in $\mathrm{Ho}(\mathscr{C})$. The proof that ε' is an isomorphism of functors is similar.

□

VI.5 Construction of model structures

We fix a universe $\mathscr{U}$ and a $\mathscr{U}$-category $\mathscr{C}$ that has all $\mathscr{U}$-small colimits.

For simplicity, we will limit ourselves in this section to compositions of sequences of morphisms indexed by $\mathbb{N}$, so we will not get the most general definition of a cofibrantly generated model category. It is relatovely easy to generalize everything to arbitrary ordinals $\alpha \in \mathscr{U}$ (see for example Section 2.1 of [5]).

VI.5.1 Transfinite composition

Definition VI.5.1.1. An *ω-sequence* (of morphisms of $\mathscr{C}$) is a functor $X : \mathbb{N} \to \mathscr{C}$. We often write the functor X as

$$X_0 \to X_1 \to X_2 \to \dots.$$

The *composition* of the ω-sequence X is the morphism $X_0 \to \varprojlim_{r \geq 0} X_r$.

Note that the composition of an ω-sequence is only defined up to (unique) isomorphism.

Definition VI.5.1.2. Let I be a set of morphisms of $\mathscr{C}$. If $X : \mathbb{N} \to \mathscr{C}$ is an ω-sequence that sends every morphism in $\mathbb{N}$ to an element of I, we call its composition a *transfinite composition* of morphisms of I.

Definition VI.5.1.3. Let I be a set of morphisms of $\mathscr{C}$ and $A \in \mathrm{Ob}(\mathscr{C})$.

(i). We say that A is *small relative to I* if, for every ω-sequence $X : \mathbb{N} \to \mathscr{C}$ sending every morphism of $\mathbb{N}$ to an element of I, the canonical map

$$\varinjlim_{r \geq 0} \mathrm{Hom}_{\mathscr{C}}(A, X_r) \to \mathrm{Hom}_{\mathscr{C}}(A, \varinjlim_{r \geq 0} X_r)$$

is bijective. [2]

(ii). We say that A is *small* if it is small relative to the set of all morphisms of $\mathscr{C}$.

[2] If we were working with arbitrary ordinals, we could κ-smallness for κ a cardinal. What we are calling "small" here would then be called "0-small".

VI.5.2 Sets of morphisms defined by lifting properties

Proposition VI.5.2.1. *Let S be a set of morphisms in $\mathscr{C}$, and let T be the set of morphisms that have the left lifting property relatively to every element of S. Then T is stable by pushouts, by ($\mathscr{U}$-small) coproducts, by transfinite compositions and by retracts.*

Of course, we have a dual result saying that the set of morphisms that has the right lifting property relatively to every element of S is stable by all the pullbacks that exist in $\mathscr{C}$, by products, by compositions (even by infinite compositions "in the other direction", that is, limits of functors $\mathbb{N}^{\mathrm{op}} \to \mathscr{C}$) and by retracts.

Proof. The stability by pushout is proved exactly as in Corollary VI.1.2.4: Consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] & A' \arrow[r, "u"] \arrow[d, "t'"] & X \arrow[d, "s"] \\
B \arrow[r, "g"'] \arrow[urr, dashed, "h"] & B' \arrow[r, "v"'] & Y
\end{tikzcd}$$

where the first square is cocartesian, $t \in T$ and $s \in S$. By definition of T, there exists a morphism $h : B \to X$ making the diagram commute. As $h \circ t = u \circ f$, the pair $(h : A \to X, u : A' \to X)$ induce a (unique) morphism $h' : B' \to X$ such that $h' \circ t' = u$ and $h' \circ g = h$. The fact that $s \circ h = v$ follows from the universal property of the pushout.

We show that T is stable by coproducts. Let $(t_i : A_i \to B_i)_{i \in I}$ be a family of morphisms of T, with I a $\mathscr{U}$-set, let $t = \coprod t_i : A = \coprod_{i \in I} A_i \to B = \coprod_{i \in I} B_i$, and consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "t"'] & X \arrow[d, "s"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

with $s \in S$. We want to find a morphism $h : B \to X$ that makes the diagram commute. For every $i \in I$, restricting f and g to A_i and B_i gives a commutative diagram

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "t_i"'] & X \arrow[d, "s"] \\
B_i \arrow[r] \arrow[ur, dashed, "h_i"] & Y
\end{tikzcd}$$

and we can a find a morphism $h_i : B_i \to X$ that makes the diagram commute because $t_i \in T$. So we can take $h = \coprod h_i : B \to X$.

Now consider a ω-sequence $A_0 \overset{t_0}{\to} A_1 \overset{t_1}{\to} A_2 \to \dots$ such that $t_r \in T$ for every $r \geq 0$, let

$t : A_0 \to B$ be its composition, and consider a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "t"'] & X \arrow[d, "s"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

with $s \in S$. We want to find a morphism $h : B \to X$ that makes the diagram commute. For every $r \geq 0$, we denote by u_r the composition of the ω-sequence $A_r \overset{t_r}{\to} A_{r+1} \overset{t_{r+1}}{\to} A_{r+2} \to \ldots$, so that $u_r \circ t_{r-1} \circ \ldots \circ t_0 = t$. Let $h_0 = f$. As t_r in in T for every r, we can construct by induction on $r \geq 1$ a morphism $h_r : A_r \to X$ that makes the following diagram commute

$$\begin{tikzcd} A_{r-1} \arrow[r, "h_{r-1}"] \arrow[d, "t_{r-1}"'] & X \arrow[d, "s"] \\ A_r \arrow[r, "g \circ u_r"'] \arrow[ur, "h_r"] & Y \end{tikzcd}$$

Then we can take $h = \varinjlim_{r \geq 0} h_r$.

Finally, we show that T is stable by retracts. Suppose that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g_A"'] \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[d, "t"'] & A' \arrow[r, "f_A"'] \arrow[d, "t'"'] & A \arrow[d, "t"'] \\ B \arrow[r, "g_B"] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r, "f_B"] & B \end{tikzcd}$$

with $t' \in T$, and consider a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "t"'] & X \arrow[d, "s"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

with $s \in S$. As $t' \in T$, there exists a morphism $h' : B' \to X$ such that $s \circ h' = v \circ f_B$ and $h' \circ t' = u \circ f_A$.

$$\begin{tikzcd} A' \arrow[r, "f_A"] \arrow[d, "t'"'] & A \arrow[r, "u"] & X \arrow[d, "s"] \\ B' \arrow[r, "f_B"'] \arrow[urr, dashed, "h'"] & B \arrow[r, "v"'] & Y \end{tikzcd}$$

Let $h = h' \circ g_B$. Then $h \circ t = h' \circ g_B \circ t = h' \circ t' \circ g_A = u \circ f_A \circ g_A = u$ and $s \circ h = s \circ h' \circ g_B = v \circ f_B \circ g_B = v$, so the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "t"'] & X \arrow[d, "s"] \\ B \arrow[r, "v"'] \arrow[ur, "h"'] & Y \end{tikzcd}$$

This shows that $t \in T$.

□

VI.5.3 The small object argument

Let I be a set of morphisms of $\mathscr{C}$.

Definition VI.5.3.1. (i). A morphism p of $\mathscr{C}$ is called *I-injective* if it has the right lifting property relatively to any element of I.

(ii). A morphism i of $\mathscr{C}$ is called a *I-cofibration* if it has the left lifting property relatively to any I-injective morphism.

(iii). A morphism f of $\mathscr{C}$ is called *I-cellular* if it is a transfinite composition of morphisms $A \to B$ that fit in a cocartesian square

$$\begin{tikzcd} \coprod_{i\in I} U_i \arrow[r] \arrow[d, "\coprod f_i"'] & A \arrow[d] \\ \coprod_{i\in I} V_i \arrow[r] & B \end{tikzcd}$$

where every $f_i : U_i \to V_i$ is an element of I.

We denote by $I - \mathrm{inf}$ (resp. $I - \mathrm{cof}$, resp. $I - \mathrm{cell}$) the set of I-injective morphisms (resp. I-cofibrations, resp. I-cellular morphisms).

Of course, we have dual notions of I-projective morphisms and I-fibrations, but we will not use them in these notes.

Corollary VI.5.3.2. *Every I-cellular morphism is a I-cofibration.*

Proof. By definition of I-cofibrations, the set $I - \mathrm{cof}$ contains I. By Proposition VI.5.2.1, the set $I - \mathrm{cof}$ is stable by pushouts, coproducts and by transfinite compositions, so it also contains $I - \mathrm{cell}$.

□

Proposition VI.5.3.3. *(The small object argument.) Suppose that I is a $\mathscr{U}$-set (i.e. isomorphic to an element of $\mathscr{U}$),* [3] *for every morphism $\gamma : U \to V$ in I, the source U of γ is small relative to the set of I-cellular morphisms. Then, for every morphism $f : X \to Y$ of $\mathscr{C}$, there exists a factorization*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

[3] In other words, the set I and sets built from it are small enough so that colimits indexed by them exist in $\mathscr{C}$.

of f with $i \in I - \mathrm{cell}$ and $p \in I - \mathrm{inj}$, and moreover this factorization is functorial in f.

Proof. Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Let $Z_0 = X$ and $f_0 = f$. We construct by induction on $r \geq 1$ a family of morphisms $i_{r-1} : Z_{r-1} \to Z_r$ and $f_r : Z_r \to Y$ such that $f_r \circ i_{r-1} = f_{r-1}$. Suppose that $r \geq 1$ and that we have constructed f_{r-1}. Let S_r be the set of commutative squares s of the form

$$\begin{tikzcd} U_s \arrow[r] \arrow[d, "\gamma_s"'] & Z_{r-1} \arrow[d, "f_{r-1}"] \\ V_s \arrow[r] & Y \end{tikzcd}$$

with $\gamma_s \in I$; as I is a $\mathscr{U}$-set, so is S_r. We define $i_r : Z_{r-1} \to Z_r$ by the commutative diagram

$$\begin{tikzcd} \coprod_{s \in S_r} U_s \arrow[r] \arrow[d, "\coprod \gamma_s"'] \arrow[dr, phantom, "(*)"] & Z_{r-1} \arrow[d, "i_{r-1}"] \arrow[ddr] & \\ \coprod_{s \in S_r} V_s \arrow[r] \arrow[drr] & Z_r \arrow[dr, "f_r"] & \\ & & Y \end{tikzcd}$$

where the square (*) is cocartesian. (That is, we take Z_r to be the pushout of the morphisms $\coprod_{s \in S} U_s \to Z_{r-1}$ and $\coprod \gamma_s : \coprod_{s \in S} U_s \to \coprod_{s \in S} V_s$, and $f_r : Z_r \to Y$ to be the morphism induced by the morphisms $f_{r-1} : Z_{r-1} \to Y$ and $\coprod_{s \in S} V_s \to Y$.)

The morphisms $(i_r)_{r \geq 0}$ define a ω-sequence, and we take $i : X \to Z = \varinjlim_{r \geq 0} Z_r$ to be its composition and $p : Z \to Y$ to be the morphism $\varinjlim_{r \geq 0} f_r$. This will be our factorization. If we make make colimits into a functor from ω-sequences to $\mathscr{C}$ (using Proposition I.5.1.4), then the construction of i and p is functorial in f. Also, the morphism i is I-cellular by definition of I-cellular.

It remains to show that p is I-injective. Consider a commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\gamma"'] & Z \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$

with $\gamma \in I$. We want to find a morphism $h : V \to Z$ making the following diagram commute:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\gamma"'] & Z \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dashed, "h"'] & Y \end{tikzcd}$$

As U is small relative to $I - \mathrm{cell}$, the morphism $U \to Z$ has a factorization $U \to Z_r \to Z$ for some $r \geq 0$. But then the square

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\gamma"'] & Z_r \arrow[d, "f_r"] \\ V \arrow[r] & Y \end{tikzcd}$$

is in S_{r+1} by definition of S_{r+1}, so, by definition of Z_{r+1}, there exists a morphism $h' : V \to Z_{r+1}$ making the following diagram commute:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\gamma"'] & Z_r \arrow[r, "i_r"] \arrow[d, "f_r"] & Z_{r+1} \arrow[d, "f_{r+1}"] \\ V \arrow[r] \arrow[urr, dashed, "h"] & Y \arrow[r, equal] & Y \end{tikzcd}$$

We get the desired morphism $h : V \to Z$ by taking the composition $V \stackrel{h'}{\to} Z_{r+1} \to Z$.

□

Corollary VI.5.3.4. *Under the hypotheses of Proposition VI.5.3.3, every I-cofibration is a retract of a I-cellular morphism.*

Proof. Let $f : X \to Y$ be an element of $I-\mathrm{cof}$. By Proposition VI.5.3.3, we can write $f = p \circ i$, with $i : X \to Z$ in $I - \mathrm{cell}$ and $p : Z \to X$ in $I - \mathrm{inj}$. By definition of $I - \mathrm{cof}$, there exists a morphism $h : Y \to Z$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "h"] & Y \end{tikzcd}$$

So we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"'] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

which shows that f is a retract of i.

□

VI.5.4 Cofibrantly generated model categories

Definition VI.5.4.1. Let $(\mathscr{C}, W, \mathrm{Fib}, \mathrm{Cof})$ be a model category. We say that $\mathscr{C}$ is *cofibrantly generated* if there exist $\mathscr{U}$-small sets of morphisms I and J in $\mathscr{C}$ such that:

(i). The sources of the morphisms of I are small relatively to $I-\mathrm{cell}$.

(ii). The sources of the morphisms of J are small relatively to $J-\mathrm{cell}$.

(iii). $\mathrm{Fib} = J - \mathrm{inj}$.

(iv). $W \cap \mathrm{Fib} = I - \mathrm{inj}$.

We call I the set of *generating cofibrations* and J the set of *generating acyclic cofibrations*.

Remark VI.5.4.2. Because we restricted ourselves to colimits index by $\mathbb{N}$ in the definition of small objects, the categories of the previous definition are actually a special class of cofibrantly generated model categories, called *finitely generated* model catgeories in Definition 2.1.17 of [5].

Corollary VI.5.4.3. *Let $\mathscr{C}$ be a cofibrantly generated model category, with generating set of cofibrations (resp. acyclic cofibrations) I (resp. J). Then* $\mathrm{Cof} = I-\mathrm{cof}$ *and* $W\cap\mathrm{Cof} = J-\mathrm{cof}$.

Proof. This follows immediately from the characterization in Proposition VI.1.2.2 of cofibrations (resp. acyclic cofibrations) as the morphisms having the left lifting property relatively to all acyclic fibrations (resp. all fibrations), and from conditions (iii) and (iv) in Definition VI.5.4.1.

□

Corollary VI.5.4.4. *If $\mathscr{C}$ is a cofibrantly generated model category, then the factorizations of axiom MC5 are functorial in the morphism f in $\mathscr{C}$.*

Theorem VI.5.4.5. *Let $\mathscr{C}$ be a $\mathscr{U}$-category having all $\mathscr{U}$-small limits and colimits and W a set of morphisms of $\mathscr{C}$ containing all identity morphisms, stable by retracts and satisfying the two out of three property. Let I and J be two $\mathscr{U}$-small sets of morphisms of $\mathscr{C}$ such that:*

(a) the sources of the morphisms of I are small relatively to $I-\mathrm{cell}$ *and the sources of the morphisms of J are small relatively to* $J-\mathrm{cell}$*;*

(b) $J-\mathrm{cof} \subset W \cap I-\mathrm{cof}$*;*

(c) $I-\mathrm{inj} \subset W \cap J-\mathrm{inj}$*;*

(d) one of the inclusions in (b) and (c) is an equality.

Then there exists a model structure on $\mathscr{C}$ with set of weak equivalences W that makes $\mathscr{C}$ a cofibrantly generated model category with generating set of cofibrations (resp. acyclic cofibrations) I (resp. J). In particular, we have $\mathrm{Fib} = J-\mathrm{inj}$, $W\cap\mathrm{Fib} = I-\mathrm{inj}$, $\mathrm{Cof} = I-\mathrm{cof}$ *and* $W\cap\mathrm{Cof} = J-\mathrm{cof}$*, hence the inclusions in (b) and (c) are both equalities.*

Proof. We set $\mathrm{Fib} = J-\mathrm{inj}$ and $\mathrm{Cof} = I-\mathrm{cof}$. By Proposition VI.5.2.1 (applied to $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$), the sets Fib and Cof are stable by composition and by retracts; they also clearly contain the identity morphisms. So far, we have checked axioms MC1, MC2 and MC3.

Let f be a morphism of $\mathscr{C}$, and let $f = p_1 \circ i_1 = p_2 \circ i_2$ be the two factorizations of f that we obtain by applying the small object argument (Proposition VI.5.3.3) to I and J. Then $p_1 \in I - \mathrm{inj} \subset W \cap J - \mathrm{inf} = W \cap \mathrm{Fib}$ and $i_1 \in I - \mathrm{cell} \subset I - \mathrm{cof} = \mathrm{Cof}$ (the first inclusion is Corollary VI.5.3.2). On the other hand, we have $p_2 \in J - \mathrm{inj} = \mathrm{Fib}$ and $i_2 \in J - \mathrm{cell} \subset J - \mathrm{cof} \subset W \cap I - \mathrm{cof} = W \cap \mathrm{Cof}$. This gives axiom MC5.

It remains to check axiom MC4. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

We want to show that there exists a diagonal morphism $h : B \to X$ making the diagram commute under some conditions on i and p. There are two cases:

(1) Suppose that $J - \mathrm{cof} = W \cap I - \mathrm{cof} = W \cap \mathrm{Cof}$. If i is an acyclic cofibration and p is a fibration, then $i \in W \cap \mathrm{Cof} = J - \mathrm{cof}$ and $p \in J - \mathrm{inj}$, so h exists by definition of $J - \mathrm{cof}$.

Suppose that p is an acyclic fibration, that is, $p \in W \cap J - \mathrm{inj}$, and that $i \in \mathrm{Cof} = I - \mathrm{cof}$. Write $p = q \circ j$, where $q \in I - \mathrm{inj}$ and $j : X \to Z$ is cofibration; as $p \in W$ and W has the two out of three property, the morphism j is actually an acyclic cofibration. By the previous paragraph, the morphism j has the left lifting property relatively to p. So, by considering the commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

we get a morphism $h : Z \to X$ such that $p \circ h = q$ and $h \circ j = \mathrm{id}_X$. So we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] \arrow[rr, bend left, "\mathrm{id}_X"] & Z \arrow[r, "h"] \arrow[d, "q"'] & X \arrow[d, "p"'] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

showing that p is a retract of q. As $q \in I - \mathrm{inj}$, another application of Proposition VI.5.2.1 shows that $p \in I - \mathrm{inj}$, so i has the left lifting property relatively to p. Note that we have also proved that $W \cap J - \mathrm{inj} \subset I - \mathrm{inj}$, so we get $I - \mathrm{inj} = W \cap J - \mathrm{inj} = W \cap \mathrm{Fib}$.

(2) If $I - \mathrm{inj} = W \cap J - \mathrm{inj}$, the proof is similar. If i is a cofibration and p is an acyclic fibration, then $i \in I - \mathrm{cof}$ and $p \in W \cap J - \mathrm{inj} = I - \mathrm{inj}$, so h exists by definition of $I - \mathrm{cof}$.

Suppose that i is an acyclic cofibration, that is, $i \in W \cap I-\mathrm{cof}$, and that $p \in \mathrm{Fib} = J-\mathrm{inj}$. We want to show that p has the right lifting property relatively to i. By Proposition VI.5.3.3, we can write $i = q \circ j$, with $j : A \to C$ in $J - \mathrm{cell} \subset J - \mathrm{cof}$ and $q : C \to B$ in $J - \mathrm{inj}$. As $j \in J - \mathrm{cof} \subset W$ and $i \in W$, the two out of three property for W implies that $q \in W \cap J - \mathrm{inj} = I - \mathrm{inj}$. By the previous paragraph, the morphism i has the left lifting property relatively to q, so there exists $h : B \to C$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "q"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "h"] & B \end{tikzcd}$$

So we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[r, equal] \arrow[d, "j"'] & A \arrow[d, "i"'] \\ B \arrow[r, "h"'] \arrow[rr, bend right, "\mathrm{id}_B"'] & C \arrow[r, "q"'] & B \end{tikzcd}$$

which shows that i is a retract of j, hence that $i \in J - \mathrm{cof}$ has the left lifting property relatively to $p \in J - \mathrm{inj}$. Note that we have also proved that $W \cap I - \mathrm{cof} \subset J - \mathrm{cof}$, so we get $J - \mathrm{cof} = W \cap I - \mathrm{cof}$.

$\square$

Example VI.5.4.6. (1) Let R be a ring. We use the notation of problem A.10.2. In that problem, we proved that the category $\mathscr{C} = \mathcal{C}^-({}_R\mathbf{Mod})$ has a model structure with the set of quasi-isomorphisms as weak equivalences. By problem A.11.1, this is a cofibrantly generated model structure, with set of generating cofibrations

$$I = \{S^n \to D^{n-1},\ n \in \mathbb{Z}\},$$

and set of generating acyclic cofibrations

$$J = \{0 \to D_n,\ n \in \mathbb{Z}\}.$$

On $\mathscr{C}' = \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$, we also have a cofibrantly generated model structure with the set of quasi-isomorphisms as weak equivalences. The set of generating cofibrations is

$$I' = \{S^n \to D^{n-1},\ n \leq 0\} \cup \{0 \to S^0\},$$

and the set of generating acyclic cofibrations is

$$J' = \{0 \to D_n,\ n \leq -1\}.$$

(2) The classical Quillen model structure on **Top** (defined after Definition VI.1.3.6) is cofibrantly generated, with
$$I = \{S^{n-1} \to D^n,\ n \geq 0\}$$
and
$$J = \{D^n \to D^n \times [0,1],\ n \geq 0\},$$
where $D^n \subset \mathbb{R}^n$ is the closed unit ball, $S^{n-1} \subset D^n$ is the unit sphere, $S^{-1} = \varnothing$, and the map $D^n \to D^n \times [0,1]$ in J send $x \in D^n$ to $(x, 0)$. The morphisms in I-cell are called relative cell complexes.

Proposition VI.5.4.7. *Let $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors, with $\mathscr{C}$ and $\mathscr{D}$ model categories. Suppose that $\mathscr{C}$ is cofibrantly generated, with set of generating cofibrations I and set of generating acyclic cofibrations J. Then (F, G) is a Quillen adjunction if and only if $F(I) \subset \mathrm{Cof}_{\mathscr{D}}$ and $F(J) \subset W_{\mathscr{D}} \cap \mathrm{Cof}_{\mathscr{D}}$.*

Proof. If (F, G) is a Quillen adjunction, then $F(I) \subset F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathscr{C}_D$ and $F(J) \subset F(W_{\mathscr{C}} \cap \mathrm{Cof}_{\mathscr{C}}) \subset W_{\mathscr{D}} \cap \mathrm{Cof}_{\mathscr{D}}$.

Conversely, suppose that $F(I) \subset \mathrm{Cof}_{\mathscr{D}}$ and $F(J) \subset W_{\mathscr{D}} \cap \mathrm{Cof}_{\mathscr{D}}$. By Proposition VI.4.2.2, if g is a morphism of $\mathscr{D}$, we have $g \in F(I) - \mathrm{inj}$ if and only if $G(g) \in I - \mathrm{inj}$; in particular, we have $G(F(I) - \mathrm{inj}) \subset I - \mathrm{inj}$. So, if $f \in \mathrm{Cof}_{\mathscr{C}} = I - \mathrm{cof}$, then f has the left lifting property relatively to every element of $G(F(I) - \mathrm{inj})$, so, by Proposition VI.4.2.2 again, the morphism $F(f)$ has the left lifting property relatively to every element of $F(I) - \mathrm{inj}$; as $F(I) \subset \mathrm{Cof}_{\mathscr{D}}$, we have $F(I) - \mathrm{inj} \supset W_{\mathscr{D}} \cap \mathrm{Fib}_{\mathscr{D}}$, so we finally get that $F(f) \in \mathrm{Cof}_{\mathscr{D}}$. This shows that $F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}}$. If we run through the same proof using J instead of I, we get that $F(W_{\mathscr{C}} \cap \mathrm{Cof}_{\mathscr{C}}) \subset W_{\mathscr{D}} \cap \mathrm{Cof}_{\mathscr{D}}$. □

VI.5.5 Promoting a model structure

Theorem VI.5.5.1. *Let $(\mathscr{C}, W_{\mathscr{C}}, \mathrm{Fib}_{\mathscr{C}}, \mathrm{Cof}_{\mathscr{C}})$ be a cofibrantly generated model category, with set of generating cofibrations I and set of generating acyclic cofibrations J. Let $\mathscr{D}$ be a $\mathscr{U}$-category that has all $\mathscr{U}$-small limits and colimits, and let $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ be a pair of adjoint functors. Suppose that:*

(1) the sources of the morphisms of $F(I)$ are small relatively to $F(I) - \mathrm{cell}$ and the sources of the morphisms of $F(J)$ are small relatively to $F(J) - \mathrm{cell}$;

(2) $G(F(J) - \mathrm{cell}) \subset W_{\mathscr{C}}$.

Then the sets $F(I)$ and $F(J)$ are the sets of generating cofibrations and generating acyclic cofibrations of a cofibrantly generated model structure $(W_{\mathscr{D}}, \mathrm{Fib}_{\mathscr{D}}, \mathrm{Cof}_{\mathscr{D}})$ on $\mathscr{D}$ such that $W_{\mathscr{D}} = G^{-1}(W_{\mathscr{C}})$, $\mathrm{Fib}_{\mathscr{D}} = G^{-1}(\mathrm{Fib}_{\mathscr{C}})$ and $F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}}$. Moreover, the pair (F, G) is a Quillen adjunction.

Proof. We check the hypotheses of Theorem VI.5.4.5 for $\mathscr{D}$, $W_{\mathscr{D}}$, $F(I)$ and $F(J)$. The fact that $W_{\mathscr{D}}$ contains all identity morphisms, has the two out of three property and is stable by retracts follows immediately from its definition (and from these properties for $W_{\mathscr{C}}$). As $G(F(J) - \mathrm{cell}) \subset W_{\mathscr{C}}$, we have $F(J) - \mathrm{cell} \subset W_{\mathscr{D}}$, so, by Corollary VI.5.3.4 and the fact that $W_{\mathscr{D}}$ is stable by retracts, we get that $F(J) - \mathrm{cof} \subset W_{\mathscr{D}}$.

Condition (a) of Theorem VI.5.4.5 is just property (1). Also, by that Theorem, we know that $J - \mathrm{cof} = W \cap I - \mathrm{cof}$ and $I - \mathrm{inj} = W_{\mathscr{C}} \cap J - \mathrm{inj}$. By Proposition VI.4.2.2, if p is a morphism of $\mathscr{D}$, then $p \in F(I) - \mathrm{inj}$ if and only if $G(p) \in I - \mathrm{inj}$; in other words, we have $F(I) - \mathrm{inj} = G^{-1}(I - \mathrm{inj})$. Similary, we have $F(J) - \mathrm{inj} = G^{-1}(J - \mathrm{inj})$. As $I - \mathrm{inj} = W_{\mathscr{C}} \cap J - \mathrm{inj}$, we get that $F(I) - \mathrm{inj} = W_{\mathscr{D}} \cap F(J) - \mathrm{inj}$. In particular, we have $F(I) - \mathrm{inj} \subset F(J) - \mathrm{inj}$, so $F(J) - \mathrm{cof} \subset F(I) - \mathrm{cof}$; we already proved that $F(J) - \mathrm{cof} \subset W_{\mathscr{D}}$, so we get that $F(J) - \mathrm{cof} \subset W_{\mathscr{D}} \cap F(I) - \mathrm{cof}$. This finishes the proof of conditions (b), (c) and (d) of Theorem VI.5.4.5. Also, we have $\mathrm{Fib}_{\mathscr{D}} = F(J) - \mathrm{inj} = G^{-1}(J - \mathrm{inj}) = G^{-1}(\mathrm{Fib}_{\mathscr{C}})$. Moreover, if $i \in \mathrm{Cof}_{\mathscr{C}} = I - \mathrm{cof}$, then i has the left lifting property relatively to every element of $I - \mathrm{inj}$; as $I - \mathrm{inj} \supset G(F(I) - \mathrm{inj})$, Proposition VI.4.2.2 implies that $F(i)$ has the left lifting property relatively to every element of $F(I) - \mathrm{inj}$, i.e. that $F(i) \in \mathrm{Cof}_{\mathscr{D}}$.

Finally, as $F(\mathrm{Cof}_{\mathscr{C}}) \subset \mathrm{Cof}_{\mathscr{D}}$ and $G(\mathrm{Fib}_{\mathscr{D}}) \subset \mathrm{Fib}_{\mathscr{C}}$, the pair (F, G) is a Quillen adjunction.

$\square$

A Problem sets

A.1 Problem set 1

A.1.1

(a). In the category $\mathbf{Set}$, show that a morphism is a monomorphism (resp. an epimorphism) if and only it is injective (resp. surjective).

(b). Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to \mathbf{Set}$ be a *faithful* functor, show that any morphism f of $\mathscr{C}$ whose such that $F(f)$ is injective (resp. surjective) is a monomorphism (resp. an epimorphism).

(c). What are the monomorphisms and epimorphisms in ${}_R\mathbf{Mod}$?

(d). What are the monomorphisms in $\mathbf{Top}$? Give an example of a continuous morphism with dense image that is not an epimorphism in $\mathbf{Top}$. [1]

(e). Find a category $\mathscr{C}$, a faithful $F : \mathscr{C} \to \mathbf{Set}$ and a monomorphism f in $\mathscr{C}$ such that $F(f)$ is not injective.

(f). Find an epimorphism in $\mathbf{Ring}$ that is not surjective.

(g). The goal of this question is to show that any epimorphism in $\mathbf{Grp}$ is a surjective map. Let $\phi : G \to H$ be a morphism of groups, and suppose that it is an epimorphism in $\mathbf{Grp}$. Let $A = \mathrm{Im}(\phi)$. Let $S = \{*\} \sqcup (H/A)$, where $\{*\}$ is a singleton, and let $\mathfrak{S}$ be the group of permutations of S. We denote by σ the element of $\mathfrak{S}$ that switches $*$ and A and leaves the other elements of H/A fixed. For every $h \in H$, we denote by $\psi_1(h)$ the element of $\mathfrak{S}$ that leaves $*$ fixed and acts on H/A by left translation by H; this defines a morphism of groups $\psi_1 : H \to \mathfrak{S}$. We denote by $\psi_2 : H \to \mathfrak{S}$ the morphism $\sigma\psi_1\sigma^{-1}$.

   (i) Show that $\psi_1 = \psi_2$.

   (ii) Show that $A = H$.

Solution.

(a). Let X, Y be sets and $f : X \to Y$ be a map.

[1] In fact, the epimorphisms in $\mathbf{Top}$ are the surjective continuous maps.

Suppose that f is injective. If $g_1, g_2 : Z \to X$ are maps such that $f \circ g_1 = f \circ g_2$, then, for every $z \in Z$, we have $f(g_1(z)) = f(g_2(z))$, hence $g_1(z) = g_2(z)$; so $g_1 = g_2$. This shows that f is a monomorphism.

Conversely, suppose that f is a monomorphism. Let $x, x' \in X$ such that $x \neq x'$. Let $\{*\}$ be a singleton, and consider the maps $g_1, g_2 : \{*\} \to X$ defined by $g_1(*) = x$ and $g_2(*) = x'$. As $g_1 \neq g_2$, we have $f \circ g_1 \neq f \circ g_2$, so $f(x) \neq f(x')$. This shows that f is injective.

Suppose that f is surjective. If $h_1, h_2 : Y \to Z$ are maps such that $h_1 \circ f = h_2 \circ f$, then, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$, and then $h_1(y) = h_1(f(x)) = h_2(f(x)) = h_2(y)$; so $h_1 = h_2$. This shows that f is a monomorphism.

Conversely, suppose that f is an epimorphism. Let $y_0 \in Y$, let $Z = \{a, b\}$ be a set with two distinct elements, and define $h_1, h_2 : Y \to Z$ by $h_1(y) = a$ for every $y \in Y$, $h_2(y) = a$ for every $y \in Y - \{y_0\}$ and $h_2(y_0) = b$. We have $h_1 \neq h_2$, so $h_1 \circ f \neq h_2 \circ f$. As h_1 and h_2 coincide on $Y - \{y_0\}$, this implies that $y_0 \in \mathrm{Im}(f)$. So f is surjective.

(b). Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Suppose that $F(f)$ is injective. Let $g_1, g_2 : Z \to X$ be morphisms of $\mathscr{C}$ such that $f \circ g_1 = f \circ g_2$. Then $F(f) \circ F(g_1) = F(f) \circ F(g_2)$, so $F(g_1) = F(g_2)$ by a). As F is faithful, this implies that $g_1 = g_2$. So f is a monomorphism.

Suppose that $F(f)$ is surjective. Let $h_1, h_2 : Y \to Z$ be morphisms of $\mathscr{C}$ such that $h_1 \circ f = h_2 \circ f$. Then $F(h_1) \circ F(f) = F(h_2) \circ F(f)$, so $F(h_1) = F(h_2)$ by a). As F is faithful, this implies that $h_1 = h_2$. So f is an epimorphism.

(c). By b), any R-linear that is injective (resp. surjective) is a monomorphism (resp. epimorphism) in ${}_R\mathbf{Mod}$.

Conversely, let $f : M \to N$ be a monomorphism in ${}_R\mathbf{Mod}$. Consider the inclusion map $g_1 : \mathrm{Ker}(f) \to M$ and the map $g_2 = 0 : \mathrm{Ker}(f) \to M$. By definition of the kernel, we have $f \circ g_1 = f \circ g_2 = 0$, so $g_1 = g_2$, so $\mathrm{Ker}(f) = 0$, so f is injective.

Now let $f : M \to N$ be an epimorphism in ${}_R\mathbf{Mod}$. Consider the obvious surjection $h_1 : N \to \mathrm{Coker}(f)$ and the zero map $h_2 : N \to \mathrm{Coker}(f)$. By definition of the cokernel, we have $h_1 \circ f = h_2 \circ f = 0$, so $h_1 = h_2$, so $\mathrm{Coker}(f) = 0$, so f is surjective.

(d). By b), we know that any (continuous) injection is a monomorphism in $\mathbf{Top}$. Conversely, let $f : X \to Y$ be a monomorphism in $\mathbf{Top}$. Let $x, x' \in X$ such that $x \neq x'$. Let $\{*\}$ be a singleton with the discrete topology, and consider the maps $g_1, g_2 : \{*\} \to X$ defined by $g_1(*) = x$ and $g_2(*) = x'$; these maps are continuous, hence morphisms in $\mathbf{Top}$. As $g_1 \neq g_2$, we have $f \circ g_1 \neq f \circ g_2$, so $f(x) \neq f(x')$. This shows that f is injective.

Let $X = \{s, \eta\}$ be a set with two distinct points. We put the topology on X for which the open sets are $\varnothing$, X and $\{\eta\}$. Note that $\{\eta\}$ is dense in X. Let $f : X \to X$ be the map sending every point of X to η. Then f has dense image, but f is not an epimorphism, because $\mathrm{id}_X \circ f = f \circ f$, while $f \neq \mathrm{id}_X$.

(e). Let $\mathscr{C}$ be the subcategory of $\mathbf{Set}$ whose objects are $\{0\}$ and $\{0,1\}$, and whose morphisms are the identities and the unique map f from $\{0,1\}$ to $\{0\}$. Then f is a monomorphism in $\mathscr{C}$, but it is not injective. (And the inclusion is a faithful functor from $\mathscr{C}$ to $\mathbf{Set}$.)

(f). Consider the inclusion $f : \mathbb{Z} \to \mathbb{Q}$. It is an epimorphism in $\mathbf{Ring}$. Indeed, let R be a ring and let $h_1, h_2 : \mathbb{Q} \to R$ are morphisms of rings such that $h_1 \circ f = h_2 \circ f$. For every $m \in \mathbb{Z} - \{0\}$, the image of m in $\mathbb{Q}$ is invertible, so $h_1(m), h_2(m) \in R^\times$. For every $x \in \mathbb{Q}$, we can write $x = nm^{-1}$ with $n \in \mathbb{N}$ and $m \in \mathbb{Z} - \{0\}$, and then $h_1(x) = h_1(n)h_1(m)^{-1} = h_2(n)h_2(m)^{-1} = h_2(x)$.

More generally, if A is a commutative ring and S is a multiplicative subset of A, then the canonical map $A \to S^{-1}A$ is an epimorphism in $\mathbf{Ring}$.

(g). [2]

(i) Note that $\psi_1(h)_{|S-\{*,A\}} = \psi_2(h)_{S-\{*,A\}}$ for every $h \in H$.

Let $h \in A = \mathrm{Im}(\phi)$. We have $\psi_1(h)(*) = *$. On the other hand, the action of h on H/A by left translation fixes A, so $\psi_1(h)(A) = A$. So $\psi_1(h)_{\{*,A\}}$ is the identity morphism of $\{*, A\}$. This implies that $\psi_2(h)_{\{*,A\}}$ is also the identity morphism of $\{*, A\}$, hence that $\psi_1(h) = \psi_2(h)$. So ψ_1 and ψ_2 are equal on the image of ϕ, which implies that $\psi_1 \circ \phi\psi_2 \circ \phi$. As ϕ is an epimorphism, we deduce that $\psi_1 = \psi_2$.

(ii) Let $h \in A$. Then $\psi_1(h)(*) = *$, and $\psi_2(h)(*) = \sigma \circ \psi_1(h)(A) = \sigma(hA)$. By (i), we know that $\psi_1(h) = \psi_2(h)$, so $* = \sigma(hA)$. This is only possible if $hA = A$, i.e. if $h \in A$. So $H = A = \mathrm{Im}(\phi)$, and ϕ is surjective.

□

A.1.2

Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor.

(a). If F has a quasi-inverse, show that it is fully faithful and essentially surjective.

(b). If F is fully faithful and essentially surjective, construct a functor $G : \mathscr{C}' \to \mathscr{C}$ and isomorphisms of functors $F \circ G \simeq \mathrm{id}_{\mathscr{C}}$ and $G \circ F \simeq \mathrm{id}_{\mathscr{C}'}$.

Solution.

(a). Let $G : \mathscr{C}' \to \mathscr{C}$ be a quasi-inverse of F, and let $u : G \circ F \overset{\sim}{\to} \mathrm{id}_{\mathscr{C}}$ and $v : F \circ G \overset{\sim}{\to} \mathrm{id}_{\mathscr{C}'}$ be isomorphisms of functors.

Let $X, Y \in \mathrm{Ob}(\mathbb{C})$. We denote by β the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ given by F. Consider the map $\alpha : \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$ that we get

[2] This proof comes from [9].

by composing $G : \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(G \circ F(X), G \circ F(Y))$ and the map $\mathrm{Hom}_{\mathscr{C}}(G \circ F(X), G \circ F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$, $g \longmapsto u(Y) \circ g \circ u(X)^{-1}$. We claim that $\alpha \circ \beta$ is the identity on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. Indeed, let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. As u is a morphism of functors, the following diagram is commutative :

$$\begin{tikzcd} G \circ F(X) \arrow[r, "G \circ F(f)"] \arrow[d, "u(X)"'] & G \circ F(Y) \arrow[d, "u(Y)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

This shows that $u(Y) \circ G \circ F(f) \circ u(X)^{-1} = f$, i.e. that $\alpha \circ \beta(f) = f$. In particular, the map β is injective and the map α is surjective. This shows that F is faithful. Applying this result to G (which is also an equivalence of categories, with quasi-inverse F), we see that the map α is also injective, hence it is bijective, hence β is also bijective. This shows that F is fully faithful.

Let $X' \in \mathrm{Ob}(\mathscr{C}')$. Then $v : F(G(X')) \xrightarrow{\sim} X'$ is an isomorphism, and $G(X') \in \mathrm{Ob}(\mathscr{C})$. This shows that F is essentially surjective.

(b). We construct the functor G. Let $X' \in \mathrm{Ob}(\mathscr{C}')$; we choose an object X of $\mathscr{C}$ and an isomorphism $u(X') : F(X) \xrightarrow{\sim} X'$, and we set $G(X) = X'$. Let $X', Y' \in \mathrm{Ob}(\mathscr{C}')$, and let $X = G(X')$ and $Y = G(Y')$. We define a map $\mathrm{Hom}_{\mathscr{C}'}(X', Y') \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ by $f' \longmapsto u(Y')^{-1} \circ f' \circ u(X')$. Composing this with the inverse of the bijection $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$, we get a map $\mathrm{Hom}_{\mathscr{C}'}(X', Y') \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$, which we denote by G.

Next we show that G is a functor. If $X' \in \mathrm{Ob}(\mathscr{C}')$, then $u(X')^{-1} \circ \mathrm{id}_{X'} \circ u(X') = \mathrm{id}_{F(G(X'))}$, so $G(\mathrm{id}_{X'}) = \mathrm{id}_{G(X')}$. Let $f' : X' \to Y'$ and $g' : Y' \to Z'$ be two morphisms of $\mathscr{C}'$, and let $f : X \to Y$ and $g : Y \to Z$ be their images by G. By definition of G on morphisms, we have $F(f) = u(Y')^{-1} \circ f' \circ u(X')$ and $F(g) = u(Z')^{-1} \circ g' \circ u(Y')$, so $F(g \circ f) = F(g) \circ F(f) = u(Z')^{-1} \circ (g' \circ f') \circ u(X') = F(G(g' \circ f'))$. As F is faithful, this implies that $g \circ f = G(g' \circ f')$, i.e. that $G(g') \circ G(f') = G(g' \circ f')$. So G is a functor.

Finally, we show that G is a quasi-inverse of F. For every $X' \in \mathrm{Ob}(\mathscr{C}')$, we have by definition of $G(X')$ an isomorphism $u(X') : F(G(X')) \xrightarrow{\sim} X'$. We need to show that this defines an isomorphism of functors $F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}'}$. So let $f' : X' \to Y'$ be a morphism of $\mathscr{C}'$. By definition of $G(f')$, we have $u(Y') \circ F(G(f')) = f' \circ u(X')$, which is what we wanted. We still need to define an isomorphism of functors $v : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$. Let $X \in \mathrm{Ob}(\mathscr{C})$. By definition of G, we have an isomorphism $u(F(X)) : F(G(F(X))) \xrightarrow{\sim} F(X)$. As F is fully faithful, there is a unique $v(X) \in \mathrm{Hom}_{\mathscr{C}}(G(F(X)), X)$ such that $F(v(X)) = u(F(X))$, and $v(X)$ is an isomorphism because a fully faithful functor is conservative. It remains to show that this defines a morphism of functors. So let $f : X \to Y$ be a morphism of $\mathscr{C}$.

Then $F(G(F(f))) = u(F(Y))^{-1} \circ F(f) \circ u(F(X))$, so

$$F(f) \circ F(v(X)) = F(f) \circ u(F(X)) = u(F(Y)) \circ F(G(F(f))) = F(v(Y)) \circ F(G(F(f))).$$

Using the fact that F is faithful (and is a functor), we get $f \circ v(X) = v(Y) \circ G(F(f))$, which is what we wanted.

$\square$

A.1.3

Let $\mathscr{C}$ be the full subcategory of $\mathbf{Ab}$ whose objects are finitely generated abelian groups.

(a). Show that every natural endomorphism of $\mathrm{id}_{\mathscr{C}}$ is multiplication by some $n \in \mathbb{Z}$.

(b). Consider the functor $F : \mathscr{C} \to \mathscr{C}$ that sends an abelian group A to $A_{\mathrm{tor}} \oplus (A/A_{\mathrm{tor}})$ (and acts in the obvious way on morphisms), where A_{tor} is the torsion subgroup of A. Show that there is no natural isomorphism $F \overset{\sim}{\to} \mathrm{id}_{\mathscr{C}}$.

Solution.

(a). Let $u : \mathrm{id}_{\mathscr{C}} \to \mathrm{id}_{\mathscr{C}}$ be a morphism of functors. Then $u(\mathbb{Z}) \in \mathrm{End}_{\mathbf{Ab}}(\mathbb{Z})$, so $u(\mathbb{Z})$ is of the form $n\mathrm{id}_{\mathbb{Z}}$ for some $n \in \mathbb{Z}$. Let A be an arbitrary abelian group. We want to show that $u(A) = n\mathrm{id}_A$. Let $a \in A$. We consider the morphism of groups $f : \mathbb{Z} \to A$ sending 1 to a. As u is a morphism of functors, we have a commutative diagram :

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "u(\mathbb{Z})"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "f"] \\ A \arrow[r, "u(A)"'] & A \end{tikzcd}$$

In particular, $u(A)(a) = u(A)(f(1)) = f(u(\mathbb{Z})(1)) = f(n) = na$. So $u(A) = n\mathrm{id}_A$.

(b). Suppose that $u : F \overset{\sim}{\to} \mathrm{id}_{\mathscr{C}}$ is a natural isomorphism. For every abelian groups A, consider the morphism $v(A) : A \to A/A_{\mathrm{tor}} \oplus A_{\mathrm{tor}}$ that is the composition of the canonical surjection $A \to A/A_{\mathrm{tor}}$ and of the injection $A/A_{\mathrm{tor}} \to A/A_{\mathrm{tor}} \oplus A_{\mathrm{tor}}$. It is easy to see that this defines a morphism of functors $v : \mathrm{id}_{\mathscr{C}} \to F$. So $u \circ v$ is an endomorphism of $\mathrm{id}_{\mathscr{C}}$, and, by a), there exists $n \in \mathbb{Z}$ such that $u \circ v$ is the multiplication by n. As $v(\mathbb{Z}) = \mathrm{id}_{\mathbb{Z}}$ by definition of v and $u(\mathbb{Z})$ is an isomorphism, we must have $n = \pm 1$. Now take $A = \mathbb{Z}/2\mathbb{Z}$. Then $v(A) = 0$, so $u \circ v(A) = 0$, so n is divisible by 2. This is a contradiction.

$\square$

A.1.4

Let k be a field, and let $F : \mathbf{Mod}_k \to \mathbf{Mod}_k$ be the functor sending a k-vector space V to $V \otimes_k V$ and a k-linear transformation f to $f \otimes f$. Show that the only morphism of functors from $\mathrm{id}_{\mathbf{Mod}_k}$ to F is the zero one, i.e. the morphism $u : \mathrm{id}_{\mathbf{Mod}_k} \to F$ such that $u(V) = 0$ for every k-vector space V.

Solution. Let $u : \mathrm{id}_{\mathbf{Mod}_k} \to F$ be a morphism of functors. Then $u(k)$ is a k-linear map from k to $k \otimes_k k$, so there exists a unique $\lambda \in k$ such that $u(k)(1) = \lambda(1 \otimes 1)$.

Let V be a k-vector space, and let $v \in V$. We denote by $f; k \to V$ the unique k-linear map such that $f(1) = v$. As u is a morphism of functors, we have $u(V) \circ f = (f \otimes f) \circ u(k)$, and in particular $u(V)(v) = u(V)(f(1)) = (f \otimes f)(\lambda(1 \otimes 1)) = \lambda(v \otimes v)$.

Take $V = k^2$, and let (e_1, e_2) be the canonical basis of V. We know that $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ is a basis of $V \otimes_k V$. Using the previous paragraph, we see that

$$u(V)(e_1 + e_2) = \lambda(e_1 + e_2) \otimes (e_1 + e_2) = \lambda(e_1 \otimes e_1) + \lambda(e_1 \otimes e_2) + \lambda(e_2 \otimes e_1) + \lambda(e_2 \otimes e_2).$$

On the other hand, as $u(V)$ is k-linear, we have

$$u(V)(e_1 + e_2) = u(V)(e_1) + u(V)(e_2) = \lambda(e_1 \otimes e_1) + \lambda(e_2 \otimes e_2).$$

This is only possible if $\lambda = 0$. But then, by the calculation of the previous paragraph, we have $u(W) = 0$ for every k-vector space W.

Note that we did not use the fact that k is a field, so the result is also true for the catgeory of modules over a commutative ring.

$\square$

A.1.5

Let $\mathscr{C}$ be a category. Remember that the category $\mathrm{PSh}(\mathscr{C})$ of presheaves on $\mathscr{C}$ is the category $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$.

Let F be a presheaf on $\mathscr{C}$ and X be an object of $\mathscr{C}$. Let $\Phi : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F) \to F(X)$ be the map defined by $\Phi(u) = u(X)(\mathrm{id}_X)$. Let $\Psi : F(X) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$ be the map sending $x \in F(X)$ to the morphism of functors $\Psi(x) : h_X \to F$ such that $\Psi(x)(Y) : h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X) \to F(Y)$ sends $f : Y \to X$ to $F(f)(x) \in F(Y)$. Show that Φ and Ψ are bijections that are inverses of each other.

Solution. We show that $\Psi \circ \Phi$ is the identity of $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$. Let $u \in \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$.

Let Y be an object of $\mathscr{C}$. As u is a morphism of functors, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[r, "u(X)"] \arrow[d, "h_X(f)"'] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Hom}_{\mathscr{C}}(Y,X) \arrow[r, "u(Y)"'] & F(Y) \end{tikzcd}$$

In particular, we have

$$F(f)(\Phi(u)) = F(f)(u(X)(\mathrm{id}_X)) = u(Y)(h_X(f)(\mathrm{id}_X)) = u(Y)(f).$$

As $F(f)(\Phi(u)) = \Psi(\Phi(u))(Y)(f)$ by definition of Ψ, this shows that $\Psi(\Phi(u))(Y) = u(Y)$, hence that $\Psi(\Phi(u)) = u$.

Now we show that $\Phi \circ \Psi$ is the identity of $F(X)$. Let $x \in F(X)$. Then $\Phi(\Psi(x)) = \Psi(x)(X)(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = \mathrm{id}_{F(X)}(x) = x$.

□

A.1.6

(a). Show that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent. (Hint : If $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is an equivalence of categories, show that $F(\varnothing)$ is a singleton and that $F(X) = \varnothing$ for X a singleton.)

(b). Let $\mathscr{C}$ be the full subcategory of $\mathbf{Set}$ whose objects are finite sets. Show that $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are not equivalent.

(c). Show that $\mathbf{Rel}$ and $\mathbf{Rel}^{\mathrm{op}}$ are equivalent.

(d). Let $\mathscr{D}$ be the full subcategory of $\mathbf{Ab}$ whose objects are finite abelian groups. Show that $\mathscr{D}$ and $\mathscr{D}^{\mathrm{op}}$ are equivalent.

Solution.

(a). Suppose that there exists an equivalence of categories $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$. For every set X, the set

$$\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X), F(\varnothing)) = \mathrm{Hom}_{\mathbf{Set}}(F(\varnothing), F(X)) \simeq \mathrm{Hom}_{\mathbf{Set}}(\varnothing, X)$$

is a singleton (because there is a unique map from the empty set into X). So $F(\varnothing)$ is a singleton.

Similarly, if X is a singleton, then, for every set Y, the set

$$\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X), F(Y)) = \mathrm{Hom}_{\mathbf{Set}}(F(Y), F(X)) \simeq \mathrm{Hom}_{\mathbf{Set}}(Y, X)$$

is a singleton. So $F(X)$ is the empty set.

Now let X be a singleton and Y be a set with two elements. Then $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$ is a set with two elements. But on the other hand, we have

$$\mathrm{Hom}_{\mathbf{Set}}(X,Y) \simeq \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X),F(Y)) = \mathrm{Hom}_{\mathbf{Set}}(F(Y),\varnothing),$$

and $\mathrm{Hom}_{\mathbf{Set}}(F(Y),\varnothing)$ has at most one element (it is empty if $F(Y) \neq \varnothing$, and it only contains $\mathrm{id}_{\varnothing}$ if $F(Y) = \varnothing$). This is a contradiction.

(b). The proof of a) works just as well.

(c). Let $F : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ be defined by $F(X) = X$ for every set X and, for all sets X, Y and every subset f of $X \times Y$, $F(f) = \{(y,x) \mid (x,y) \in f\}$. We want to show that F is a functor. (Then it will clearly be an equivalence, and even an isomorphism of categories.) Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be morphisms in $\mathbf{Rel}$; that is, f is a subset of $X \times Y$ and g is a subset of $Y \times Z$. Then, in $\mathbf{Rel}$, we have $g \circ f = \{(x,z) \mid \exists y \in Y,\ (x,y) \in f \text{ and } (y,z) \in g\}$. On the other hand, in $\mathbf{Rel}^{\mathrm{op}}$, we have $F(f) \circ F(g) = \{(z,x) \in Z \times X \mid \exists y \in Y,\ (y,x) \in F(f) \text{ and } (z,y) \in F(g)\}$. This is clearly equal to $F(g \circ f)$.

(d). Consider the functor $F = \mathrm{Hom}_{\mathbf{Ab}}(\cdot, \mathbb{Q}/\mathbb{Z}) : \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$. If A is a finite abelian group, then so is $F(A)$. So F induces a functor $\mathscr{D}^{\mathrm{op}} \to \mathscr{D}$, which we still denote by F. We can also see F as a functor from $\mathscr{D}$ to $\mathscr{D}^{\mathrm{op}}$. We claim that F is an equivalence of categories, and in fact that it is its own quasi-inverse. To show this, it suffices to construct an functorial isomorphism $\mathrm{id}_{\mathscr{D}} \xrightarrow{\sim} F \circ F$. For every finite abelian group A, we consider the map $u : A \to F(F(A)) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, $a \longmapsto (f \longmapsto f(a))$. The fact that this defines a morphism of functors is a straightforward verification. The fact that is an isomorphism if Pontrjagin duality for finite abelian groups. (By the structure theorem for finite abelian groups, it suffices to check that $u(A)$ is an isomorphism for A of the form $\mathbb{Z}/n\mathbb{Z}$, which is easy.)

$\square$

A.1.7

Let $\mathscr{C}$ and $\mathscr{C}'$ and $F : \mathscr{C} \to \mathscr{C}'$, $G : \mathscr{C}' \to \mathscr{C}$ be two functors. We consider the two bifunctions $H_1, H_2 : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ defined by $H_1 = \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot)$ and $H_2 = \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$. Suppose that we are given, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $Y \in \mathrm{Ob}(\mathscr{C}')$, a bijection $\alpha(X,Y) : H_1(X,Y) \xrightarrow{\sim} H_2(X,Y)$. Show that the two following statements are equivalent :

(i) The family of bijections $(\alpha(X,Y))_{X \in \mathrm{Ob}(\mathscr{C}), Y \in \mathrm{Ob}(\mathscr{C}')}$ defines an isomorphism of functors $H_1 \xrightarrow{\sim} H_2$.

(ii) For every morphism $f : X_1 \to X_2$ in $\mathscr{C}$, every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, and for all $u \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$ and $v \in \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2)$, the square

$$\begin{tikzcd} F(X_1) \arrow[r, "u"] \arrow[d, "F(f)"'] & Y_1 \arrow[d, "g"] \\ F(X_2) \arrow[r, "v"'] & Y_2 \end{tikzcd}$$

is commutative if and only if the square

$$\begin{tikzcd} X_1 \arrow[r, "\alpha(X_1,Y_1)(u)"] \arrow[d, "f"'] & G(Y_1) \arrow[d, "G(g)"] \\ X_2 \arrow[r, "\alpha(X_2,Y_2)(v)"'] & G(Y_2) \end{tikzcd}$$

is commutative.

Solution. The key is to write explicitly what it means for the $(\alpha(X, Y))$ to define a morphism of functors. It means that, for every morphism $f : X_1 \to X_2$ in $\mathscr{C}$ (that is, a morphism $X_2 \to X_1$ in $\mathscr{C}^{\mathrm{op}}$) and for every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, the following square commutes :

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_1) \arrow[r, "H_1(f,g)"] \arrow[d, "\alpha(X_2,Y_1)"'] & \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_2) \arrow[d, "\alpha(X_1,Y_2)"] \\ \mathrm{Hom}_{\mathscr{C}}(X_2, G(Y_1)) \arrow[r, "H_2(f,g)"'] & \mathrm{Hom}_{\mathscr{C}}(X_1, G(Y_2)) \end{tikzcd}$$

The fact that the square commutes says exactly that, for every morphism $w : F(X_2) \to Y_1$ in $\mathscr{C}'$, we have

$$\alpha(X_1, Y_2)(g \circ w \circ F(f)) = G(g) \circ \alpha(X_2, Y_1)(w) \circ f.$$

Suppose that (i) holds. Using the calculation of the previous, we get :

(a) Taking $X_1 = X_2$, $f = \mathrm{id}_{X_1}$ and $g : Y_1 \to Y_2$ arbitrary : for every $u : F(X_1) \to Y_1$, we have

$$\alpha(X_1, Y_2)(g \circ u) = G(g) \circ \alpha(X_1, Y_1)(u).$$

(b) Taking $f : X_1 \to X_2$ arbitrary, $Y_1 = Y_2$ and $g = \mathrm{id}_{Y_2}$: for every $v : F(X_2) \to Y_2$, we have

$$\alpha(X_1, Y_2)(v \circ F(f)) = \alpha(X_2, Y_2)(v) \circ f.$$

Suppose that we are in the situation of (ii), that is, we are given morphisms $f : X_1 \to X_2$ in $\mathscr{C}$, $g : Y_1 \to Y_2$ in $\mathscr{C}'$, and $u \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$ and $v \in \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2)$. We want to show that the top square of (ii) commutes if and only if the bottom squqre commutes.

Suppose that the top square commutes, that is, that $v \circ F(f) = g \circ u$. Applying (a) and (b), we get

$$G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_1, Y_2)(g \circ u) = \alpha(X_1, Y_2)(v \circ F(f)) = \alpha(X_2, Y_2)(v) \circ f.$$

This shows that the bottom square commutes.

Conversely, suppose that the bottom square commutes, that is, that $G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_2, Y_2)(v) \circ f$. Again, applying (a) and (b), we get

$$\alpha(X_1, Y_2)(g \circ u) = G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_2, Y_2)(v) \circ f = \alpha(X_1, Y_2)(v \circ F(f)).$$

As $\alpha(X_1, Y_2)$ is bijective, this implies that $g \circ u = v \circ F(f)$, which means that the top square commutes.

Now we assume that (ii) holds, and we want to show that (i) also holds. Let $f : X_1 \to X_2$ be a morphism in $\mathscr{C}$, $g : Y_1 \to Y_2$ be a morphism in $\mathscr{C}'$, and $w : F(X_2) \to Y_1$ be a morphism in $\mathscr{C}'$. We want to show that $\alpha(X_1, Y_2)(g \circ w \circ F(f)) = G(g) \circ \alpha(X_2, Y_1)(w) \circ f$. We apply (i) to $u = w \circ F(f) : F(X_1) \to Y_1$ and $v = g \circ w : F(X_2) \to Y_2$. We obviously have $g \circ u = v \circ F(f)$, so, by (i), this implies that

(*) $$\alpha(X_2, Y_2)(g \circ w) \circ f = G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)).$$

Applying (*) to the particular case where $Y_1 = Y_2$ and $g = \mathrm{id}_{Y_1}$, we get:

(**) $$\alpha(X_2, Y_1)(w) \circ f = \alpha(X_1, Y_1)(w \circ F(f)).$$

Applying (**) with w replaced by $g \circ w : F(X_2) \to Y_2$, we get

(***) $$\alpha(X_2, Y_2)(g \circ w) \circ f = \alpha(X_1, Y_2)(g \circ w \circ F(f)).$$

Putting (*), (**) and (***) together gives

$$\begin{aligned}\alpha(X_1, Y_2)(g \circ w \circ F(f)) = \alpha(X_2, Y_2)(g \circ w) \circ f &= G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)) \\ &= G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)),\end{aligned}$$

which is what we wanted to prove.

□

A.1.8

Remember that a functor $F : \mathscr{C} \to \mathbf{Set}$ is called representable if there exists an object X of $\mathscr{C}$ and an element x of $F(X)$ such that the morphism of functors $u : \mathrm{Hom}_{\mathscr{C}}(X, \cdot) \to F$ defined by $u(Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to F(Y)$, $(f : X \to Y) \longmapsto F(f)(x)$ is an isomorphism. The couple (X, x) is then said to represent F.

The following functors are representable. For each of them, give a couple representing the functor. (If the functor is only defined on objects, it is assumed to act on morphisms in the obvious way.)

(a). The identity endofunctor of $\mathbf{Set}$.

(b). The functor $F : \mathbf{Grp} \to \mathbf{Set}$, $G \longmapsto G^n$, where $n \in \mathbb{N}$.

(c). The forgetful functor $\mathbf{Mod}_R \to \mathbf{Set}$, where R is a ring.

(d). The forgetgul functor $\mathbf{Ring} \to \mathbf{Set}$.

(e). The functor $\mathbf{Ring} \to \mathbf{Set}$, $R \longmapsto R^\times$.

(f). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of objects.

(g). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of morphisms (i.e. $\bigcup_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$).

(h). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of isomorphisms.

(i). The functor $F : \mathbf{Top}_* \to \mathbf{Set}$ that takes a pointed topological space (X,x) to the set of continuous loops on X with base point x.

(j). The functor $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ such that $F(X) = \mathfrak{P}(X)$ and, for every map $f : X \to Y$, $F(f) : \mathfrak{P}(Y) \to \mathfrak{P}(X)$ is the map $A \longmapsto f^{-1}(A)$.

(k). The functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ that sends a topological space to its set of open subsets. (If $f : X \to Y$ is a continuous map, $F(f) : F(Y) \to F(X)$ is the map $U \longmapsto f^{-1}(U)$.)

(l). If k is a field, the functor $F : \mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Set}$ that sends a k-vector space to the underlying set of V^* (so F is the composition of the duality functor $\mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Mod}_k$ and of the forgetful functor from $\mathbf{Mod}_k$ to $\mathbf{Set}$.)

Solution.

(a). Take $X = \{x\}$ to be a singleton and x to be the unique element of $F(X) = X$. Then, for every set Y, $u(Y) : \mathrm{Hom}_{\mathbf{Set}}(X,Y) \to F(Y) = Y$ sends $f : X \to Y$ to $f(x) \in F(Y) = Y$; it is clearly bijective.

(b). Let $X = F_n$ be the free group on n generators $(x_1, \ldots, x_n)$, and $x = (x_1, \ldots, x_n) \in F(F_n) = (F_n)^n$. For every group G, the map $u(G) : \mathrm{Hom}_{\mathbf{Grp}}(F_n, G) \to G^n$ sends $f : F_n \to G$ to $(f(x_1), \ldots, f(x_n)) \in G^n$. The fact that this is bijective is the universal property of the free group F_n.

(c). Take $X = R$ with the obvious right R-action, and $x = 1 \in F(R) = R$. Then, for every right R-module M, the map $u(M) : \mathrm{Hom}_R(R, M) \to F(M) = M$ sends $f : R \to M$ to $f(1)$. This is bijective because R is a free R-module with base $\{1\}$.

(d). Take X equal to the polynomial ring $\mathbb{Z}[x]$ and $x \in F(X) = X$ to be the indeterminate. For every ring R, the map $u(R) : \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \to F(R) = R$ sends $f : \mathbb{Z}[x] \to R$ to $f(x) \in R$. The fact that this is bijective is the universal property of the polynomial ring.

(e). Take $X = \mathbb{Z}[x, x^{-1}]$ (the polynomial ring $\mathbb{Z}[x]$ localized at the indeterminaye x)

and x to be the indeterminate in $F(X) = \mathbb{Z}[x, x^{-1}]^\times$. For every ring R, the map $u(R) : \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \to F(R) = R^\times$ sends $f : \mathbb{Z}[x] \to R$ to $f(x) \in R^\times$. The fact that this is bijective follows from the universal properties of the polynomial ring of the localization.

(f). Let X be the category with only one object $*$ and such that $\mathrm{End}_X(*) = \{\mathrm{id}_*\}$, and let $x \in F(X) = \{*\}$ be the unique object. (Note that X is the category corresponding to the poset $[0]$.) If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$ takes a functor $G : X \to \mathscr{C}$ to $G(*) \in \mathrm{Ob}(\mathscr{C})$. This map is bijective, with inverse the map $v(\mathscr{C}) : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Func}(X, \mathscr{C})$ sending $c \in \mathrm{Ob}(\mathscr{C})$ to the functor $G : X \to \mathscr{C}$ defined by $G(*) = c$ and $G(\mathrm{id}_*) = \mathrm{id}_c$.

(g). Let X be the category corresponding to the poset $[1]$, that is, X has two objects 0 and 1, and a unique non-identity morphism $\alpha : 0 \to 1$. Let $x \in F(X)$ be the morphism α. If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(X)$ sends a functor $G : X \to \mathscr{C}$ to $G(\alpha) \in \mathrm{Hom}_{\mathscr{C}}(F(0), F(1))$. Let $v(\mathscr{C}) : F(X) \to \mathrm{Func}(X, \mathscr{C})$ be defined as follows : if $f : c_0 \to c_1$ is a morphism of $\mathscr{C}$, that is, an element of $F(\mathscr{C})$, we defined a functor $G : X \to \mathscr{C}$ by $G(0) = c_0$, $G(1) = c_1$ and $G(\alpha) = f$. Then $v(\mathscr{C})$ is an inverse of $u(\mathscr{C})$, so $u(\mathscr{C})$ is bijective.

Let X be the category such that $\mathrm{Ob}(X) = \{0, 1\}$, and such that the only two non-identity morphisms of X are morphisms $\alpha : 0 \to 1$ and $\beta : 1 \to 0$ such that $\alpha \circ \beta = \mathrm{id}_1$ and $\beta \circ \alpha = \mathrm{id}_0$. If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(X)$ sends a functor $G : X \to \mathscr{C}$ to $G(\alpha) \in \mathrm{Hom}_{\mathscr{C}}(F(0), F(1))$, which is an isomorphism with inverse $G(\beta)$. Let $v(\mathscr{C}) : F(X) \to \mathrm{Func}(X, \mathscr{C})$ be defined as follows : if $f : c_0 \to c_1$ is an isomorphism of $\mathscr{C}$, that is, an element of $F(\mathscr{C})$, we defined a functor $G : X \to \mathscr{C}$ by $G(0) = c_0$, $G(1) = c_1$, $G(\alpha) = f$ and $G(\beta) = f^{-1}$. Then $v(\mathscr{C})$ is an inverse of $u(\mathscr{C})$, so $u(\mathscr{C})$ is bijective.

(h). Remember that a loop on a topological space Y with base point y is just a continuous map γ from S^1 (the unit circle in $\mathbb{C}$) to Y such that $\gamma(1) = y$. In other words, it is a morphism from $(S^1, 1)$ to (Y, y) in the category $\mathbf{Top}_*$. So we can take $X = (S^1, 1)$ and $x = \mathrm{id}_{S^1} \in F(X)$.

(i). For every set Y, we have a bijection $v(Y) : \mathfrak{P}(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Set}}(Y, \{0, 1\})$ sending a subset A of Y to its characteristic function. So we can take $X = \{0, 1\}$ and $x = \{1\} \in \mathfrak{P}(X)$. Indeed, if Y is a set, then the map $u(Y) : \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(Y, X) \to \mathfrak{P}(Y)$ sends $f : Y \to \{0, 1\}$ to $f^{-1}(\{1\})$, which is the inverse of the bijection $v(Y)$.

(j). Let X be the Sierpinski space, that is, the topological space $\{s, \eta\}$ where the open subsets are $\varnothing$, $\{\eta\}$ and $\{s, \eta\}$, and let $x = \{\eta\} \in F(X)$. Then, if Y is a topological space, the map $u(Y) : \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(Y, X) \to \mathfrak{P}(Y)$ sends $f : Y \to \{s, \eta\}$ to the open subset $f^{-1}(\{\eta\})$ of Y. Conversely, if U is an open subset of Y, then the map $f : Y \to \{s, \eta\}$ such that $f(y) = \eta$ for $y \in Y$ and $f(y) = s$ for $y \in Y - U$ is continuous. So $u(Y)$ is bijective.

(k). For every k-vector space V, we have $F(V) = \mathrm{Hom}_k(V,k)$. So we can take $X = k$ (with the obvious action of k) and $x = \mathrm{id}_k \in \mathrm{Hom}_k(k,k)$. Indeed, for every k-vector space V, the map $u(V) : \mathrm{Hom}_{\mathbf{Mod}_k^{\mathrm{op}}}(k,V) = \mathrm{Hom}_k(V,k) \to F(V) = \mathrm{Hom}_k(V,k)$ sends $f : V \to k$ to $\mathrm{id}_k \circ f = f$. This is the identity of $F(V)$, so it is obviously bijective.

□

A.1.9

The simplicial category Δ is defined in Example I.2.1.8(5) of the notes. It is the category whose objects are the finite sets $[n] = \{0,1,\ldots,n\}$ with their usual order and whose morphisms are the nondecreasing maps between these sets.

The category $\mathbf{sSet}$ of *simplicial sets* if $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$. So a simplicial set is by definition a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$; in that case, we write X_n for $X([n])$ and, if $f : [n] \to [m]$, we often write $f^* : X_m \to X_n$ for $X(f)$. For example, for each $n \in \mathbb{N}$, the *standard simplex* of dimension n is the simplicial set $\mathrm{Hom}_\Delta(\cdot, [n])$.

If X is a simplicial set, a simplicial subset Y of X is the data of a subset Y_n of X_n, for every $n \in \mathbb{N}$, such that $\alpha^*(Y_m) \subset Y_n$ for every morphism $\alpha : [n] \to [m]$ in Δ. We can form images of morphisms of simplicial sets, and unions and intersections of simplicial subsets, in the obvious way.

If we see each poset $[n]$ as a category in the usual way, then the morphisms of Δ become functors, so this allows us to see Δ as a subcategory of $\mathbf{Cat}$.

Let $\mathscr{C}$ be a category. Its *nerve* $N(\mathscr{C})$ is the restriction to Δ^{op} of the functor $\mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathscr{C})$ on $\mathbf{Cat}^{\mathrm{op}}$; it is a functor from Δ^{op} to $\mathbf{Set}$, i.e. a simplicial set. As $\mathrm{Hom}_{\mathbf{Cat}}$ is a bifunctor, this construction is functorial in $\mathscr{C}$, and we get a nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$.

(a). If $\mathscr{C}$ is a category, show that $N(\mathscr{C})_0 \simeq \mathrm{Ob}(\mathscr{C})$ and $N(\mathscr{C})_1 \simeq \coprod_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$. Can you give a similar description of $N(\mathscr{C})_n$ for $n \geq 2$?

(b). Let $n \in \mathbb{N}$. Show that the nerve of $[n]$ is isomorphic to Δ_n.

(c). Let $n \in \mathbb{N}$. Show that there exists $e_n \in \Delta_n([n])$ such that, for every simplicial set X, the map $\mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \overset{\sim}{\to} X_n$ sending u to $u_n(e_n)$ is bijective.

(d). For every category $\mathscr{C}$ and every simplicial set X, if $u, v : X \to N(\mathscr{C})$ are two morphisms of simplicial sets such that $u_i, v_i : X_i \to N(\mathscr{C})_i$ are equal for $i \in \{0,1\}$, show that $u = v$.

(e). We denote by $\Delta_{\leq 2}$ the full subcategory of Δ whose objects are $[0]$, $[1]$ and $[2]$; if X is a simplicial set, we denote by $X_{\leq 2}$ its restriction to $\Delta_{\leq 2}$ (which is a functor $\Delta_{\leq 2}^{\mathrm{op}} \to \mathbf{Set}$).

Let X be a simplicial set and $\mathscr{C}$ be a category. Show that every morphism $X_{\leq 2} \to N(\mathscr{C})_{\leq 2}$ extends to a morphism $X \to N(\mathscr{C})$.

(f). Show that the functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful.

Let $n \in \mathbb{N}$ For every $k \in [n]$, we denote by δ_k the unique injective increasing map $[n-1] \to [n]$ such that $k \notin \mathrm{Im}(\delta_k)$. This induces a map $\Delta_{n-1} \to \Delta_n$, that we also denote by δ_k; the image of this map is called the kth facet of Δ_n.

If $k \in [n]$, the *horn* Λ^n_k is the union of all the facets of Δ_n except for the kth one; in other words, it is the simplicial subset of Δ_n defined by

$$\Lambda^n_k([m]) = \{f \in \mathrm{Hom}_\Delta([m],[n]) \mid \exists l \in [n]-\{k\} \text{ and } g \in \mathrm{Hom}_\Delta([m],[n-1]) \text{ with } f = \delta_l \circ g\}.$$

(g). (1 point) Let $\mathscr{C}$ be a category. If $n \geq 3$ and $k \in [n] - \{0, n\}$, show that every morphism of simplicial sets $\Lambda^n_k \to X$ extends uniquely to a morphism $\Delta_n \to X$.

(h). Let $\mathscr{C}$ be a category. Show that every morphism of simplicial sets $\Lambda^2_1 \to X$ extends uniquely to a morphism $\Delta_2 \to X$.

(i). Show that a simplicial set X is the nerve of a category if and only if, for every $n \in \mathbb{N}$, every $0 < k < n$ and every morphism of simplicial sets $u : \Lambda^n_k \to X$, the morphism u extends uniquely to a morphism $\Delta_n \to X$.

Solution.

(a). By problem A.1.8(f), the functor $\mathbf{Cat} \to \mathbf{Set}$, $\mathscr{C} \longmapsto \mathrm{Ob}(\mathscr{C})$ is represented by $[0]$. As $N(\mathscr{C})_0 = \mathrm{Hom}_{\mathbf{Cat}}([0], \mathscr{C})$, this gives an isomorphism $N(\mathscr{C})_0 \simeq \mathrm{Ob}(\mathscr{C})$, natural in $\mathscr{C}$. Similarly, by A.1.8(g), the functor $\mathbf{Cat} \to \mathbf{Set}$, $\mathscr{C} \longmapsto \coprod_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$ is represented by $[1]$. As $N(\mathscr{C})_1 = \mathrm{Hom}_{\mathbf{Cat}}([1], \mathscr{C})$, this gives an isomorphism $N(\mathscr{C})_1 \simeq \coprod_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$, also natural in $\mathscr{C}$. Note that, if $\delta_0, \delta_1 : [0] \to [1]$ are the two maps defined by $\delta_0(0) = 1$ and $\delta_1(0) = 0$, then $\delta_1^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism to its source and $\delta_0^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism to its target.

Let $\mathscr{C}$ be category. For $n \geq 1$, consider the set M_n of sequences of n composable morphisms $c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} c_n$ of $\mathscr{C}$, which we will also write as $(f_1, \ldots, f_n)$. We have a map $\alpha : N(\mathscr{C})_n \to M_n$ sending a functor $F : [n] \to \mathscr{C}$ to the sequence $F(0) \to F(1) \to \ldots \to F(n)$, where the morphism $F(i) \to F(i+1)$ is the image by F of the unique morphism $i \to i+1$ in $[n]$. This uniquely determines the functor F, because, for $i \leq j$ in $[n]$, the unique morphism $i \to j$ is the composition of $i \to i+1 \to i+2 \to \ldots \to j$. For the same reason, every element of M_n comes from a functor $F : [n] \to \mathscr{C}$. So we get a bijection $N(\mathscr{C})_n \overset{\sim}{\to} M_n$. (We can easily make M_n into a functor $\mathbf{Cat} \to \mathbf{Set}$, and then this bijection is an isomorphism of functors.)

We will identify $N(\mathscr{C})_n$ with M_n in the rest of this solution. We also write $M_0 = \mathrm{Ob}(\mathscr{C})$. (we can think of $c \in \mathrm{Ob}(\mathscr{C})$ as a length 0 sequence of composable morphisms (c).)

Let $\alpha : [m] \to [n]$ be a nondecreasing map. We can give an explicit description of the map $\alpha^* : N(\mathscr{C})_n \to N(\mathscr{C})_m$ by chasing through the identifications. If $n = 0$ and $m \geq 1$, then α^* sends $c \in \mathrm{Ob}(\mathscr{C})$ to the sequence $(\mathrm{id}_c, \ldots, \mathrm{id}_c) \in M_m$. If $n \geq 1$ and $m = 0$, let

$i = \alpha(0)$; then α^* sends the sequence $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ to c_i. Suppose that $n, m \geq 1$, let $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ be an element of M_n, and let $d_0 \xrightarrow{g_1} d_1 \xrightarrow{g_2} \dots \xrightarrow{g_m} d_m$ be its image by α^*. For $i \in \{1, \dots, m\}$, we have :

- if $\alpha(i-1) = \alpha(i)$, then $d_{i-1} = d_i = c_{\alpha(i)}$ and $g_i = \mathrm{id}_{c_{\alpha(i)}}$;
- if $\alpha(i-1) < \alpha(i)$, then $g_i = f_{\alpha(i)} \circ \dots \circ f_{\alpha(i-1)+1}$.

(b). As we have identified Δ to a subcategory of $\mathbf{Cat}$, this is just the definition of Δ_n.

(c). Let $e_n = \mathrm{id}_{[n]} \in \Delta_n([n]) = \mathrm{Hom}_\Delta([n],[n])$. The fact that the map of the statement is bijective is exactly the Yoneda lemma (Theorem I.3.2.2).

(d). Suppose that $u, v : X \to N(\mathscr{C})$ satisfy the condition of the question. Let $n \geq 2$, and let $x \in X_n$. Write $u(x) = (c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n)$ and $v(x) = (d_0 \xrightarrow{g_1} d_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} d_n)$. We want to show that $u(x) = v(x)$, that is, that $f_i = g_i$ for every $i \in \{1, \dots, n\}$. Fix $i \in \{1, \dots, n\}$, and consider the map $\alpha : [1] \to [n]$ sending 0 to $i-1$ and 1 to i. Then α is a morphism in Δ, so we have a commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[d, "\alpha^*"'] & N(\mathscr{C})_n \arrow[d, "\alpha^*"] \\ X_1 \arrow[r, "u_1"'] & N(\mathscr{C})_1 \end{tikzcd}$$

and a similar commutative diagram for v. By definition of the bijection $N(\mathscr{C})_n \xrightarrow{\sim} M_n$, the map α^* sends a sequence $e_0 \xrightarrow{h_1} e_1 \xrightarrow{h_2} \dots \xrightarrow{h_n} e_n$ to $h_i : e_{i-1} \to e_i$. So we get $f_i = \alpha^*(u_n(x)) = u_1(\alpha^*(x)) = v_1(\alpha^*(x)) = \alpha^*(v_n(x)) = g_i$.

(e). Let $u : X_{\leq 2} \to N(\mathscr{C})_{\leq 2}$. We want to show that u extends to a morphism of simplicial sets $v : X \to N(\mathscr{C})$. The solution of question (d) tells us how we must extend u: Let $n \geq 2$, and, for $i \in \{1, \dots, n\}$, let $\alpha_i^n = \alpha_i : [1] \to [n]$ be the map $m \longmapsto m + i - 1$. Then, for every $x \in X_n$, $v_n(x)$ must be the sequence $(u_1(\alpha_1^*(x)), \dots, u_1(\alpha_n^*(x)))$ of morphisms of $\mathscr{C}$. These morphisms are composable : indeed, if we denote by $\delta_0, \delta_1 : [0] \to [1]$ the two maps defined by $\delta_0(0) = 1$ and $\delta_1(0) = 0$, then $\alpha_i \circ \delta_0 = \alpha_{i+1} \circ \delta_1$ for $1 \leq i \leq n-1$, so the target $u_0(\delta_0^* \alpha_i^*(x))$ of $u_1(\alpha_i^*(x))$ is equal to the source $u_0(\delta_1^* \alpha_{i+1}^*(x))$ of $u_1(\alpha_{i+1}^*(x))$.

We have to check that $v_2 = u_2$ and that v is a morphism of simplicial sets. The proof that $v_2 = u_2$ is exactly as in the solution of (d). To show that v is a morphism of simplicial sets, we take a nondecreasing map $\alpha : [m] \to [n]$ and we show that $v_m \circ \alpha^* = \alpha^* \circ v_n$. We can write $\alpha = \alpha' \circ \alpha''$ with α', α'' both nondecreasing, α' injective and α'' surjective, and it suffices to show the statement for α' and α''. Moreover, we can write α' (resp. α'') as a composition of injective (resp. surjective) nondecreasing maps $[p] \to [p+1]$ (resp. $[p+1] \to [p]$). So we may assume that α is injective or surjective and that $n = m \pm 1$.

Suppose first that $\alpha : [n+1] \to [n]$ is a surjective nondecreasing map. Then there is a unique $i \in [n]$ such that $\alpha(i) = \alpha(i+1) = i$, $\alpha(j)$ for $0 \leq j < i$ and

$\alpha(j) = j - 1$ for $i + 1 < j \leq n + 1$. Let $x \in X_n$, and let $(f_1, \dots, f_n) = v_n(x)$. The map $\alpha^* : N(\mathscr{C})_n \to N(\mathscr{C})_{n+1}$ sends the sequence of composable morphisms $(f_1, \dots, f_n)$ to $(f_1, \dots, f_i, \mathrm{id}_c, f_{i+1}, \dots, f_n)$, where c is the target of f_i. By definition, $v_{n+1}(\alpha^*(x)) = (g_1, \dots, g_{n+1})$, with $g_j = u_1(\alpha_j^{n+1*}\alpha^*(x))$. If $1 \leq j \leq i$, then $\alpha \circ \alpha_j^{n+1} = \alpha_j^n$, so $g_j = f_j$. If $i + 2 \leq j \leq n + 1$, then $\alpha \circ \alpha_j^{n+1} = \alpha_{j-1}^n$, so $g_j = f_{j-1}$. Finally, $\alpha_{i+1}^n \circ \alpha : [1] \to [n]$ is the map sending every element of $[1]$ to i, so it is equal to $\alpha' \circ \alpha''$, where $\alpha' : [0] \to [n]$ sends 0 to i and $\alpha'' : [1] \to [0]$ is the unique map; so $g_{i+1} = u_1(\alpha''^* \circ \alpha'^*(x)) = \alpha''^* u_0(\alpha'^*(x))$ is $\mathrm{id}_{c'}$, where $c' = u_0(\alpha'^*(x))$; as $\alpha' = \alpha_i^n \circ \delta_0$, we have $c' = \delta_0^*(f_i)$, that is, c' is the target c of f_i, as we wanted.

Now we take $\alpha : [n-1] \to [n]$ injective and increasing; we may also assume $n \geq 3$, as we already the result for $n \leq 2$. There exists $i \in [n]$ such that $\mathrm{Im}(\alpha) = [n] - \{i\}$, that is, such that α is the map δ_i defined before (g). Let $x \in X_n$, and let $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$ be $v_n(x)$ and $(g_1, \dots, g_{n-1}$ be $v_{n-1}(\alpha^*(x))$. As we saw in the solution of (a), the map $\alpha^* : M_n \to M_{n-1}$ sends the sequence of composable morphisms $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$ to the sequence :

- $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} c_{n-1}$ if $i = n$;
- $c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ if $i = 0$;
- $c_0 \xrightarrow{f_1} \dots c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \dots \xrightarrow{f_n} c_n$ if $1 \leq i \leq n - 1$.

If $1 \leq j \leq i-1$, we have $\alpha \circ \alpha_j^{n-1} = \alpha_j^n$, which implies that $g_j = f_j$. If $i+1 \leq j \leq n-1$, we have $\alpha \circ \alpha_j^{n-1} = \alpha_{j+1}^n$, which implies that $g_j = f_{j+1}$. To finish the proof that $\alpha^*(v_n(x)) = v_{n-1}(\alpha^*(x))$, it remains to consider the case $j = i \in \{1, \dots, n-1\}$. Then $\alpha \circ \alpha_j^{n-1} = \alpha' \circ \delta_1$, where $\alpha' : [2] \to [n]$ is the map $x \longmapsto x + i - 1$ and $\delta_1 : [1] \to [2]$ is the map sending 0 to 0 and 1 to 2. Hence $g_j = u_1(\delta_1^* \alpha'^*(x)) = \delta_1^* u_2(\alpha'^* x)$, so if $u_2(\alpha'^*(x)) = (h_1, h_2)$ then $g_j = h_2 \circ h_1$; but it is easy to see that $u_2(\alpha'^*(x)) = (f_i, f_{i+1})$ (by looking at the composition of α' with $\alpha_1^2, \alpha_2^2 : [1] \to [2]$), so we are done.

(f). Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. We want to show that the map $N : \mathrm{Func}(\mathscr{C}, \mathscr{C}') \to \mathrm{Hom}_{\mathbf{sSet}}(N(\mathscr{C}), N(\mathscr{C}'))$ is bijective, so we try to construct an inverse of this map.

Let $u : N(\mathscr{C}) \to N(\mathscr{C}')$ be a morphism of simplicial sets. We denote by F the map $\mathrm{Ob}(\mathscr{C}) \simeq N(\mathscr{C})_0 \xrightarrow{u_0} N(\mathscr{C}')_0 \simeq \mathrm{Ob}(\mathscr{C}')$. Let $f : c_0 \to c_1$ be a morphism of $\mathscr{C}$. We saw in (a) that this morphism corresponds to a functor $T : [1] \to \mathscr{C}$, that is, an element of $N(\mathscr{C})_1$. We denote by $F(f) : d_0 \to d_1$ the morphism of $\mathscr{C}'$ corresponding to $u_1(T) \in N(\mathscr{C}')_1$. We want to show that $d_0 = F(c_0)$ and $d_1 = F(c_1)$. Let $i \in \{0, 1\}$, and consider the map $\alpha : [0] \to [1]$ sending 0 to i. This is a morphism of Δ, and $\alpha^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism of $\mathscr{C}$ to its source if $i = 0$ and its target if $i = 1$. Using the commutativity of the diagram

$$\begin{tikzcd} N(\mathscr{C})_1 \arrow[r, "u_1"] \arrow[d, "\alpha^*"'] & N(\mathscr{C}'_1) \arrow[d, "\alpha^*"] \\ N(\mathscr{C}_0) \arrow[r, "u_0"'] & N(\mathscr{C}'_0) \end{tikzcd}$$

we see that $d_0 = F(c_0)$ and $d_1 = F(c_1)$. Now we show that F is a functor. There are two conditions to check :

(1) Consider the unique map $\alpha : [1] \to [0]$. This is a morphism of Δ, and $\alpha^* : N(\mathscr{C})_0 \to N(\mathscr{C})_1$ sends the element of $N(\mathscr{C})_0$ corresponding to an object c of $\mathscr{C}$ to the element of $N(\mathscr{C})_1$ corresponding to id_c. As $u_1 \circ \alpha^* = \alpha^* \circ u_0$, we get that, for every $c \in \mathrm{Ob}(\mathscr{C})$, $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

(2) Consider the map $\alpha : [1] \to [2]$ sending 0 to 0 and 1 to 2, and the map $\sigma_i : [1] \to [2]$, $m \longmapsto m+i$, for $i \in \{0, 1\}$. Then α^* (resp. σ_0^*, resp. σ_1^*) sends the element of $N(\mathscr{C})_2$ corresponding to a sequence $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2$ to the element of $N(\mathscr{C})_1$ corresponding to $f_2 \circ f_1$ (resp. f_1, resp. f_2). (This is clear on the identifications of (a).)

   Let $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2$ be a sequence of composable morphisms of $\mathscr{C}$. Using the previous paragraph and the fact that u is a morphism of functors, we see that the image by u of the element of $N(\mathscr{C})_2$ corresponding to this sequence is the sequence $F(c_0) \xrightarrow{F(f_1)} F(c_1) \xrightarrow{F(f_2)} F(c_2)$, and using this and the fact that $\alpha^* \circ u_2 = u_1 \circ \alpha^*$, we finally get $F(f_2) \circ F(f_1) = F(f_2 \circ f_1)$.

So we have constructed a map $\Phi : \mathrm{Hom}_{\mathbf{sSet}}(N(\mathscr{C}), N(\mathscr{C}')) \to \mathrm{Func}(\mathscr{C}, \mathscr{C}')$, and it is clear on the construction that, for every functor $F : \mathscr{C} \to \mathscr{C}'$, we have $\Phi(N(F)) = F$. Now let $u : N(\mathscr{C}) \to N(\mathscr{C}')$ be a morphism of simplicial sets, and let $F = \Phi(u)$. We want to show that $N(F) = u$. Again, it is clear from the construction of Φ that $N(F)_0 = u_0$ and $N(F)_1 = u_1$. But then the fact that $N(F) = u$ follows from (c).

(g). If $n \geq 4$, then, for every $k \in [n]$, the morphism $\Lambda^n_{k,\leq 2} \to \Delta_{n,\leq 2}$ induced by the inclusion $\Lambda^n_k \subset \Delta_n$ is the identity morphism. So, by (d) and (e), every morphism $\Lambda^n_k \to N(\mathscr{C})$ extends uniquely to a morphism $\Delta_n \to N(\mathscr{C})$.

We still need to treat the case $n = 3$. Note that the uniqueness of the extension will follow from the fact that $\Lambda^3_{k,\leq 1} = \Delta_{3,\leq 1}$.

Let $\partial\Delta_3$ be the union of all the faces of Δ_3. Then the inclusion $\partial\Delta_3 \subset \Delta_3$ induces an equality $\partial\Delta_{3,\leq 2} = \Delta_{3,\leq 2}$, so it suffices to show that the morphism $u : \Lambda^3_k \to N(\mathscr{C})$ extends to $\partial\Delta_3$. As $\partial\Delta_3 = \Lambda^3_k \cup \delta_{k*}(\Delta_2)$ and $\Lambda^3_k \cap \delta_{k*}(\Delta_2) = \delta_{k*}(\partial\Delta_2)$, it suffices to extend u from $\delta_{k*}(\partial\Delta_2)$ to $\delta_{k*}(\Delta_2)$. For $0 \leq i < j \leq 3$, let $\alpha_{i,j} : [1] \to [3]$ be the map sending 0 to i and 1 to j; note that $\alpha_{i,j} \in \Lambda^3_k([3])$. Let $f_i = u_3(\alpha_{i-1,i})$, for $1 \leq i \leq 3$. We treat the case $k = 1$, the case $k = 2$ is similar. Factoring both $\alpha_{1,2}$ and $\alpha_{2,3}$ through the morphism $\delta_0 : [2] \to [3]$, we see that f_3 and f_2 are composable, and that $f_3 \circ f_2 = u_3(\alpha_{1,3})$. Factoring both $\alpha_{0,1} = f_1$ and $\alpha_{1,3}$ through the morphism $\delta_2 : [2] \to [3]$, we see that

$f_3 \circ f_2 = u_3(\alpha_{1,3})$ and f_1 are composable, and that $f_3 \circ f_2 \circ f_1 = u_3(\alpha_{0,3})$. Similary, using $\delta_3 : [2] \to [3]$, we show that $f_2 \circ f_1 = u_3(\alpha_{0,2})$.

In particular, we see that $u_3(\alpha_{0,3}) = f_3 \circ f_2 \circ f_1 = u_3(\alpha_{3,2}) \circ u_3(\alpha_{0,2})$. This is exactly the condition that we need to extend u from $\delta_{1*}(\partial\Delta_2)$ to $\delta_{1*}(\Delta_2)$. (See the solution of the next question.)

(h). Let $u : \Lambda_1^2 \to N(\mathscr{C})$. We want to extend u to a morphism $v : \Delta_2 \to N(\mathscr{C})$. Remember that, by the Yoneda lemma, giving v is the same as giving an element e of $N(\mathscr{C})_2$; the fact that v extends u then says that, for every $\alpha : [m] \to [2]$ such that $\alpha \in \Lambda_1^2([n])$, we have $\alpha^*(e) = u_m(\alpha)$.

Note that the maps δ_2 and δ_0 from $[1]$ to $[2]$ are in $\Lambda_1^2([1])$ by definition of the horn Λ_1^2. We set $f_1 = u_1(\delta_2)$ and $f_2 = u_1(\delta_0)$. Comparing the compositions of δ_0 and δ_2 with the two maps $[0] \to [1]$, we see that (f_1, f_2) is a sequence of composable morphisms of $\mathscr{C}$, hence an element of e of $N(\mathscr{C})_2$; we denote by $v : \Delta_2 \to N(\mathscr{C})$ the corresponding morphism, that is, the unique morphism such that $v_2(e_2) = (f_1, f_2)$. Using the method of the solution of (d), we see that this is the only possibility for a morphism extending u (such a morphism must send $e_2 \in \Delta_2([2])$ to (f_1, f_2)).

It remains to show that v does extend u. Let $\alpha : [m] \to [2]$ be an element of $\Lambda_1^2([2])$; by definition of the horn, this means that we can write $\alpha = \delta_i \circ \beta$, with $\beta : [m] \to [1]$ nondecreasing and $i \in \{0, 2\}$. Then $v_2(\alpha) = \alpha^*(e) = \beta^*(\delta_i^*(e))$ and $u_2(\alpha) = \beta^* u_1(\delta_i)$, so it suffices to show that $\delta_i^*(e) = u_1(\delta_i)$; but this follows from the definition of f_1 and f_2 and the description of $\delta_i^* : N(\mathscr{C})_2 \to N(\mathscr{C})_1$ in (a).

(i). Let X be a simplicial set, and suppose that every morphism $u : \Lambda_k^n \to X$ with $0 < k < n$ extends uniquely to Δ_n. We denote by $d_0, d_1 : [0] \to [1]$ the two maps sending 0 to 0 and 1 respectively, and by s the unique map from $[1]$ to $[0]$. We construct a category $\mathscr{C}$ in the following way :

(1) We take $\mathrm{Ob}(\mathscr{C}) = X_0$.

(2) If $c, d \in X_0$, we have $\mathrm{Hom}_{\mathscr{C}}(c, d) = \{f \in X_1 \mid d_0^*(f) = c \text{ and } d_1^*(f) = d\}$.

(3) For every $c \in X_0$, we denote by id_c the element $s^*(c)$ of X_1.

(4) Let $c, d, e \in X_0$ and $f \in \mathrm{Hom}_{\mathscr{C}}(c, d)$, $g \in \mathrm{Hom}_{\mathscr{C}}(d, e)$. We want to construct a morphism $u : \Lambda_1^2 \to X$. Let $\alpha : [m] \to [2]$ be an element of $\Lambda_1^2([m])$. By definition of Λ_1^2, there exists $\beta : [m] \to [1]$ and $j \in \{0, 2\}$ such that $\alpha = \delta_j \circ \beta$. We set $u_m(\alpha) = \beta^*(f_j)$, with $f_j = f$ if $j = 2$ and $f_j = g$ if $j = 0$. We must check that this is well-defined; if α can be written as $\beta \circ \delta_0$ and $\beta' \circ \delta_2$, with $\beta : [m] \to [1]$, this means that $\mathrm{Im}(\alpha) = \{1\}$, so $\mathrm{Im}(\beta) = \{0\}$ and $\mathrm{Im}(\beta') = \{1\}$, so there exists $\gamma : [m] \to [0]$ such that $\beta = d_0 \circ \gamma$ and $\beta' = d_1 \circ \gamma$, hence $\beta^*(g) = \gamma^*(d) = \beta'^*(f)$. We now check that u is a morphism of simplicial sets. If $\alpha : [m] \to [2]$ is an element of $\Lambda_1^2([m])$, write $\alpha = \delta_j \circ \beta$, with $\beta : [m] \to [1]$ and $j \in \{0, 2\}$; then, for every $\gamma : [m'] \to [m]$, we have $\alpha \circ \gamma = \delta_j \circ (\beta \circ \gamma)$, so $u_{m'}(\gamma) = (\beta \circ \gamma)^*(f_j) = \gamma^*(\beta^*(f_j)) = \gamma^*(u_m(\beta))$.

So u is a morphism of simplicial sets, and, by assumption, it extends uniquely to a morphism $v : \Delta_2 \to X$. We take $g \circ f = v_1(\delta_1)$. It is easy to check that $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(c, e)$.

It is easy to check that the identity morphisms are unit elements for the composition.

We check that the composition law of $\mathscr{C}$ is associative. Let (f_1, f_2, f_3) be a sequence of composable morphisms in $\mathscr{C}$. Remember that we have maps $\delta_i : [2] \to [3]$, inducing morphisms of simplicial sets $\delta_{i*} : \Delta_2 \to \Delta_3$. As in the construction of the composition in (4), we use the pair (f_1, f_2) to construct a morphism $\delta_{3*}(\Delta_2) \to X$, the pair (f_2, f_3) to construct a morphism $\delta_{1*}(\Delta_2) \to X$, and the pair $(f_1, f_3 \circ f_2)$ to construct a morphism $\delta_{2*}(\Delta_2) \to X$. These three morphisms glue to a morphism $\Lambda_1^3 \to X$, which extends uniquely to $v : \Delta_3 \to X$. In particular, if we define maps $\alpha_{i,j} : [1] \to [3]$ as in (g), we see as in that question that

$$v_1(\alpha_{0,3}) = v_1(\alpha_{3,2}) \circ v_1(\alpha_{2,0}) = f_3 \circ (f_2 \circ f_1) = v_1(\alpha_{3,1} \circ \alpha_{1,0}) = (f_3 \circ f_2) \circ f_1.$$

For $n \geq 1$ and $1 \leq i \leq n$, let the $\alpha_i^n : [1] \to [n]$ be as in the solution of (e).

Let $n \geq 2$. If $1 \leq m \leq n$ $0 \leq i_0 < i_1 < \ldots < i_m \leq n$, we denote by $\alpha_{i_0,\ldots,i_m} : [1] \to [n]$ the map sending $r \in [m]$ to $i_r \in [n]$. If $x \in X_n$, we define morphisms $f_{1,x}, \ldots, f_{n,x}$ in $\mathscr{C}$ by $f_i = \alpha_i^{n*}(x)$. As $\alpha_i^n \circ d_0 = \alpha_{i+1}^n \circ d_1$ for $1 \leq i \leq n-1$, the f_i form a sequence of composable morphisms, so $(f_1, \ldots, f_n) \in N(\mathscr{C})_n$. We claim that :

(A) For every $(f_1, \ldots, f_n)$ ni $N(\mathscr{C})_n$, there exists $x \in X$ such that $(f_{1,x}, \ldots, f_{n,x}) = (f_1, \ldots, f_n)$.

(B) If $x, y \in X_n$ are such that $f_{i,x} = f_{i,y}$ for $1 \leq i \leq n$, then $x = y$.

We prove (A). Let $(f_1, \ldots, f_n)$ be a sequence of composable morphisms in $\mathscr{C}$. For $0 \leq i_0 < i_1 \leq n$, we define a morphism $u_{i_0,i_1} : \alpha_{i_0,i_1,*}(\Delta_1) \to X$ by sending $\alpha_{i_0,i_1,*}(e_1)$ to $f_{i_1} \circ \ldots f_{i_0+1} \in X_1$, where $e_1 \in \Delta_1([1])$ is the element defined in (c). Suppose that $0 \leq i_0 < i_1 < i_2 \leq n$. Then the morphisms u_{i_0,i_1}, u_{i_1,i_2} and u_{i_0,i_2} agree on the intersections of their domains (because the f_i are composable), so they glue to a morphism $u'_{i_0,i_1,i_2} : \alpha_{i_0,i_1,i_2,*}(\partial\Delta_2) \to X$; by the property of X, this morphism extends uniquely to a morphism $u_{i_0,i_1,i_2} : \alpha_{i_0,i_1,i_2,*}(\Delta_2) \to X$. Now take $0 \leq i_0 < i_1 < i_2 < i_3 \leq n$. Then the morphisms u_{i_0,i_1,i_2}, u_{i_0,i_1,i_3}, u_{i_0,i_2,i_3} and u_{i_1,i_2,i_3} agree on the intersections of their domains (we just recover one of the u_{i_r,i_s} on such an intersection), so they glue to a morphism $u'_{i_0,i_1,i_2,i_3} : \alpha_{i_0,i_1,i_2,i_3,*}(\partial\Delta_3) \to X$; by the property of X, this morphism extends uniquely to a morphism $u_{i_0,i_1,i_2,i_3} : \alpha_{i_0,i_1,i_2,i_3,*}(\Delta_3) \to X$. We continue in this way until we get a morphism $u_{0,1,\ldots,n} : \Delta_n \to X$ extending the original u_{i_0,i_1}; the corresponding element of X_n has the required property.

Now we prove (B). By the Yoneda lemma, the elements $u, v \in X_n$ correspond to two morphisms $u_x, u_y : \Delta_n \to X$, and the condition of (B) says that u_x and u_y agree on $\alpha_{i_0,i_1,*}(\Delta_1)$ for all $i_0, i_1 \in [n]$ such that $i_0 < i_1$. But we saw in the proof of (A) that there is a unique way to extend a family of morphisms $\alpha_{i_0,i_1,*}(\Delta_1) \to X$ (agreeing on the intersections of the $\alpha_{i_0,i_1,*}(\Delta_1)$) to a morphism $\Delta_n \to X$. So $u_x = u_y$, that is, $x = y$.

Finally, we define $u : X \to N(\mathscr{C})$ by taking u_1 and u_0 to be the obvious bijections and by sending $x \in X_n$ to the sequence of maps $(f_{1,x}, \dots, f_{n,x})$, for every $n \geq 2$. This induces a morphism of functors from $X_{\leq 2}$ to $N(\mathscr{C})_{\leq 2}$ by the definition of the composition and the description of the maps between the $N(\mathscr{C})_n$ in (a). Then the solution of (e) shows that u is a morphism of simplicial sets. Points (A) and (B) imply that u is an isomorphism.

□

A.2 Problem set 2

A.2.1 Monoidal categories

A *monoidal category* is a category $\mathscr{C}$ equipped with a bifunctor $(\cdot) \otimes (\cdot) : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (the tensor product or monoidal functor), with an identity (or unit) object $\mathbb{1}$ and with three natural isomorphisms $\alpha(A,B,C) : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $\lambda(A) : \mathbb{1} \otimes A \xrightarrow{\sim} A$ and $\rho_A : A \otimes \mathbb{1} \xrightarrow{\sim} A$, satisfying the following conditions :

- for all $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes \mathrm{id}_D"] \arrow[d, "\alpha(A\otimes B,C,D)"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathrm{id}_A \otimes \alpha(B,C,D)"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

- for all $A, B \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha(A,\mathbb{1},B)"] \arrow[dr, "\rho(A)\otimes \mathrm{id}_B"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

Here are some examples :

- $\mathscr{C} = \mathbf{Set}$ or $\mathbf{Top}$, $\otimes = \times$, $\mathbb{1}$ is a singleton;
- $\mathscr{C} = \mathbf{Grp}$, $\otimes = \times$, $\mathbb{1} = \{1\}$;
- $\mathscr{C} = {}_R\mathbf{Mod}$ with R a commutative ring, $\otimes = \otimes_R$, $\mathbb{1} = R$;
- $\mathscr{C} = \mathrm{Func}(\mathscr{D}, \mathscr{D})$ with $\mathscr{D}$ a category, $\otimes = \circ$, $\mathbb{1} = \mathrm{id}_{\mathscr{D}}$.

A *monoid* in $\mathscr{C}$ is an object M of $\mathscr{C}$ together with two morphisms $\mu : M \otimes M \to M$ (multiplication) and $\eta : \mathbb{1} \to M$ (unit), such that the two following diagrams commute :

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "\mathrm{id}_M \otimes \mu"] & M \otimes M \arrow[r, "\mu"] & M \\
(M \otimes M) \otimes M \arrow[u, "\alpha(M,M,M)"] \arrow[r, "\mu \otimes \mathrm{id}_M"'] & M \otimes M \arrow[ur, "\mu"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes M \arrow[dr, "\mu"] & \mathbb{1} \otimes M \arrow[l, "\eta \otimes \mathrm{id}_M"'] \arrow[d, "\lambda(M)"] \\
M \otimes \mathbb{1} \arrow[u, "\mathrm{id}_M \otimes \eta"] \arrow[r, "\rho(M)"'] & M
\end{tikzcd}$$

(We can also define morphisms of monoids, and monoids in $\mathscr{C}$ form a category.)

Examples :

- A monoid in $(\mathbf{Set}, \times)$ is a monoid (in the usual sense).
- A monoid in $(\mathbf{Top}, \times)$ is a topological monoid.
- If R is a commutative ring, a monoid in $({}_R\mathbf{Mod}, \otimes)$ is a R-algebra. (In particular, a monoid in $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$ is a ring.)
- A monoid in $(\mathrm{Func}(\mathscr{D}, \mathscr{D}), \circ)$ is called a *monad on* $\mathscr{D}$.

(a). Let $\mathbf{Mon}$ be the category of (usual) monoids. It is a monoidal category, with the monoidal functor given by $\times$ and the unit object $\{1\}$. If (M, μ, η) is a monoid in $\mathbf{Mon}$, show that M is a commutative monoid and μ is equal to the multiplication of M.

(b). Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors such that (F, G) is a pair of adjoint functors, and let $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ be the counit and unit of the adjunction. Define a morphism of functors $\mu : (G \circ F) \circ (G \circ F) \to G \circ F$ by $\mu(X) = G(\varepsilon(F(X))) : G(F \circ G(F(X))) \to G(F(X))$. Show that $(G \circ F, \mu, \eta)$ is a monad on $\mathscr{C}$.

Solution.

(a). We denote the monoid operation of M by $(a, b) \longmapsto a \cdot b$ and its unit element by 1. We also denote the map $\mu : M^2 \to M$ by $(a, b) \longmapsto a * b$. The fact that μ is a morphism of monoids says that, for all $a, b, c, d \in M$, we have

(*) $$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d).$$

As $\eta : \{1\} \to M$ is a morphism of monoids, it sends 1 to $1 \in M$, so $\cdot$ and $*$ have the same unit. [3] So, if $a, d \in M$, we have

$$a \cdot d = (a * 1) \cdot (1 * d) = (a \cdot 1) * (1 \cdot d) = a * d,$$

and also

$$a \cdot d = (1 * a) \cdot (d * 1) = (1 \cdot d) * (a \cdot 1) = d * a.$$

This proves both statements. [4]

(b). Note that the operations $(\cdot) \circ \mathrm{id}_{\mathscr{C}}$ and $\mathrm{id}_{\mathscr{C}} \circ (\cdot)$ are the identity functor of the category $\mathrm{Func}(\mathscr{C}, \mathscr{C})$, so the functorial isomorphisms ρ and λ are just the identity in that case; similarly, as $(H \circ H') \circ H'' = H \circ (H' \circ H'')$ for any $H, H', H'' \in \mathrm{Func}(\mathscr{C}, \mathscr{C})$, the functorial isomorphism α is also the identity. So we have three things to prove :

(1) $\mu \circ (\mathrm{id}_{G \circ F} \otimes \eta) = \mathrm{id}_{G \circ F}$;

(2) $\mu \circ (\eta \otimes \mathrm{id}_{G \circ F}) = \mathrm{id}_{G \circ F}$;

(3) $\mu \circ (\mu \otimes \mathrm{id}_{G \circ F}) = \mu \circ (\mathrm{id}_{G \circ F} \otimes \mu)$.

To prove (1), we note that, by definition of $\otimes$ and μ, for every $X \in \mathrm{Ob}(\mathscr{C})$, the left-hand side of (1) applied to X is the image by G of the composition

$$F(X) \xrightarrow{F(\eta(X))} F(G(F(X))) \xrightarrow{\varepsilon(F(X))} F(X) .$$

So (1) follows from the first statement of Proposition I.4.4 . The proof of (2) is similar : by definition of $\otimes$ and μ, for every $X \in \mathrm{Ob}(\mathscr{C})$, the left-hand side of (2) applied to X is the composition

$$G(F(X)) \xrightarrow{\eta(G(F(X)))} G(F(G(F(X)))) \xrightarrow{G(\varepsilon(F(X)))} G(F(X)) ,$$

and we can apply the second statement of Proposition I.4.4 .

It remains to prove (3). Let $X \in \mathrm{Ob}(\mathscr{C})$. Then, when applied to X, the square

$$\begin{tikzcd}
(G \circ F) \circ (G \circ F) \circ (G \circ F) \arrow[r, "\mu \otimes \mathrm{id}_{G \circ F}"] \arrow[d, "\mathrm{id}_{G \circ F} \otimes \mu"'] & (G \circ F) \circ (G \circ F) \arrow[d, "\mu"] \\
(G \circ F) \circ (G \circ F) \arrow[r, "\mu"'] & (G \circ F)
\end{tikzcd}$$

becomes

(*)
$$\begin{tikzcd}
G(F(G(F(G(F(X)))))) \arrow[r, "G(\varepsilon(F(G(F(X)))))"] \arrow[d, "G(F(G(\varepsilon(F(X)))))"'] & G(F(G(F(X)))) \arrow[d, "G(\varepsilon(F(X)))"] \\
G(F(G(F(X)))) \arrow[r, "G(\varepsilon(F(X)))"'] & G(F(X))
\end{tikzcd}$$

[3] This would be automatic even if we did not assume that η is a morphism of monoids : Let e be the unit of $*$. Then $1 = 1 \cdot 1 = (e * 1) \cdot (1 * e) = (e \cdot 1) * (1 \cdot e) = e * e = e$.

[4] Note that we did not use the associativity of $\cdot$ and $*$. In fact, we could deduce the associativity of $\cdot$ and $*$ from property (*).

Let $Y = F(X)$ and $u = \varepsilon(Y) : F(G(Y)) \to Y$. As $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ is a morphism of functors, the following square is commutative

$$\begin{tikzcd} F(G(F(G(Y)))) \arrow[r, "\varepsilon(F(G(Y)))"] \arrow[d, "F(G(u))"'] & F(G(Y)) \arrow[d, "u"] \\ F(G(Y)) \arrow[r, "\varepsilon(Y)"'] & Y \end{tikzcd}$$

Applying the functor F to this square, we recover the square (*), so (*) is also commutative.

□

A.2.2 Geometric realization of a simplicial set

Remember that the simplicial category Δ is the subcategory of $\mathbf{Set}$ whose objects are the sets $[n] = \{0, 1, \ldots, n\}$, for $n \in \mathbb{N}$, and whose morphisms are nondecreasing maps (where we put the usual order on $[n]$). The category of simplicial sets $\mathbf{sSet}$ is defined by $\mathbf{sSet} = \mathrm{PSh}(\Delta) = \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$; if X is a simplicial set, we write X_n for $X([n])$ and $\alpha^* : X_m \to X_n$ for $X(\alpha) : X([m]) \to X([n])$ (if $\alpha : [n] \to [m]$ is a nondecreasing map). The standard n-simplex Δ is the simplicial set represented by $[n]$, i.e. $\mathrm{Hom}_\Delta(\cdot, [n])$.

(a). Let $\mathscr{C}$ be a category and $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$. We consider the category $\mathscr{C}/F$ whose objects are pairs (X, x), with $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$, and such that a morphism $(X, x) \to (Y, y)$ is a morphism $f : X \to Y$ in $\mathscr{C}$ with $F(f)(y) = x$. Note that we have an obvious faithful functor $G_F : \mathscr{C}/F \to \mathscr{C}$ (forgetting the second entry in a pair), so we get a functor $h_{\mathscr{C}} \circ G_F : \mathscr{C}/F \to \mathrm{PSh}(\mathscr{C})$.

   (i) When does $\mathscr{C}/F$ have a terminal object ?

   (ii) Show that $\varinjlim(h_{\mathscr{C}} \circ G_F) = F$. (Hint : Use the second entries of the pairs to construct a morphism from $\varinjlim(h_{\mathscr{C}} \circ G_F)$ to F.) [5]

For every $n \in \mathbb{N}$, let $|\Delta_n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid x_0 + \ldots + x_n = 1\}$ with the subspace topology. If $f : [n] \to [m]$ is a map, we define $|f| : |\Delta_n| \to |\Delta_m|$ by $|f|(x_0, \ldots, x_n) = (\sum_{i \in f^{-1}(j)} x_i)_{0 \le j \le m}$. (With the convention that an empty sum is equal to 0.) Consider the functor $|.| : \Delta \to \mathbf{Top}$ sending $[n]$ to $|\Delta_n|$ and $f : [n] \to [m]$ to $|f|$.

Let X be a simplicial set, and consider the functor $G_X : \Delta/X \to \Delta$ of (a). The *geometric realization of* X is by definition the topological space $|X| = \varinjlim(|.| \circ G_X)$.

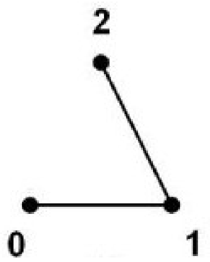
(b). Show that this construction upgrades to a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$. [6]

[5] So every presheaf is a colimit of representable presheaves.

[6] This functor is called the *left Kan extension* of $|.| : \Delta \to \mathbf{Top}$ along the Yoneda embedding $\Delta \to \mathbf{sSet}$.

(c). Show that, if X is Δ_n, then $|X| = |\Delta_n|$.

(d). Give a simplicial set whose geometric realization is $\{(x_0, x_1, x_2) \in [0,1]^2 \mid x_0 = 0 \text{ or } x_2 = 0\}$. (Hint: why are the horns called horns ?)



(e). Consider the functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ given by $\mathrm{Sing}(X) = \mathrm{Hom}_{\mathbf{Top}}(|.|, X) : \Delta^{\mathrm{op}} \to \mathbf{Set}$. (That is, if X is a topological space, then $\mathrm{Sing}(X)$ is the simplicial set such that $\mathrm{Sing}(X)_n$ is the set of continuous maps from $|\Delta_n|$ to X, and, if $f : [n] \to [m]$ is nondecreasing, then $f^* : \mathrm{Sing}(X)_m \to \mathrm{Sing}(X)_n$ sends a continuous map $u : |\Delta_m| \to X$ to $u \circ |f|$.) The simplicial set $\mathrm{Sing}(X)$ is called the *singular simplicial complex of* X of X.

Show that $(|.|, \mathrm{Sing})$ is a pair of adjoint functors.

Solution.

(a). (i) Suppose that (X, x) is a terminal object of $\mathscr{C}/F$. Let Y be an object of $\mathscr{C}$, and consider the map $\phi : \mathrm{Hom}_{\mathscr{C}}(Y, X) \to F(Y)$ sending $f : Y \to X$ to $F(f)(x) \in F(Y)$. (Remember that F is a contravariant functor on $\mathscr{C}$.) We claim that ϕ is bijective. Indeed, if $f, g : Y \to X$ are two morphisms such that $F(f)(x) = F(g)(x)$, then they define morphisms from $(Y, F(f)(x))$ to (X, x) in the category $\mathscr{C}/F$, hence must be equal; so ϕ is injective. Also, if $y \in F(Y)$, then (Y, y) is an object of $\mathscr{C}/F$, so there exists a morphism $h : (Y, y) \to (X, x)$ in $\mathscr{C}/F$, that is, a morphism $h : Y \to X$ in $\mathscr{C}$ such that $F(h)(x) = y$; so ϕ is surjective.

This proves that a terminal object in $\mathscr{C}/F$ is exactly a pair representing the functor F, so such a terminal object exists if and only if F is representable.

(ii) If $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$, then, by the Yoneda lemma, there is unique morphism $u_x : h_X \to F$ in $\mathrm{PSh}(\mathscr{C})$ such that $u_x(X)(\mathrm{id}_X) = x$. We claim that the family of these morphisms defines a cone under $h_{\mathscr{C}} \circ G_F$ with nadir F. This claim means that, for any two objects (X, x) and (Y, y) in $\mathscr{C}/F$ and any morphism $f : (X, x) \to (Y, y)$, the following diagram commutes :

$$\begin{tikzcd} h_X \arrow[r, "h_f"] \arrow[d, "u_x"'] & h_Y \arrow[dl, "u_y"] \\ F & \end{tikzcd}$$

As the morphism $u_y \circ h_f : h_X \to F$ sends $\mathrm{id}_X \in h_X(X)$ to $u_y(X)(f \circ \mathrm{id}_X) = F(f)(y) = F(x) = u_x(X)(\mathrm{id}_X)$, we have $u_y \circ h_f = u_x$ by the Yoneda lemma, so the diagram commutes, as desired.

By the universal property of the colimit, this gives a morphism $\phi : \varinjlim(h_{\mathscr{C}} \circ G_F) \to F$ in $\mathrm{PSh}(\mathscr{C})$.

Now we show that ϕ is an isomorphism. Let $F' = \varinjlim(h_{\mathscr{C}} \circ G_F)$. This is a colimit in the category of presheaves on $\mathscr{C}$, so we can use Proposition I.5.3.1 to compute it. Let Z be an object of $\mathscr{C}$. Then $F'(Z) = \varinjlim_{(X,x)\in\mathrm{Ob}(\mathscr{C}/F)} \mathrm{Hom}_{\mathscr{C}}(Z,X)$, and the map $\phi(Z) : F'(Z) \to F(Z)$ sends a morphism $f : Z \to X$ to $F(f)(x) \in F(Z)$. If $z \in F(Z)$, then (Z,z) is an object of $\mathscr{C}/F$, and $\phi(Z)(\mathrm{id}_Z) = z$; this shows that $\phi(Z)$ is surjective. Let (X,x) and (Y,y) be two objects of $\mathscr{C}/F$, let $f : Z \to X$ and $g : Z \to Y$ be morphisms of $\mathscr{C}$, and suppose that $F(f)(x) = F(g)(y)$. Let $z = F(f)(x)$. Then (Z,z) is an object of $\mathscr{C}/F$, the morphisms f and g induce morphisms $(Z,z) \to (X,x)$ and $(Z,z) \to (Y,y)$ in $\mathscr{C}/F$, and, in the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(Z,Z) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(Z,f)"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(Z,g)"'] & \mathrm{Hom}_{\mathscr{C}}(Z,X) \arrow[d, "\phi(Z)"] \\ \mathrm{Hom}_{\mathscr{C}}(Z,Y) \arrow[r, "\phi(Z)"'] & F(Z) \end{tikzcd}$$

the element id_Z of $\mathrm{Hom}_{\mathscr{C}}(Z,Z)$ is sent to the same element z of Z by both paths. So the images of f and g in $F'(Z)$ are equal, which proves that $\phi(Z)$ is injective.

(b). For X a simplicial set, we set

$$L(X) = \coprod_{n\in\mathbb{N}} \coprod_{x\in X_n} |\Delta_n|,$$

so that $|X|$ is the quotient of $L(X)$ by the equivalence relation $\sim$ of Theorem I.5.2.1 , with the quotient topology. If $f : X \to Y$ is a morphism of simplicial sets, we denote by $L(f)$ a continuous map $L(X) \to L(Y)$ that, for each $n \in \mathbb{N}$ and each $x \in X_n$, sends the component $|\Delta_n|$ of $L(X)$ corresponding to (n,x) to the component $|\Delta_n|$ of $L(Y)$ corresponding to $(n, f_n(x))$ by $\mathrm{id}_{|\Delta_n|}$. This clearly defines a functor $L : \mathbf{sSet} \to \mathbf{Top}$. To show that $|.|$ upgrades to a functor, it suffices to show that, for every morphism $f : X \to Y$ in $\mathbf{sSet}$, the map $f' : L(X) \overset{L(f)}{\to} L(Y) \to |Y|$ factors through the quotient map $L(X) \to |X|$.

Fix f, let $n, m \in \mathbb{N}$, $x \in X_n$, $y \in X_m$, $s \in |\Delta_n|$ and $t \in |\Delta_m|$ such that the images of $(n, x, s), (m, y, t) \in L(X)$ in $|X|$ are equal; we want to show that the images of $(n, f_n(x), s), (m, f_m(y), t) \in L(Y)$ in $|Y|$ are also equal. We may assume that there exists $\alpha : [n] \to [m]$ such that $x = \alpha^*(y)$ and $t = |\alpha|(s)$. Then $f_n(x) = f_n(\alpha^*(y)) = \alpha^*(f_m(y))$, so $(n, f_n(x), s)$ and $(m, f_m(y), t)$ have the same image in $|Y|$.

(c). By (a)(i), the category Δ/Δ_n has a terminal object, which is $([n], \mathrm{id}_{[n]})$. It follows immediately from the definition of a cone under a functor that a cone $(S, (u_{m,x})_{m\in\mathbb{N}, x\in\Delta_n([m])})$ under $|.| \circ G_{\Delta_n}$ is uniquely determined by the continuous map $u_{n,\mathrm{id}_{[n]}} : |\Delta_n| \to S$, and that this map can be arbitrary. In other words, the functor sending a topological space S to the space of cones under $|.| \circ G_{\Delta_n}$ with nadir S is representable by $|\Delta_n|$. This means that $|\Delta_n| = \varinjlim(|.| \circ G_{\Delta_n}) = |\Delta_n|$.

(d). Let's take $X = \Lambda^2_1$ (see problem A.1.9). The geometric relaization $|X|$ is the quotient of $\coprod_{n\in\mathbb{N}} \coprod_{x\in X_n} |\Delta_n|$ by the equivalence relation $\sim$ of Theorem I.5.2.1 .

By definition, for every $n \in \mathbb{N}$, the set X_n is the set of nondecreasing maps $\alpha : [n] \to [2]$ such that $\{0, 2\} \not\subset \mathrm{Im}(\alpha)$. In particular, such a map always factors as $\alpha = \beta \circ \gamma$ with $\gamma : [n] \to [1]$ and $\beta : [1] \to [2]$ two nondecreasing maps such that $\beta \in X_1$, so $\alpha = \gamma^*(\beta)$, so, for every $s \in |\Delta_n|$, we have $(n, \alpha, s) \sim (1, \beta, |\gamma|(s))$. This means that $|X|$ is homeomorphic to the quotient of $\coprod_{n\in\{0,1\}} \coprod_{x\in X_n} |\Delta_n|$ by the relation of $\sim$.

For every $i \in [2]$, let $\alpha_i : [0] \to [2]$ be the map $0 \longmapsto i$, and $\delta_i : [1] \to [2]$ be the unique increasing map such that $\mathrm{Im}(\delta_i) = [2] - \{i\}$. Let β be the unique map from $[1]$ to $[0]$. Then $X_0 = \{\alpha_0, \alpha_1, \alpha_2\}$ and $X_1 = \{\delta_0, \delta_2, \alpha_0 \circ \beta, \alpha_1 \circ \beta, \alpha_2 \circ \beta\}$. Also, for every $i \in [2]$ and every $s \in |\Delta_1|$, we have $(1, \alpha_i \circ \beta, s) \sim (0, \alpha_i, |\beta|(s))$. So $|X|$ is the quotient of the disjoint union of three points corresponding to $\alpha_0, \alpha_1, \alpha_2$, say 0, 1 and 2, and of two line segments (homeomorphic to $[0, 1]$) corresponding to δ_0, δ_2, say I_0 and I_2, by the restriction of $\sim$. It is easy to see that this equivalence relation identifies the two extremities of I_0 (resp. I_2) with 1 and 2 (resp. 0 and 1), so $|X|$ is homeomorphic to the space of the figure.

(e). Let X be a simplicial set and Y be a topological space. By definition, we have $|X| = \varinjlim_{\Delta/X}(|.| \circ G_X)$, so, by Proposition I.5.3.2 , we have an isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Hom}_{\mathbf{Top}}(|\Delta_n|, Y) = \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Sing}(Y)_n.$$

Also, by question (a)(ii), we have $X = \varinjlim_{\Delta/X} G_X$, so, by the same proposition, we have

$$\mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y)) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, \mathrm{Sing}(Y)) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Sing}(Y)_n$$

(the last isomorphism comes from the Yoneda lemma). So we get an isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \simeq \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y)),$$

and checking that it is an isomorphism of functors is straightforward.

□ m

A.2.3 Yoneda embedding and colimits

Let k be a field, and let $\mathscr{C}$ be the category of k-vector spaces.

(a). For every $n \in \mathbb{N}$, let $k[x]_{\leq n}$ be the vector space of polynomials of degree $\leq n$ in $k[x]$. Using the inclusions $k[x]_{\leq n} \subset k[x]_{\leq m}$ for $n \leq m$, we get a functor $F : \mathbb{N} \to \mathscr{C}$, $n \longmapsto k[x]_{\leq n}$. Show that $\varinjlim F = k[x]$.

(b). Show that $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ does not commute with all colimits.

Solution.

(a). Note that the colimit is filtrant, because $\mathbb{N}$ is a directed poset. By an easy analogue Proposition I.5.6.2 to conclude that the $\varinjlim F$ is the quotient of $\bigoplus_{n\in\mathbb{N}} k[x]_{\leq n}$ by the subspace generated by the images of all the maps $u_{m,i} : k[x]_{\leq m} \to \bigoplus_{n\in\mathbb{N}} k[x]_{\leq n}$ sending $f \in k[x]_{\leq m}$ to $(f, -f)$, where the first entry is in the summand $k[x]_{\leq m}$ and the second entry is in the summand $k[x]_{\leq m+i}$, for every $m \in \mathbb{N}$ and every $i \geq 1$. [7] So the sum map from $\bigoplus_{n\in\mathbb{N}} k[x]_{\leq n} \to k[x]$ (sending a family $(f_0, f_1, \ldots)$ with finite support to $f_0 + f_1 + \ldots$) factors through $\varinjlim F$ and induces an isomorphism $\varinjlim F \xrightarrow{\sim} k[x]$.

(b). Let $V = k[x]$. We have seen in (a) that $V = \varinjlim_{n\in\mathbb{N}} k[x]_{\leq n}$, so we get a morphism of presheaves $u : \varinjlim_{n\in\mathbb{N}} h_{k[x]_{\leq n}} \to h_V$. If W is a k-vector space, $u(W)$ is the map from $(\varinjlim_{n\in\mathbb{N}} h_{k[x]_{\leq n}})(W) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_k(W, k[x]_{\leq n})$ to $\mathrm{Hom}_k(W, V)$ induced by the obvious injections $\mathrm{Hom}_k(W, k[x]_{\leq n}) \subset \mathrm{Hom}_k(W, V)$. So the image of $u(W)$ is the set of k-linear maps from W to V whose image is contained in one of the subspaces $k[x]_{\leq n}$ of V. In particular, $\mathrm{id}_V \in h_V(V)$ is not in the image of $u(V)$, so u is not an isomorphism.

□

A.2.4 Filtrant colimits of modules

Let R be a ring, let $\mathscr{I}$ be a filtrant category and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be a functor. For every $i \in \mathrm{Ob}(\mathscr{I})$, we write $M_i = F(i)$. Let $\sim$ be the equivalence relation on $\coprod_{i\in\mathrm{Ob}(\mathscr{I})} M_i$ defined in Proposition I.5.6.2 of the notes; so $(i, x) \sim (j, y)$ if there exist morphisms $\alpha : i \to k$ and $\beta : j \to k$ in $\mathscr{I}$ such that $F(\alpha)(x) = F(\beta)(y)$). Let $M = \coprod_{i\in\mathrm{Ob}(\mathscr{I})} M_i/\sim$; this is the colimit of the composition $\mathscr{I} \xrightarrow{F} {}_R\mathbf{Mod} \xrightarrow{\mathrm{For}} \mathbf{Set}$. Denote by $q_i : M_i \to M$ the obvious maps.

Show that there exists a unique structure of left R-module on M such that all the q_i are R-linear maps, and that this structure makes $(M, (q_i))$ into a colimit of F.

Solution. Let $X = \coprod_{i\in I} M_i$. If (i, m) and (i, n) are elements of X such that $(i, m) \sim (j, n)$, and

[7] We could also use problem A.2.4 to calculate the colimit.

if $a \in R$, then $(i, m) \sim (j, n)$ (because the maps $F(\alpha)$ are all R-linear). So the action of R by left multiplication on X descends to an action on M. Now let (i_1, m_1) and (i_2, m_2) be elements of X. Choose morphisms $\alpha_1 : i_1 \to j$ and $\alpha_2 : i_2 \to i$ in $\mathscr{I}$. Then $(i_1, m_1) \sim (i, F(\alpha_1)(m_1))$ and $(i_2, m_2) \sim (i, F(\alpha_1)(m_2))$, so, if M has a structure of abelian group such that the map $M_i \to M$ is additive, this forces the image of $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2))$ in M to be the sum of the images of (i_1, m_1) and (i_2, m_2) in M. We must check that this definition of addition does not depend on the choices, so we take $(j_1, n_1), (j_2, n_2) \in X$ such that $(j_1, n_1) \sim (i_1, m_1)$ and $(j_2, n_2) \sim (i_2, m_2)$. Choose morphisms $\alpha'_1 : j_1 \to j$ and $\alpha'_2 : j_2 \to j$. We want to check that $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2)) \sim (j, F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))$. The hypothesis on (j_1, n_1) and (j_2, n_2) means that there exist morphisms $\beta_1 : i_1 \to k_1$, $\gamma_1 : j_1 \to k_1$, $\beta_2 : i_2 \to k_2$ and $\gamma_2 : j_2 \to k_2$ in $\mathscr{I}$ such that $F(\beta_1)(m_1) = F(\gamma_1)(n_1)$ and $F(\beta_2)(m_2) = F(\gamma_2)(n_2)$. As $\mathscr{I}$ is filtrant, we can find an object l of $\mathscr{I}$ and morphisms $\delta : i \to l$, $\delta_1 : k_1 \to l$, $\delta_2 : k_2 \to l$ and $\delta' : j \to l$, and then we can find a morphism $\epsilon : l \to l'$ such that

$$\epsilon \circ \delta \circ \alpha_1 = \epsilon \circ \delta_1 \circ \beta_1 : i_1 \to l',$$

$$\epsilon \circ \delta \circ \alpha_2 = \epsilon \circ \delta_2 \circ \beta_2 : i_2 \to l',$$

$$\epsilon \circ \delta' \circ \alpha'_1 = \epsilon \circ \delta_1 \circ \gamma_1 : j_1 \to l',$$

and

$$\epsilon \circ \delta' \circ \alpha'_2 = \epsilon \circ \delta_2 \circ \gamma_2 : i_1 \to l'.$$

Then

$$\begin{aligned}(l', F(\epsilon \circ \delta)(F(\alpha_1)(m_1) + F(\alpha_2)(m_2))) &= (l', F(\epsilon)(F(\delta_1 \circ \beta_1)(m_1) + F(\delta_2 \circ \beta_2)(m_2))) \\ &= (l', F(\epsilon)(F(\delta_1 \circ \gamma_1)(n_1) + F(\delta_2 \circ \gamma_2)(n_2))) \\ &= (l', F(\epsilon \circ \delta')(F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))),\end{aligned}$$

which implies that $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2)) \sim (j, F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))$.

The fact that these two operations define a left R-module structure no M follows easily from their definition and from the fact that the M_i are left R-modules.

The obvious R-module maps $q_i : M_i \to M$ define a cone under F with apex M in the category ${}_R\mathbf{Mod}$. Let $(N, (v_i)_{I \in \mathrm{Ob}(\mathscr{I})})$ be another cone under F in ${}_R\mathbf{Mod}$. In particular, this defines a cone under $\mathrm{For} \circ F$ in $\mathbf{Set}$, where $\mathrm{For} : {}_R\mathbf{Mod} \to \mathbf{Set}$ is the forgetful functor. So there is a unique map $f : M \to N$ such that $f \circ q_i = v_i$ for every $i \in \mathrm{Ob}(\mathscr{I})$. We need to show that f is R-linear. Let $x_1, x_2 \in M$ and $a \in R$. We choose elements (i_1, m_1) and (i_2, m_2) of $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} M_i$ representing x_1 and x_2; as we have seen in the definition of the addition on M, we may assume that $i_1 = i_2$. Then ax_1 is represented by (i_1, am_1), so $f(ax_1) = v_{i_1}(am_1) = av_{i_1}(m_1) = af(x_1)$, and $x_1 + x_2$ is represented by $(i_1, m_1 + m_2)$, so $f(x_1 + x_2) = v_{i_1}(m_1 + m_2) = v_{i_1}(m_1) + v_{i_1}(m_2) = f(x_1) + f(x_2)$.

$\square$

A.2.5 Filtrant colimits are exact

Let R be a ring and $\mathscr{I}$ be a filtrant category. Show that the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is exact, i.e. that if $u : F \to G$ and $v : G \to H$ are morphism of functors from $\mathscr{I}$ to ${}_R\mathbf{Mod}$ such that the sequence $0 \to F(i) \overset{u(i)}{\to} G(i) \overset{v(i)}{\to} H(i) \to 0$ is exact for every $i \in \mathrm{Ob}(\mathscr{I})$, then the sequence $0 \to \varinjlim F \overset{\varinjlim u}{\to} \varinjlim G \overset{\varinjlim v}{\to} \varinjlim H \to 0$ is exact. (Remember that we say that a sequence of R-modules $0 \to M \overset{f}{\to} N \overset{g}{\to} P \to 0$ is exact if $\mathrm{Ker}\, f = 0$, $\mathrm{Ker}\, g = \mathrm{Im}\, f$ and $\mathrm{Im}\, g = P$.)

Solution. First we note that, if $f : M \to N$ is a morphism of ${}_R\mathbf{Mod}$, then $\mathrm{Ker}(f) = \mathrm{Ker}(f, 0)$ is a finite limit in ${}_R\mathbf{Mod}$ and $\mathrm{Coker}(f) = \mathrm{Coker}(f, 0)$ is a (finite colimit). Also, we have $\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$, and so $\mathrm{Im}(f) = N$ if and only if $\mathrm{Coker}(f) = 0$.

By Subsection I.5.4.1 and Corollary I.5.6.5 , we have (with the notation of the problem)

$$\mathrm{Ker}(\varinjlim u) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(u(i)) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} 0 = 0$$

and

$$\mathrm{Coker}(\varinjlim v) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(v(i)) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} 0 = 0.$$

Also,

$$\mathrm{Coker}(\varinjlim u) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Coker}(u(i)),$$

so

$$\begin{aligned}\mathrm{Im}(\varinjlim u) = \mathrm{Ker}(\mathrm{Coker}(\varinjlim u)) &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(\mathrm{Coker}(u(i))) \\ &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Im}(u(i)) \\ &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(v(i)) \\ &= \mathrm{Ker}(\varinjlim v).\end{aligned}$$

□

A.2.6 Objects of finite type and of finite presentation

Let $\mathscr{C}$ a category that admits all filtrant colimits (indexed by small enough categories). An object X of $\mathscr{C}$ is called *of finite type* (resp. *of finite presentation* or *compact*) if, for every filtrant category $\mathscr{I}$ and every functor $F : \mathscr{I} \to \mathscr{C}$, the canonical map

$$\varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, F(i)) \to \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim F)$$

(see the beginning of Subsection I.5.4.2 of the notes) is injective (resp. bijective).

(a). Let R be a ring and M be a left R-module.

(i) If M is free of finite type as a R-module, show that it is of finite presentation as an object of ${}_R\mathbf{Mod}$.

(ii) If M is of finite type (resp. of finite presentation) as a R-module, show that it is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$.

(iii) Let $\mathscr{I}$ the poset of R-submodules of M that are of finite type, ordered by inclusion, and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be the functor sending $N \subset M$ to M/N; if $N \subset N' \subset M$, we send the unique morphism $N \to N'$ in $\mathscr{I}$ to the canonical projection $M/Ncat \to M/N'$. Show that $\varinjlim F = 0$.

(iv) If M is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$, show that it is of finite type (resp. of finite presentation) as an R-module.

(b). Let R be a commutative ring and S be a commutative R-algebra. Show that S is finitely presented as an R-algebra if and only if it is of finite presentation as an object of $R\!-\!\mathbf{CAlg}$.

(c). (i) If X is a finite set with the discrete topology, show that X is of finite presentation as an object of $\mathbf{Top}$.

(ii) Let X be a topological space. Let $\mathscr{I}$ be the poset of finite sets of X ordered by inclusion; wee see $\mathscr{I}$ as a subcategory of $\mathbf{Top}$ (we use the subset topology on each finite $Y \subset X$), and we denote by $F : \mathscr{I} \to \mathbf{Top}$ the inclusion functor. Show that $X = \varinjlim F$ if the topology on X is the indiscrete (= coarse) topology.

(iii) Let X be a topological space. If X is of finite presentation as an object of $\mathbf{Top}$, show that it is finite.

(iv) For $n \in \mathbb{N}$, let $X_n = \mathbb{N}_{\geq n} \times \{0,1\}$, with the topology for which the open subsets are $\varnothing$ and $(\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})$, for $m \geq n$. Define $f_n : X_n \to X_{n+1}$ by $f_n(n,a) = (n+1,a)$ and $f_n(m,a) = (m,a)$ if $m > n$. Show that the X_n are topological spaces and that the maps f_n are continuous.

(v) Show that $\varinjlim_{n \in \mathbb{N}} X_n$ is $\{0,1\}$ with the indiscrete topology. By $\varinjlim_{n \in \mathbb{N}} X_n$, we mean the colimit of the functor $F : \mathbb{N} \to \mathbf{Top}$ such that $F(n) = X_n$ and that, for each non-identity morphism $\alpha : n \to m$ in $\mathbb{N}$, that is, for $n < m$ in $\mathbb{N}$, $F(\alpha) = f_{m-1} \circ f_{m-2} \circ \ldots \circ f_n : X_n \to X_m$.

(vi) Let X be a topological space. If X is of finite presentation as an object of $\mathbf{Top}$, show that X is finite and has the discrete topology.

(d). Let X be a topological space, and let $\mathrm{Open}(X)$ be the set of open subsets of X, ordered by inclusion. Show that X is compact if and only if X is of finite presentation as an object of $\mathrm{Open}(X)$.

Solution.

(a). (i) We can deduce this from the facts that :

- $\mathrm{Hom}_R(R, N) = N$ for every left R-module N (so R is of finite presentation as an object of ${}_R\mathbf{Mod}$);
- $\mathrm{Hom}_R(M_1 \oplus M_2, \cdot) = \mathrm{Hom}_R(M_1, \cdot) \oplus \mathrm{Hom}_R(M_2, \cdot)$ (so the direct sum of two objects of ${}_R\mathbf{Mod}$ of finite type (resp. of finite presentation) is also of finite type (resp. of finite presentation)).

Alternately, here is a very categorical way to answer the question. Let (F, G) be a pair of adjoint functors, with $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. Suppose that all filtrant colimits exist in $\mathscr{C}$ and $\mathscr{D}$ and that G commutes with filtrant colimits. Then we claim that F sends objects of finite presentation in $\mathscr{C}$ to objects of finite presentation in $\mathscr{D}$. Indeed, let $X \in \mathrm{Ob}(\mathscr{C})$. Then, for every functor $\alpha : \mathscr{I} \to \mathscr{D}$, with $\mathscr{I}$ filtrant, we have a commutative diagram :

$$\begin{tikzcd}
\varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{D}}(F(X), \alpha(i)) \arrow[rr, "\sim"] \arrow[d] & & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, G(\alpha(i))) \arrow[d] \\
\mathrm{Hom}_{\mathscr{D}}(F(X), \varinjlim \alpha) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(\varinjlim \alpha)) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim (G \circ \alpha))
\end{tikzcd}$$

If X is of finite presentation, then the rigth vertical morphism is an isomorphism, so the left vertical morphism also is.

We apply this to the pair of adjoint functors (Φ, For), where $\mathrm{For} : {}_R\mathbf{Mod} \to \mathbf{Set}$ is the forgetful functor and $\Phi : \mathbf{Set} \to {}_R\mathbf{Mod}$ sends a set X to the free left R-module on X. The fact that For commutes with filtrant colimits is Corollary I.5.6.3 . So it suffices to prove that finite sets are objects of finite presentation in $\mathbf{Set}$. This follows from the fact that $\mathrm{Hom}_{\mathbf{Set}}(X, \cdot) = (\cdot)^X$ for every set X, and from Proposition I.5.6.4. (It is also easy to see directly.)

(ii) Suppose that M is of finite type. Then we have an exact sequence $0 \to P \to N \to M \to 0$, with N free of finite type. Let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be a functor, with $\mathscr{I}$ filtrant. By problem A.2.5 and the exactness properties of Hom_R, we have a commutative diagram with exact columns :

(*)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(M, F(i)) \arrow[r, "(1)"] \arrow[d] & \mathrm{Hom}_R(M, \varinjlim F) \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(N, F(i)) \arrow[r, "(2)"] \arrow[d] & \mathrm{Hom}_R(N, \varinjlim F) \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(P, F(i)) \arrow[r, "(3)"] & \mathrm{Hom}_R(P, \varinjlim F)
\end{tikzcd}$$

By question (i), the arrow labeled (2) is an isomorphism, so the arrow labeled (1) is injective, which is what we wanted to prove.

Now assume that M is of finite presentation. Then we have an exact sequence $0 \to P \to N \to M \to 0$, with N free of finite type and P of finite type. So, if we write the diagram (*) again, the arrow labeled (2) is an isomorphism by (i), and the arrow labeled (3) is injective by the previous paragraph. This implies that the arrow labeled (1) is an isomorphism,[8] which is what we wanted.

(iii) Note that $\mathscr{I}$ is a filtrant category, because it comes from a directed poset. (If N and N' are two submodules of finite type of M, then they are both contained in $N + N'$, which is also of finite type.) So we can use problem A.2.4 to calculate $\varinjlim F$. Let $x \in \varinjlim F$, and let (N, m) be an element of $\coprod_{N\in\mathrm{Ob}(\mathscr{I})}(M/N)$ representing it (so N is a submodule of M of finite type, and $m \in M/N$). Then there exists a submodule N' of M of finite type such that $N \subset N'$ and that the image of m in M/N' is 0 (just take the submodule N' generated by N and by a preimage of m in M), so $(N, m) \sim (N', 0)$ in $\coprod_{N\in\mathrm{Ob}(\mathscr{I})}(M/N)$, and so $x = 0$. This shows that $\varinjlim F = 0$.

(iv) Suppose that M is of finite type as an object of ${}_R\mathbf{Mod}$. Using the functor $F : \mathscr{I} \to {}_R\mathbf{Mod}$ of (iii), we see that the canonical morphism

$$\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(M, M/N) \to \mathrm{Hom}_R(M, 0) = 0$$

is injective, which means that $\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(M, M/N) = 0$. Consider $\mathrm{id}_M \in \mathrm{Hom}_R(M, M/0)$. Its image in the filtrant colimit $\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_R(M, M/N)$ is 0, so there exists a morphism $0 \to N$ in $\mathscr{I}$ (that is, an object N of $\mathscr{I}$) such that the image of id_M in $\mathrm{Hom}_R(M, M/N)$ is 0. In other words, there exists a submodule N of M of finite type such that $M = N$, which means that M is of finite type.

[8] By the 4 lemma in the category $\mathbf{Ab}$, which I am assuming that you have seen in a previous class. This also follows from an easy diagram chase.

Now suppose that M is of finite presentation as an object of ${}_R\mathbf{Mod}$. By the previous paragraph, M is a R-module of finite type, so there exists an exact sequence $0 \to P \to N \to M \to 0$ with N a free R-module of finite type. We want to show that the R-module P is also of finite type. As in (iii), we consider the category $\mathscr{I}$ associated to the poset of finite type R-submodules of P, and the functor $F.G : \mathscr{I} \to {}_R\mathbf{Mod}$ defined by $F(Q) = P/Q$ and $G(Q) = N/Q$. For every $Q \in \mathrm{Ob}(\mathscr{I})$, we have an exact sequence $0 \to F(Q) \to G(Q) \to N/P \to 0$. Using problem A.2.5 and (iii), we get an exact sequence $0 \to 0 \to \varinjlim G \to N/P \to 0$. In other words, the canonical morphism $\varinjlim_{Q\in\mathrm{Ob}(\mathscr{I})} N/Q \to N/P$ (induced by the projections $N/Q \to N/P$, for $Q \subset P$) is an isomorphism. Using the isomorphism $N/P \xrightarrow{\sim} M$, we get an isomorphism $f : M \xrightarrow{\sim} \varinjlim_{Q\in\mathrm{Ob}(\mathscr{I})} N/Q$. As M is of finite presentation as an object of ${}_R\mathbf{Mod}$, there exists $Q \in \mathrm{Ob}(\mathscr{I})$ and a morphism $g : M \to N/Q$ such that f is the composition $M \xrightarrow{f} N/Q \to N/P$, where the second map is the canonical projection. This implies that the kernel of the morphism $N \to M$ is contained in Q, hence that $P = Q$ is of finite type.

(b). First we show that polynomial rings over R on finitely many indeterminates are of finite presentation as objects of $R - \mathbf{CAlg}$. For this, we apply the second proof of (a)(i) to the pair of adjoint functors (Φ, For), where $\mathrm{For} : R-\mathbf{CAlg} \to \mathbf{Set}$ is the forgetful functor and $\Phi : \mathbf{Set} \to R - \mathbf{CAlg}$ sends a set X to the free commutative R-algebra on X, that is, the polynomial ring $R[X]$. We already know that finite sets are objects of finite presentation in $\mathbf{Set}$. So it remains to check that $\mathrm{For} : R - \mathbf{CAlg} \to \mathbf{Set}$ commutes with filtrant colimits. The proof is exactly the same as for R-modules : using the procedure of problem A.2.4, we show that, if $F : \mathscr{I} \to R - \mathbf{CAlg}$ is a functor with $\mathscr{I}$ filtrant, then there is a unique R-algebra structure on $\varinjlim(\mathrm{For} \circ F)$ that makes all the canonical morphisms $F(i) \to \varinjlim(\mathrm{For} \circ F)$ into R-algebra morphisms, and that $\varinjlim(\mathrm{For} \circ F)$ with this R-algebra structure satisfies the universal property characterizing the colimit of F. (We already know how to define the addition and the action of R, and we define the multiplication using the same trick as for the addition. See the solution of problem A.2.4.)

Let S be a commutative finitely presented R-algebra. We show that S is of finite presentation as an object of $R - \mathbf{CAlg}$. Choose a surjective R-algebra morphism $f : S_0 := R[x_1, \dots, x_n] \to S$ whose kernel is finitely generated; write $\mathrm{Ker}(f) = (a_1, \dots, a_m)$ with $a_1, \dots, a_m \in S_0$, and let $g : S_1 := R[y_1, \dots, y_m] \to S_0$ be the unique R-algebra morphisms such that $g(y_j) = a_j$ for $1 \le j \le m$. For any commutative R-algebra T, we denote by $e_T : S_1 \to T$ the unique R-algebra morphism sending every y_j to 0. Then, if T is a commutative R-algebra, we have a sequence of maps

$$\mathrm{Hom}_{R-\mathbf{CAlg}}(S, T) \xrightarrow{u_T} \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0, T) \overset{v_T}{\rightrightarrows} \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1, T),$$

where $u_T(h) = h \circ f$ and $v_T(h') = h' \circ g$. As $f : S_0 \to S$ is surjective, the map u_T is injective. As the image of $g : S_1 \to S_0$ generates the ideal $\mathrm{Ker}(f)$, a morphism $h' : S_0 \to T$ factors as $S_0 \xrightarrow{f} S \xrightarrow{h} T$ if and only if it is zero on the image of g; in other words, the image of u_T is exactly the set of $h' \in \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0, T)$ such that $v_T(h') = e_T$.

In other words. we have just proved that the map u_T identifies the set $\mathrm{Hom}_{R-\mathbf{CAlg}}(S,T)$ with the fiber product of the diagram :

$$\begin{tikzcd} & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,T) \arrow[d, "v_T"] \\ \{e_T\} \arrow[r] & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,T) \end{tikzcd}$$

Let $* : R-\mathbf{CAlg} \to \mathbf{Set}$ be the functor sending T to the singleton $\{e_T\}$. The inclusion $\{e_T\} \subset \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,T)$ defines a morphism of functors $e : * \to \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,\cdot)$. Note also that u_T and v_T define morphisms of functors u and v. So u identifies the functor $\mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,\cdot)$ with the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,\cdot) \arrow[d, "v"] \\ * \arrow[r, "e"'] & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,\cdot) \end{tikzcd}$$

(in the category $\mathrm{Func}(R-\mathbf{CAlg},\mathbf{Set})$). As the three functors in this diagram commute with filtrant colimits by the first paragraph, as filtrant colimits commute with finite limits in $\mathbf{Set}$ (Proposition I.5.6.4), the functor $\mathrm{Hom}_{R-\mathbf{CAlg}}(S,\cdot)$ also commutes with filtrant colimits.

It remains to show that a commutative R-algebra S that is of finite presentation as an object of $R-\mathbf{CAlg}$ is a finitely presented R-algebra. First, consider the poset $\mathscr{I}$ of finitely generated sub-R-algebras $S' \subset S$, seen as a category, and the obvious (inclusion) functor from $\mathscr{I}$ to $R-\mathbf{CAlg}$. The category $\mathscr{I}$ is clearly filtrant (because the union of two finitely generated subslalgebras of S is contained in a finitely generated subalgebra), and $\varinjlim F = S$ because we saw in the first paragraph that the forgetful functor $R-\mathbf{CAlg} \to \mathbf{Set}$ commutes with filtrant colimits. So the canonical map $\varinjlim_{S'\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{R-\mathbf{CAlg}}(S,S') \to \mathrm{Hom}_{R-\mathbf{CAlg}}(S,S)$ is bijective, which implies that there exists a finitely generated subalgebra S' of S such that the identity of S factors through the inclusion $S' \subset S$, i.e. such that $S' = S$. So S is a finitely generated R-algebra. We write $S = R[x_1,\dots,x_n]/I$, with I an ideal of $R[x_1,\dots,x_n]$. Let $\mathscr{I}'$ be the poset of finite generated ideals $J \subset I$, seen as category; again, this is a clearly a filtrant category. Define a functor $G : \mathscr{I}' \to R-\mathbf{CAlg}$ by sending J to $R[x_1,\dots,x_n]/J$. For every $J \in \mathrm{Ob}(\mathscr{I}')$, let $u_J : G(J) = R[x_1,\dots,x_n] \to S$ be the quotient morphism. Then $(S,(u_J))$ is a cone under G, and we claim that it is a colimit of G. Indeed, let $(T,(v_J))$ be another cone under G. In particular, all the morphisms $R[x_1,\dots,x_n] \to R[x_1,\dots,x_n]/J \overset{v_J}{\to} T$ are equal, so we get a morphism $f : R[x_1,\dots,x_n] \to T$. Also, $\mathrm{Ker}(f)$ contains every finitely generated subideal of I, so it contains every element of I, so $I \subset \mathrm{Ker}(f)$, so f factors as $R[x_1,\dots,x_n] \to S \overset{g}{\to} T$. The morphism g is clearly a morphism of cones, and it is the only possible morphism of cones from $(S,(u_J))$ to $(T,(v_j))$ because all the maps u_J are surjective. As S is of finite presentation as an object of $R-\mathbf{CAlg}$, the canonical map

$$\varinjlim_{J\in\mathrm{Ob}(\mathscr{I}')} \mathrm{Hom}_{R-\mathbf{CAlg}}(S,R[x_1,\dots,x_n]/J)\to\mathrm{Hom}_{R-\mathbf{CAlg}}(S,S)$$

is bijective. In particular, there exists a finitely generated ideal $J\subset I$ such that the identity morphism of S factors as $S\to R[x_1,\dots,x_n]/J\to S$, where the second map is the quotient map; this forces J and I to be equal, so I is a finitely generated ideal, and so S is a finitely presented R-algebra.

(c). (i) As in (a)(i), we can do this directly or categorically. If we do it directly, we use the fact that a singleton is clearly of finite presentation in **Top**, and that a finite discrete set is a finite coproduct of singletons in **Top**. If we do it categorically, we apply the fact that we proved in (a)(i) to the pair of adjoitn functors (F,For), where $\mathrm{For}:\mathbf{Top}\to\mathbf{Set}$ is the forgetful functor (which preserves all colimits by Section I.5.5.1) and F is its left adjoint, i.e. the functor that sends a set X to itself with the discrete topology (Example I.4.8). Then the result follows from the fact that a finite set is of finite presentation as an object of **Set**, which we proved in (a)(i).

(ii) Let $\mathrm{For}:\mathbf{Top}\to\mathbf{Set}$ be the forgetful functor. It is easy to see that $\mathrm{For}(X)=\varinjlim(\mathrm{For}\circ F)$ (this just says that X is the union of all its finite subsets). We use this to identify X and $\varinjlim F$ as sets. Then X and $\varinjlim F$ are isomorphic as topological spaces if and only if the original topology on X coincides with the colimiit topology. Let U be a subset of X. It is open in the colimit topology if and only $U\cap Y$ is open in Y for every finite subset Y of X (using the subset topology on Y). This is certainly true if X has the coarse topology. [9]

(iii) Let X_0 be the underlying set of X with the coarse topology. Then the identity map $i:X\to X_0$ is continuous. As X is of finite presentation, question (ii) implies that i factors through a finite subset of X, hence that X is finite.

(iv) Let $n\in\mathbb{N}$, and let $(m_i)_{i\in I}$ be a family of integers $\geq n$. Then

$$\bigcup_{i\in I}(\mathbb{N}_{\geq m_i}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{1\})=(\mathbb{N}_{\geq \inf_{i\in I}m_i}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{1\})$$

with $\inf_{i\in I}m_i\geq n$. Also, if I is finite, then

$$\bigcap_{i\in I}(\mathbb{N}_{\geq m_i}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{1\})=(\mathbb{N}_{\geq \sup_{i\in I}m_i}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{1\})$$

So the family of "open sets" of the statement does define a topology on $\mathbb{N}_{\geq n}\times\{0,1\}$.

Let $n\in\mathbb{N}$, and let $m\geq n+1$. Then

$$f_n^{-1}((\mathbb{N}_{\geq m}\times\{0\})\cup(\mathbb{N}_{\geq n+1}\times\{1\}))=\begin{cases}(\mathbb{N}_{\geq m}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{1\}) & \text{if } m\geq n+2\\ (\mathbb{N}_{\geq n}\times\{0\})\cup(\mathbb{N}_{\geq n}\times\{0\}) & \text{if } m=n+1.\end{cases}$$

[9] It is also true if X has the discrete topology...

So f_n is continuous.

(v) We put the coarse topology on $\{0,1\}$. Then the second projections maps $X_n \to \{0,1\}$, hence define a cone under the functor F. So we get a continuous map $f : \varinjlim_{n\in\mathbb{N}} X_n \to \{0,1\}$.

If $a \in \{0,1\}$, then the image of $(0,a) \in X_0$, so its image by the obvious map $X_0 \to \coprod_{n\in\mathbb{N}} X_n \to \varinjlim_{n\in\mathbb{N}} X_n$ is a preimage of a by f. So f is surjective.

We prove that f is injective. Let $(m,a) \in X_n$ and $(m',b) \in X_{n'}$, and suppose that the images of (m,a) and (m',b) by the maps $X_n \to \coprod_{i\in\mathbb{N}} X_i \to \varinjlim_{i\in\mathbb{N}} X_i \xrightarrow{f} \{0,1\}$ and $X_m \to \coprod_{i\in\mathbb{N}} X_i \to \varinjlim_{i\in\mathbb{N}} X_i \xrightarrow{f} \{0,1\}$ are equal. We want to prove that (m,a) and (m',b) have the same image in $\varinjlim_{i\in\mathbb{N}} X_i$. As the f_i do not change the second coordinate of elements of X_i, the assumption implies that $a = b$. If $m > n$, then $f_{m-1} \circ \ldots \circ f_n(m,a) = (m,a) \in X_m$ has the same image as $(m,a) \in X_n$ in $\varinjlim_{i\in\mathbb{N}} X_i$; so we may assume that $n = m$. Similarly, we may assume that $n' = m'$. Up to switching n and n', we may assume that $n' \geq n$. If $n' = n$, we are done. Otherwise, we have $(n',a) = f_{n'-1} \circ \ldots \circ f_n(n,a)$, so $(n',a) \in X_{n'}$ and $(n,a) \in X_n$ have the same image in $\varinjlim_{i\in\mathbb{N}} X_i$.

It remains to prove that f^{-1} is continuous. If it were not, this would mean that $\{0\}$ or $\{1\}$ is open in $\varinjlim_{i\in\mathbb{N}} X_i$. But the preimages of $\{0\}$ and $\{1\}$ by the continuous map $X_n \to \varinjlim_{i\in\mathbb{N}} X_i$ are $\mathbb{N}_{\geq n} \times \{0\}$ and $\mathbb{N}_{\geq n} \times \{1\}$ respectively, and these are not open subsets of X_n. So neither $\{0\}$ nor $\{1\}$ is open in $\varinjlim_{i\in\mathbb{N}} X_i$.

(vi) We already know that X is finite by (iii). Let U be a subset of X, and let $f : X \to \{0,1\}$ be the indicator map of U. Then f is continuous if we put the coarse topology on $\{0,1\}$, so, by the hypothesis on X and question (v), there exists a continuous map $X \xrightarrow{g} X_n$ such that f is the composition of g and of the second projection $X_n \to \{0,1\}$. As X is finite, there exists $m \geq n$ such that, for every $x \in X$, the first coordinate of $g(x) \in \mathbb{N}_{\geq n} \times \{0,1\}$ is $< m$. Then

$$U = g^{-1}(\mathbb{N}_{\geq n} \times \{1\}) = g^{-1}((\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})).$$

As g is continuous, this proves that U is open in X. As U was an arbitrary subset of X, this shows that the topology of X is discrete.

(d). Let $\mathscr{U} = (U_i)_{i\in I}$ be a family of open subsets of X. Let $\mathscr{I}_I$ be the category associated to the poset of finite subsets of I, ordered by inclusions; then $\mathscr{I}_I$ is filtrant. Let $F_{\mathscr{U}} : \mathscr{I}_I \to \mathrm{Open}(X)$ be the functor sending a finite subset $J \subset I$ to $\bigcup_{j\in J} U_j$; if $J \subset J'$, then the image by $F_{\mathscr{U}}$ of the corresponding morphism of $\mathscr{I}_I$ is the inclusion $\bigcup_{j\in J} U_j \subset \bigcup_{j\in J'} U_j$. Then it is easy to see that $\varinjlim F_{\mathscr{U}} = \bigcup_{i\in I} U_i$.

Suppose that X is finite presentation as an object of $\mathrm{Open}(X)$, and let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of X. Then the identity morphism $X \to \varinjlim F_{\mathscr{U}}$ comes from a morphism

$X \to \bigcup_{j \in J} U_j$ with $J \in \mathrm{Ob}(\mathscr{I}_I)$, or, in other words, there exists a finite subset J of I such that $X \subset \bigcup_{j \in J} U_j$. This means that X is compact.

Conversely, suppose that X is compact, and let $F : \mathscr{I} \to \mathrm{Open}(X)$ be a functor, with $\mathscr{I}$ filtrant. Let $U = \varinjlim F$. We claim that $U = \bigcup_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$. For every $i \in \mathrm{Ob}(\mathscr{I})$, we have a morphism $F(i) \to U$ in $\mathrm{Open}(X)$, so $F(i) \subset U$. Conversely, let $U' = \bigcup_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$. Then we have a morphism $F(i) \to U'$ in $\mathrm{Open}(X)$ for every $i \in \mathrm{Ob}(\mathscr{I})$, and this defines a cone under F with apex U', so the universal property of the colimit implies that we have a morphism $U \to U'$ in $\mathrm{Open}(X)$, that is, that $U \subset U'$.

Now we show that the map $\alpha : \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathrm{Open}(X)}(X, F(i)) \to \mathrm{Hom}_{\mathrm{Open}(X)}(X, U)$ is bijective. Note that, as all Hom sets in $\mathrm{Open}(X)$ are empty sets or singletons, and as $\mathscr{I}$ is filtrant, the source of α has at most one element. If $U \neq X$, then $\mathrm{Hom}_{\mathrm{Open}(X)}(X, U) = \varnothing$ and $\mathrm{Hom}_{\mathrm{Open}(X)}(X, F(i)) = \varnothing$ for every $i \in \mathrm{Ob}(\mathscr{I})$, so α is bijective. Suppose that $U = X$; then $\mathrm{Hom}_{\mathrm{Open}(X)}(X, U) = \{\mathrm{id}_X\}$, and we want to show that id_X has a preimage by α. This is equivalent to the fact that $X = F(i)$ for some $i \in \mathrm{Ob}(\mathscr{I})$. As X is compact and as $X = U = \bigcup_{i \in \mathrm{Ob}(\mathscr{I})} F(i)$, we know that there exist $i_1, \ldots, i_n \in \mathrm{Ob}(\mathscr{I})$ such that $X = F(i_1) \cup \ldots \cup F(i_n)$. As $\mathscr{I}$ is filtrant, there exists $j \in \mathrm{Ob}(\mathscr{I})$ and morphisms $i_1 \to j$, $\ldots$, $i_n \to j$. So we have morphisms $F(i_r) \to F(j)$ in $\mathrm{Open}(X)$ for $1 \leq r \leq n$, that is, $F(j)$ contains $F(i_1), \ldots, F(i_n)$; this implies that $F(j) = X$.

$\square$

A.3 Problem set 3

A.3.1 Free preadditive and additive categories.

Remember that $\mathbf{Cat}$ is the category of category (the objects of $\mathbf{Cat}$ are categories, and the morphisms of $\mathbf{Cat}$ are functors). Let $\mathbf{PreAdd}$ be the category whose objects are preadditive categories and whose morphisms are additive functors; let $\mathbf{Add}$ be the full subcategory of $\mathbf{PreAdd}$ whose objects are additive categories. We have a (faithful) forgetful functor $\mathrm{For} : \mathbf{PreAdd} \to \mathbf{Cat}$; we also denote the inclusion functor from $\mathbf{Add}$ to $\mathbf{PreAdd}$ by F.

(a). Show that For has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathbb{Z}[\mathscr{C}]$.

(b). Show that F has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathscr{C}^{\oplus}$. (Hint : If $\mathscr{C}$ is preadditive, consider the category $\mathscr{C}^{\oplus}$ whose objects are 0 and finite sequences $(X_1, \ldots, X_n)$ of objects of $\mathscr{C}$, where a morphism from $(X_1, \ldots, X_n)$ to $(Y_1, \ldots, Y_m)$ is a $m \times n$ matrix of morphisms $X_i \to Y_j$, and where the only from 0 to any object and from any object to 0 is 0.)

Solution. One subtlety is that the categories $\mathbf{Cat}$, $\mathbf{PreAdd}$ and $\mathbf{Add}$ are actually 2-categories, so the Homs in these categories are themselves categories, and it is not reasonable in general to expect the adjunction isomorphism to be an isomorphism of categories; it it much more natural to require it to be an equivalence that is natural in both its entries in the appropriate sense. There are several natural ways to make this precise, and it can quickly become extremely painful. See for example `https://ncatlab.org/nlab/show/2-adjunction` for a discussion and further references.

(a). Let $\mathscr{C}$ be a category. We defined the category $\mathbb{Z}[\mathscr{C}]$ in the following way :

- $\mathrm{Ob}(\mathbb{Z}[\mathscr{C}]) = \mathrm{Ob}(\mathscr{C})$;
- for all $X, Y \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}_{\mathbb{Z}[\mathscr{C}]}(X, Y) = \mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(X,Y))}$;
- the composition law of $\mathbb{Z}[\mathscr{C}]$ is deduced from that of $\mathscr{C}$ by bilinearity.

Note that $\mathscr{C}$ is naturally a subcategory of $\mathbb{Z}[\mathscr{C}]$.

This construction is functorial in $\mathscr{C}$, that is, any functor $F : \mathscr{C} \to \mathscr{D}$ defines in an obvious way a functor $\mathbb{Z}[F] : \mathbb{Z}[\mathscr{C}] \to \mathbb{Z}[\mathscr{D}]$, and we have $\mathbb{Z}[G \circ F] = \mathbb{Z}[G] \circ \mathbb{Z}[F]$. (In other words, $\mathscr{C} \longmapsto \mathbb{Z}[\mathscr{C}]$ is a strict 2-functor, see `https://ncatlab.org/nlab/show/strict+2-functor`). Let $\mathscr{C}$ be a category, $\mathscr{D}$ be a preadditive category and F be a functor. Then there is an obvious additive functor $\alpha(\mathscr{C}, \mathscr{D})(F) : \mathbb{Z}[\mathscr{C}] \to \mathscr{D}$; it is equal to F on the objects of $\mathbb{Z}[\mathscr{C}]$ and equal to the unique extension of F by linearity on the groups of morphisms. Also, any morphism $u : F \to G$ of functors $\mathscr{C} \to \mathscr{D}$ gives rise to a morphism of additive functors $\alpha(u) : \alpha(\mathscr{C}, \mathscr{D})(F) \to \alpha(\mathscr{C}, \mathscr{D})(G)$. This defines a functor $\alpha(\mathscr{C}, \mathscr{D}) : \mathrm{Func}(\mathscr{C}, \mathscr{D}) \to \mathrm{Func}_{\mathrm{add}}(\mathbb{Z}[\mathscr{C}], \mathscr{D})$, that is natural in $\mathscr{C}$ in $\mathscr{D}$. In this case, the functor $\alpha(\mathscr{C}, \mathscr{D})$ is actually an isomorphism of categories. Indeed, if $G : \mathbb{Z}[\mathscr{C}] \to \mathscr{D}$ is an functor, then its restriction F to the subcategory $\mathscr{C}$ of $\mathbb{Z}[\mathscr{C}]$ is a functor $\mathscr{C} \to \mathscr{D}$ such that $G = \alpha(\mathscr{C}, \mathscr{D})(F)$.

(b). Let $\mathscr{C}$ be a preadditive category. We show that the preadditive category $\mathscr{C}^{\oplus}$ defined in the problem is additive. It has a zero object by construction, so it suffices to show that the product of two objects always exists. Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$ be two objects of $\mathscr{C}^{\oplus}$. Let $Z = (X_1, \dots, X_n, Y_1, \dots, Y_m)$ and $p : Z \to X$, $q : Z \to Y$ be the morphisms given by the matrices $\begin{pmatrix} I_n & 0_{n,m} \end{pmatrix}$ and $\begin{pmatrix} 0_{m,n} & I_m \end{pmatrix}$, where $I_n = \begin{pmatrix} \mathrm{id}_{X_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{id}_{X_n} \end{pmatrix}$, $I_m = \begin{pmatrix} \mathrm{id}_{Y_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{id}_{Y_m} \end{pmatrix}$ and $0_{n,m}$ (resp. $0_{m,n}$) is a $n \times m$ (resp. $m \times n$) matrix with all its entries equal to 0. We claim that this makes Z into the product of X and Y. Indeed, we have morphisms $i : X \to Z$ and $j : Y \to Z$ with matrices $\begin{pmatrix} I_n \\ 0_{m,n} \end{pmatrix}$ and $\begin{pmatrix} 0_{n,m} \\ I_m \end{pmatrix}$ respectively, and it is easy to check the conditions of Proposition II.1.1.6(iii) .

Note that we have an obvious inclusion $\mathscr{C} \subset \mathscr{C}^{\oplus}$, which is fully faithful. If $\mathscr{C}$ is additive, then $(X_1, \dots, X_n)$ is isomorphic to $X_1 \oplus \ldots \oplus X_n$ for all $X_1, \dots, X_n \in \mathrm{Ob}(\mathscr{C})$ (and the zero object of $\mathscr{C}^{\oplus}$ is isomorphic to the zero object of $\mathscr{C}$), so the inclusion $\mathscr{C} \subset \mathscr{C}^{\oplus}$ is essentially surjective in this case, hence an equivalence of categories.

If $\mathscr{C}$ is a preadditive category, $\mathscr{D}$ is an additive category and $F : \mathscr{C} \to \mathscr{D}$ is an additive functor, then we can extend F to an additive functor $\alpha(\mathscr{C}, \mathscr{D})(F) : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. However, this requires the choice of a particular direct sum for every finite family of objects of $\mathscr{D}$, so this construction is not unique (just unique up to unique isomorphism); in particular, if F is obtained by restriction from an additive functor $G : \mathscr{C}^{\oplus} \to \mathscr{D}$, we can only say that $\alpha(\mathscr{C}, \mathscr{D})(F)$ and G are isomorphic (and the isomorphism between them is unique). So we still get a functor $\alpha(\mathscr{C}, \mathscr{D}) : \mathrm{Func}_{\mathrm{add}}(\mathscr{C}, \mathscr{D}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\oplus}, \mathscr{D})$ (natural in $\mathscr{C}$ and $\mathscr{D}$), but it is an equivalence of categories, not an isomorphism.

$\square$

A.3.2 Pseudo-abelian completion.

Let $\mathscr{C}$ be an additive category. If X is an object of $\mathscr{C}$, an endomorphism $p \in \mathrm{End}_{\mathscr{C}}(X)$ is called a *projector* or *idempotent* if $p \circ p = p$. A *pseudo-abelian* (or *Karoubian*) category is a preadditive category in which every projector has a kernel.

(a). Let $\mathscr{C}$ be a category and $p \in \mathrm{End}_{\mathscr{C}}(X)$ be a projector. Show that :

- $\mathrm{Ker}(p, \mathrm{id}_X)$ exists if and only if $\mathrm{Coker}(p, \mathrm{id}_X)$ exists;
- if $u : Y \to X$ is a kernel of (p, id_X) and $v : X \to Z$ is a cokernel of (p, id_X), then there exists a unique morphism $f : Z \to Y$ such that $u \circ f \circ v = p$, and this morphism f is an isomorphism.

(b). If $\mathscr{C}$ is a pseudo-abelian category, show that every projector has a kernel, a cokernel, a coimage and an image and that, if $p \in \mathrm{End}_{\mathscr{C}}(X)$ is a projector, then the canonical morphisms $\mathrm{Ker}(p) \to X$ and $\mathrm{Im}(p) \to X$ make X into a coproduct of $(\mathrm{Ker}(p), \mathrm{Im}(p))$. (In other words, the coproduct $\mathrm{Ker}(p) \oplus \mathrm{Im}(p)$ exists, and it is canonically isomorphic to X.)

(c). Let $\mathscr{C}$ be a category. Its *pseudo-abelian completion* (or *Karoubi envelope*) is the category $\mathrm{kar}(\mathscr{C})$ defined by :

- $\mathrm{Ob}(\mathrm{kar}(\mathscr{C})) = \{(X, p) \mid X \in \mathrm{Ob}(\mathscr{C}),\ p \in \mathrm{End}_{\mathscr{C}}(X) \text{ is a projector}\}$;
- $\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Y, q)) = \{f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \mid q \circ f = f \circ p = f\}$;
- the composition is given by that of $\mathscr{C}$, and the identity morphism of (X, p) is p.

Show that $\mathrm{kar}(\mathscr{C})$ is a pseudo-abelian category, and that the functor $\mathscr{C} \to \mathrm{kar}(\mathscr{C})$ sending X to (X, id_X) is additive and fully faithful.

(d). If $\mathscr{C}$ is an additive category, show that $\mathrm{kar}(\mathscr{C})$ is also additive.

(e). Let $\mathbf{PseuAb}$ be the full subcategory of $\mathbf{PreAdd}$ (see problem A.3.1) whose objects are pseudo-abelian categories. Show that the inclusion functor $\mathbf{PseuAb} \to \mathbf{PreAdd}$ has a left adjoint.

Solution.

(a). We prove the first statement. Let $u : Y \to X$ be a kernel of (p, id_X). Consider the morphism $p : X \to X$. As $p \circ p = p = p \circ \mathrm{id}_X$, there exists a unique morphism $v : X \to Y$ such that $p = u \circ v$. We claim that $v : X \to Y$ is the cokernel of (p, id_X). First, note that $u \circ v \circ p = p \circ p = p = u \circ v \circ \mathrm{id}_X$; as u is a monomorphism by Lemma II.1.3.3, we get that $v \circ p = v \circ \mathrm{id}_X$. Also, we have $u \circ v \circ u = p \circ u = u$, so $v \circ u = \mathrm{id}_Y$, again because u is a monomorphism. Let $v' : X \to Y'$ be a morphism such that $v' \circ p = v'$. Let $w = v' \circ u : Y \to Y'$. Then $w \circ v = v' \circ u \circ v = v' \circ p = v'$. Let $w' : Y \to Y'$ be another morphism such that $w' \circ v = v'$; then $w' = w' \circ v \circ u = v' \circ u = w$. This shows that $v : X \to Y$ is indeed a cokernel of (p, id_X).

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, dashed, "w"'] & X \arrow[r, "p"] \arrow[dr, "\mathrm{id}_X"] & X \\
Y' & X \arrow[ul, "v"] \arrow[u, "p"'] \arrow[l, "v'"] & X
\end{tikzcd}$$

Conversely, if $v : X \to Y$ is a cokernel of (p, id_X), then applying the previous paragraph to $\mathscr{C}^{\mathrm{op}}$ shows that (p, id_X) has a kernel.

We prove the second statement. Let $u : Y \to X$ be a kernel of (p, id_X) and $v : X \to Z$ be a cokernel of (p, id_X). By the first paragraph of the proof, there exists a morphism $v' : X \to Y$ such that v' is a cokernel of (p, id_X), $v' \circ u = \mathrm{id}_Y$ and $u \circ v' = p$. By the uniqueness of the cokernel, there exists a unique morphism $f : Z \to Y$ such that $f \circ v = v'$, and this morphism is an isomorphism. Finally, as u is a monomorphism, the condition $f \circ v = v'$ is equivalent to $u \circ f \circ v = u \circ v' = p$.

(b). Let $p \in \mathrm{End}_{\mathscr{C}}(X)$ be a projector. Then $q = \mathrm{id}_X - p$ is also a projector, because $q \circ q = \mathrm{id}_X - p - p + p \circ p = q$. As $\mathrm{Ker}(p) = \mathrm{Ker}(q, \mathrm{id}_X)$ and $\mathrm{Coker}(p) = \mathrm{Coker}(q, \mathrm{id}_X)$, question (a) implies that p has a cokernel, and that we may assume that $\mathrm{Ker}(p) = \mathrm{Coker}(p)$ (as objects of $\mathscr{C}$). Let $Y = \mathrm{Ker}(p)$, and let $u : Y \to X$ and $v : X \to Y$ be the kernel and cokernel morphisms. We saw in the solution of (a) that $u \circ v = q = \mathrm{id}_X - p$ and $v \circ u = \mathrm{id}_Y$. Similarly, let $Z = \mathrm{Ker}(q)$, and let $a : Z \to X$ and $b : X \to Z$ be the kernel and the cokernel morphisms; we have $b \circ a = \mathrm{id}_Z$ and $a \circ b = p$. We claim that $a : Z \to X$ is the kernel of $v : X \to Y$, that is, the image of p. As a is the kernel of q, we have $q \circ a = 0$, that is, $p \circ a = a$; so $v \circ a = v \circ p \circ a = 0$. Let $a' : Z' \to X$ be a morphism such that $v \circ a' = 0$. Then $q \circ a' = u \circ v \circ a' = 0$, so there exists a unique morphism $c : Z' \to Z$ such that $a \circ c = a'$. This finishes the proof that $a : Z \to X$ is the image of p. A similar proof (actually, the same proof in $\mathscr{C}^{\mathrm{op}}$) shows that $q : X \to Z$ is the coimage of p. Note in particular that the canonical morphism $\mathrm{Coim}(p) \to \mathrm{Im}(p)$ is an isomorphism.

It remains to show that X is the coproduct of $(u : Y \to X, a : Z \to X)$. Let $u' : Y \to X'$ and $a' : Z \to X'$ be morphisms. We must show that there exists a unique morphism $f : X \to X'$ such that $f \circ u = u'$ and $f \circ a = a'$. Take $f = u' \circ v + a' \circ b$. Then $f \circ u = u' \circ v \circ u + a \circ b \circ v = u' \circ \mathrm{id}_Y = u'$ (the fact that $b \circ v = 0$ follows from the previous paragraph applied to q, which shows that $v : Y \to X$ is the kernel of b); similarly, $f \circ a = u' \circ v \circ a + a' \circ b \circ a = a' \circ \mathrm{id}_Z = a'$. Let $f' : X \to X'$ be another morphism such that $f' \circ u = u'$ and $f' \circ a = a'$. Then

$$f' = f' \circ (p + q) = f' \circ (a \circ b + u \circ v) = a' \circ b + u' \circ v = .$$

(c). We first show that $\mathrm{kar}(\mathscr{C})$ is a pseudo-abelian category. First, $\mathrm{kar}(\mathscr{C})$ is clearly a preadditive category, because $\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X,p),(Y,q))$ is a subgroup of $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all $(X,p),(Y,q) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{C}))$.

Let (X,p) be an object of $\mathrm{kar}(\mathscr{C})$, and let $f \in \mathrm{End}_{\mathrm{kar}(\mathscr{C})}((X,p))$ be a projector. We need to show that f has a kernel. By definition of the morphisms and composition in $\mathrm{kar}(\mathscr{C})$, f is an endomorphism of X in $\mathscr{C}$ such that $p \circ f = f \circ p = f$, and such that $f \circ f = f$. Let $g = p - f = \mathrm{id}_{(X,p)} - f \in \mathrm{End}_{\mathrm{kar}(\mathscr{C})}((X,p))$. Then $g \in \mathrm{End}_{\mathscr{C}}(X)$ and $g \circ g = p \circ p - p \circ f - f \circ p + f \circ f = p - f = g$, so (X,g) is an object of $\mathrm{kar}(\mathscr{C})$, and $g \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X,g),(X,p))$. We claim that $g : (X,g) \to (X,p)$ is the kernel of f. First, we have $g \circ f = p \circ f - f \circ f = 0$. Let (Y,q) be another object of $\mathrm{kar}(\mathscr{C})$, and let $u : (Y,q) \to (X,p)$ be a morphism such that $f \circ u = 0$. So $u \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$ and $u \circ q = p \circ u = u$. Then have $g \circ u = p \circ u - f \circ u = p \circ u = q \circ u = u$, so u also define a morphism from (Y,q) to (X,g) in $\mathrm{kar}(\mathscr{C})$, and the following diagram commutes:

$$\begin{tikzcd}
(X,g) \arrow[r, "g"] & (X,p) \arrow[r, "f"] & (X,p) \\
& (Y,q) \arrow[ul, "u"] \arrow[u, "u"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

Suppose that $v : (Y,q) \to (X,g)$ is another morphism (in $\mathrm{kar}(\mathscr{C})$) such that $g \circ v = u$. Then $v \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$ and $g \circ v = v \circ q = v$, so we get $v = u$.

The last statement is clear.

(d). The object $(0, \mathrm{id}_0)$ of $\mathrm{kar}(\mathscr{C})$ is initial and final, so it suffices to show that the product of two objects of $\mathrm{kar}(\mathscr{C})$ always exists. (We'll get finite products by an obvious induction.) Let (X,p) and (Y,q) be two objects of $\mathrm{kar}(\mathscr{C})$, let $Z = X \oplus Y$ and $r = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathrm{End}_{\mathscr{C}}(Z)$. Then r is clearly a projector, so (Z,r) is an object of $\mathrm{kar}(\mathscr{C})$. For every object (T,s) of $\mathrm{kar}(\mathscr{C})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(Z,r)) &= \{f \in \mathrm{Hom}_{\mathscr{C}}(T,Z) \mid r \circ f = f \circ s = f\} \\
&= \{(f_1,f_2) \in \mathrm{Hom}_{\mathscr{C}}(T,X) \times \mathrm{Hom}_{\mathscr{C}}(T,Y) \mid p \circ f_1 = f_1 \circ s = f_1 \\
&\qquad \text{and } q \circ f_2 = f_2 \circ s = f_2\} \\
&\simeq \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(X,p)) \times \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((T,s),(Y,q)),
\end{aligned}$$

so (Z, r) is the product of (X, p) and (Y, q) in $\mathrm{kar}(\mathscr{C})$.

(e). We denote by $\Phi : \mathbf{PseuAb} \to \mathbf{PreAdd}$ the inclusion functor. Note that, if $\mathscr{C}$ is a preadditive category, then the construction of $\mathrm{kar}(\mathscr{C})$ is functorial in $\mathscr{C}$ and the functor $\eta(\mathscr{C}) : \mathscr{C} \to \mathrm{kar}(\mathscr{C})$ defined in (c) gives a morphism of functors $\mathrm{id}_{\mathbf{PseuAb}} \to \Phi \circ \mathrm{kar}$. Indeed, if $F : \mathscr{C} \to \mathscr{D}$ is an additive functor between preadditive categories, then we get a commutative diagram of functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\eta(\mathscr{C})"] \arrow[d, "F"'] & \mathrm{kar}(\mathscr{C}) \arrow[d, "\mathrm{kar}(F)"] \\ \mathscr{D} \arrow[r, "\eta(\mathscr{D})"'] & \mathrm{kar}(\mathscr{D}) \end{tikzcd}$$

by taking $\mathrm{kar}(F)$ to be the functor sending $(X, p) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{C}))$ to $(F(X), F(p)) \in \mathrm{Ob}(\mathrm{kar}(\mathscr{D}))$ and sending $f \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Y, q))$ to $F(f)$ (we obviously have $F(f) \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{D})}((F(X), F(p)), (F(Y), F(q))))$.

We claim that (kar, Φ) is a pair of adjoint functors. We already constructed a candidate unit morphism $\eta : \mathrm{id}_{\mathbf{PseuAb}} \to \Phi \circ \mathrm{kar}$. Let $\mathscr{C}$ be a pseudo-abelian category. We claim that $\eta(\mathscr{C}) : \mathscr{C} \to \mathrm{kar}(\mathscr{C})$ is an equivalence of categories. We already know that it is fully faithful, so it suffices to show that it is essentially surjective. Let (X, p) be an object of $\mathrm{kar}(\mathscr{C})$, and let $a : Z \to X$ be the image of X in $\mathscr{C}$. We claim that $a \in \mathrm{Hom}_{\mathrm{kar}\mathscr{C}}((Z, \mathrm{id}_Z), (X, p))$, and that it is an isomorphism. The first statement just says that $p \circ a = a$, and we proved it in (b). For the second statement, remember that we also know by (b) that Z is the cokernel of $\mathrm{id}_X - p$, and let $b : X \to Z$ be the corresponding cokernel morphism. Then $b \circ p = b$ by (a), so $b \in \mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Z, \mathrm{id}_Z))$, and we have seen in (a) that $b \circ a = \mathrm{id}_Z$ and $a \circ b = p = \mathrm{id}_{(X,p)}$.

Now let $\mathscr{C}$ be a preadditive category and $\mathscr{D}$ be a pseudo-abelian category. Then we have functors, clearly functorial in $\mathscr{C}$ and $\mathscr{D}$:

$$\mathrm{Func}(\mathscr{C}, \mathscr{D}) \xrightarrow{\mathrm{kar}} \mathrm{Func}(\mathrm{kar}(\mathscr{C}), \mathrm{kar}(\mathscr{D})) \xleftarrow{\eta(\mathscr{D}) \circ (\cdot)} \mathrm{Func}(\mathrm{kar}(\mathscr{C}), \mathscr{D}).$$

Also, the functor on the right is an equivalence of categories. To construct a quasi-inverse of this equivalence, we need a quasi-inverse of $\eta(\mathscr{D})$ that is functorial in $\mathscr{D}$. It would be painful to show by hand that we can choose a quasi-inverse of $\eta(\mathscr{D})$ in a way that is (weakly) natural in $\mathscr{D}$, unless we have the good idea of using a left (or right) adjoint of $\eta(\mathscr{D})$ as quasi-inverse, and then things are slightly less annoying. Still, the functors Φ and kar are only adjoint in the sense of 2-categories. See the discussion in the solution of problem A.3.1.

$\square$

A.3.3 Torsionfree abelian groups

Let $\mathbf{Ab}_{\mathrm{tf}}$ be the full subcategory of $\mathbf{Ab}$ whose objects are torsionfree abelian groups.

(a). Give formulas for kernels, cokernels, images and coimages in $\mathbf{Ab}_{\mathrm{tf}}$.

(b). Show that the inclusion functor $\iota : \mathbf{Ab}_{\mathrm{tf}} \to \mathbf{Ab}$ admits a left adjoint κ, and give this left adjoint.

Solution.

(a). Let $f : A \to B$ be a morphism of groups, with A and B torsionfree. Let $C = \{a \in A \mid f(a) = 0\}$ and $D = \{b \in B \mid \exists n \in \mathbb{Z} - \{0\}$ and $a \in A$ such that $f(a) = nb\}$. (This subgroup D is called the *saturation* of $f(A)$ in B.)

We claim that $i : C \to A$ is the kernel of $\mathbf{Ab}_{\mathrm{tf}}$. Indeed, C is torsionfree because it is a subgroup of A, and, as $\mathbf{Ab}_{\mathrm{tf}}$ is a full subcategory of $\mathbf{Ab}$, we have, for every torsionfree abelian group G,

$$\mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(G, C) = \{u \in \mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(G, A) \mid f \circ u = 0\}.$$

We show that B/D is torsionfree. Let x be a torsion element of B/D, and let $b \in B$ be a lift of x. Then there exists $n \in \mathbb{Z} - \{0\}$ such that $nb \in D$, and it is obvious on the definition of D that this implies that $b \in D$, hence that $x = 0$. We claim that $B \to B/D$ is the cokernel of f in $\mathbf{Ab}_{\mathrm{tf}}$. Let $p : B \to B/D$ be the canonical projection. As D contains all the $f(a)$, for $a \in A$, we clearly have $p \circ f = 0$. Let $g : B \to G$ be a morphism in $\mathbf{Ab}_{\mathrm{tf}}$ such that $g \circ f = 0$. Let $b \in D$; then there exists $a \in A$ and $n \in \mathbb{Z} - \{0\}$ such that $nb = f(a)$, so $ng(b) = g(f(a)) = 0$, so $g(b) = 0$ because G is torsionfree; this shows that $\mathrm{Ker}\, g \supset D$, so there is a unique morphism $h : B/D \to G$ such that $g = h \circ p$.

To find the image and coimage of f, we use their definitions, as well as the description of kernels and cokernels that we just obtained. The image of f is the kernel of the cokernel of f, so it is equal to D. The coimage of f is the cokernel of the kernel of f, so it is equal to the quotient A/C', where $C' = \{a \in A \mid \exists n \in \mathbb{Z} - \{0\}, na \in C\}$. Note that, in general, $f(A) := \{f(a),\ a \in A\}$ is neither the image nor the coimage of f.

(b). We define a functor $\kappa : \mathbf{Ab} \to \mathbf{Ab}_{\mathrm{tf}}$ by $\kappa(A) = A/A_{\mathrm{tor}}$, where A_{tor} is the torsion subgroup of A. If $f : A \to B$ is a morphism of abelian groups, then $f(A_{\mathrm{tor}}) \subset B_{\mathrm{tor}}$, so f induces a morphism $\kappa(f) : A/A_{\mathrm{tor}} \to B/B_{\mathrm{tor}}$. This clearly defines a functor $\mathbf{Ab}_{\mathrm{tf}} \to \mathbf{Ab}$. Let A be an abelian group and B a torsionfree abelian group. Then every group morphism $A \to B$ factors uniquely through A/A_{tor}, so we get a bijection

$$\mathrm{Hom}_{\mathbf{Ab}}(A, \iota(B)) \simeq \mathrm{Hom}_{\mathbf{Ab}}(A/A_{\mathrm{tor}}, B) = \mathrm{Hom}_{\mathbf{Ab}_{\mathrm{tf}}}(\kappa(A), B),$$

and this bijection is clearly an isomorphism of functors.

□

A.3.4 Filtered R-modules

Let R be a ring, and let $\mathrm{Fil}({}_R\mathbf{Mod})$ be the category of filtered R-modules (M, Fil_*M) (see Example II.2.1.3) such that $M = \bigcup_{n\in\mathbb{Z}} \mathrm{Fil}_n M$. [10]

(a). Give formulas for kernels, cokernels, images and coimages in $\mathrm{Fil}({}_R\mathbf{Mod})$.

(b). Let $\iota : \mathrm{Fil}({}_R\mathbf{Mod}) \to \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ be the functor sending a filtered R-module (M, Fil_*M) to the functor $\mathbb{Z} \to {}_R\mathbf{Mod}$, $n \longmapsto \mathrm{Fil}_n M$. Show that ι is fully faithful.

(c). Show that ι has a left adjoint κ, and give a formula for κ.

(d). Show that every object of the abelian category $\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ is isomorphic to the cokernel of a morphism between objects in the essential image of ι.

Solution.

(a). Let $f : (M, \mathrm{Fil}_*M) \to (N, \mathrm{Fil}_*N)$ be a morphism of filtered R-modules. Let $M' = \{x \in M \mid f(x) = 0\}$, with the filtration Fil_*M' defined by $\mathrm{Fil}_nM' = M' \cap \mathrm{Fil}_nM$. Let $N' = N/f(M)$, with the filtration Fil_*N' defined by $\mathrm{Fil}_nN' = (\mathrm{Fil}_nN + f(M))/f(M)$ (that is, Fil_nN' is the image of Fil_nN by the quotient map $N \to N'$).

The inclusion $u : M' \to M$ is a morphism in $\mathrm{Fil}({}_R\mathbf{Mod})$, by definition of the filtration on M'. We claim that (M', Fil_*M') is the kernel of f in $\mathrm{Fil}({}_R\mathbf{Mod})$. First, we have $f \circ u = 0$. Let $g : (M'', \mathrm{Fil}_*M'') \to (M, \mathrm{Fil}_*M)$ be a morphism such that $f \circ g = 0$. As the functor $\mathrm{Fil}({}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ that forgets the filtration is faithful, there is at most one morphism $h : (M'', \mathrm{Fil}_*M'') \to (M', \mathrm{Fil}_*M')$ such that $g = u \circ h$. Also, as M' is the kernel of f in ${}_R\mathbf{Mod}$, there exists $h : M'' \to M'$ such that $g = u \circ f$, and it suffices to check that this h is compatible with the filtrations. Let $n \in \mathbb{Z}$. Then $g(\mathrm{Fil}_nM'') \subset \mathrm{Fil}_nM$, so $h(\mathrm{Fil}_nM'') = M' \cap g(\mathrm{Fil}_nM'') \subset M' \cap \mathrm{Fil}_nM = \mathrm{Fil}_nM'$.

The quotient map $p : N \to N'$ is a morphism in $\mathrm{Fil}({}_R\mathbf{Mod})$, by definition of Fil_*N'. We claim that (N', Fil_*N') is the cokernel of f. Let $g : (N, \mathrm{Fil}_*N) \to (N'', \mathrm{Fil}_*N'')$ be a morphism such that $g \circ f = 0$. As in the previous paragraph, it suffices to prove that the unique morphism of R-modules $h : N' \to N''$ such that $h \circ p = g$ (given by the fact that N' is the cokernel of f in ${}_R\mathbf{Mod}$) is compatible with the filtrations. Let $n \in \mathbb{Z}$. Then $\mathrm{Fil}_n N' = p(\mathrm{Fil}_nN)$, so $h(\mathrm{Fil}_nN') = g(\mathrm{Fil}_nN) \subset \mathrm{Fil}_nN''$.

We can now calculate the image and coimage of f using our formulas for the kernel and cokernel of a morphism of $\mathrm{Fil}({}_R\mathbf{Mod})$. The image of f is the kernel of $p : (N, \mathrm{Fil}_*N) \to (N', \mathrm{Fil}_*N')$, so it is the submodule $f(M)$ of N, with the fil-

[10] We say that the filtration is *exhaustive*.

tration given by $\mathrm{Fil}_n f(M) = f(M) \cap \mathrm{Fil}_n N$. The coimage is the cokernel of $u : (M', \mathrm{Fil}_* M') \to (M, \mathrm{Fil}_* M)$, so it is the R-module $M/M' \simeq f(M)$, with the filtration image of that of M by the quotient map $M \to M/M'$. Note that, even though the image and coimage of f have the same underlying R-module, their filtrations are different in general.

(b). If $(M, \mathrm{Fil}_* M)$ is a filtered R-module and $F = \iota(M, \mathrm{Fil}_* M)$ is the associated functor $\mathbb{Z} \to {}_R\mathbf{Mod}$, then $\varinjlim_{\mathbb{Z}} F = \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n M = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n M$; if $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$, this is equal to M.

Let $f, g : (M, \mathrm{Fil}_* M) \to (N, \mathrm{Fil}_* N)$ be two morphisms in $\mathrm{Fil}({}_R\mathbf{Mod})$ such that $\iota(f) = \iota(g)$. Then $\varinjlim \iota(f) = \varinjlim \iota(g)$ as morphisms from $M = \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n M \to \varinjlim_{n \in \mathbb{Z}} \mathrm{Fil}_n N = N$; as the first of these morphisms is equal to f (because its restriction to each $\mathrm{Fil}_n M$ is equal to f) and the second is equal to g (same reason), we get that $f = g$. So the functor ι is faithful.

Let $(M, \mathrm{Fil}_* M)$, $(N, \mathrm{Fil}_* N)$ be objects of $\mathrm{Fil}({}_R\mathbf{Mod})$, and let $\alpha : \iota(M, \mathrm{Fil}_* M) \to \iota(N, \mathrm{Fil}_* N)$ be a morphism of functors. Then $f = \varinjlim \alpha$ is a morphism of R-modules from M to N. We claim that f is actually a morphism of filtered R-modules. Indeed, let $n \in \mathbb{Z}$. Then we have a commutative diagram

$$\begin{tikzcd}
\iota(M, \mathrm{Fil}_* M)(n) \arrow[r, equal] & \mathrm{Fil}_n M \arrow[r, "\alpha(n)"] \arrow[d] & \mathrm{Fil}_n N \arrow[r, equal] \arrow[d] & \iota(N, \mathrm{Fil}_* N)(n) \\
& M \arrow[r, "f"'] & N &
\end{tikzcd}$$

which shows that $f(\mathrm{Fil}_n M) \subset \mathrm{Fil}_n N$, and also that $\iota(f) = \alpha$. So the functor ι is full.

We can also calculate the essential image of ι. We claim that it is the subcategory of functors $F : \mathbb{Z} \to {}_R\mathbf{Mod}$ such that $F(u)$ is injective for every morphism u of $\mathbb{Z}$. First, the functors $\iota(M, \mathrm{Fil}_* M)$ clearly satisfy this conditions, because the mrophisms $F(u)$ are the inclusion $\mathrm{Fil}_n M \subset \mathrm{Fil}_m M$ for $n \leq m$. Conversely, suppose that F satisfies the condition above. Let $M = \varinjlim F$. For every $n \in \mathbb{Z}$, the morphism $F(n)$ is the colimit of the injections $F(n) \to F(m)$, $m \geq n$, so it is injective because filtrant colimits are exact in ${}_R\mathbf{Mod}$; let $\mathrm{Fil}_n M \subset M$ be its image. We have $M = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n M$ because $M = \varinjlim_{n \in \mathbb{Z}} F(n)$, and the isomorphisms $F(n) \overset{\sim}{\to} \mathrm{Fil}_n M$ induce an isomorphism of functors $F \overset{\sim}{\to} \iota(M, \mathrm{Fil}_* M)$.

(c). We define $\kappa : \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod}) \to \mathrm{Fil}({}_R\mathbf{Mod})$ in the following way : If $F : \mathbb{Z} \to {}_R\mathbf{Mod}$ is a functor, we set $\kappa(F) = (M, \mathrm{Fil}_* M)$, where $M = \varinjlim F$ and $\mathrm{Fil}_n M$ is the image of the canonical morphism $F(n) \to M$. If $\alpha : F \to G$ is a morphism of functors, we get a morphism of R-modules $f : \varinjlim \alpha : M = \varinjlim F \to N = \varinjlim G$, and this is a morphism of filtered R-modules because we have commutative squares

$$\begin{array}{ccc} \mathrm{Fil}(n) & \xrightarrow{\alpha(n)} & G(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array}$$

We claim that κ is left adjoint to ι. First we construct a morphism of functors $\eta : \mathrm{id}_{\mathrm{Func}(\mathbb{Z},{}_R\mathbf{Mod})} \to \iota \circ \kappa$. Let $F : \mathbb{Z} \to {}_R\mathbf{Mod}$ be a functor, let $(M, \mathrm{Fil}_* M) = \kappa(F)$, and let $G = \iota(\kappa(F))$. By construction of $\kappa(F)$, we have a morphism of R-modules $F(n) \to \mathrm{Fil}_n M = G(n)$ for every $n \in \mathbb{Z}$, and these morphisms define a morphism of functors $\eta(F) : F \to G$. The fact that the morphisms $\eta(F)$ define a morphism of functors is immediate. Also note that, if $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$ and $F = \iota(M, \mathrm{Fil}_* M)$, then $M = \varinjlim F$ and $F(n) = \mathrm{Fil}_n M$ for every $n \in \mathbb{Z}$, so $\kappa(F) = (M, \mathrm{Fil}_* M)$. This gives an isomorphism of functors $\varepsilon : \kappa \circ \iota \xrightarrow{\sim} \mathrm{id}_{\mathrm{Fil}({}_R\mathbf{Mod})}$, and it is easy to see that $\iota(\varepsilon)$ is the inverse of $\eta(\iota) : \iota \to \iota \circ \kappa \circ \iota$. We want to apply Proposition I.4.6 to show that (κ, ι) is a pair of adjoint functors. It remains to prove that the composition

$$\kappa \xrightarrow{\kappa(\eta)} \kappa \circ \iota \circ \kappa \xrightarrow{\varepsilon(\kappa)} \kappa$$

is the identity, but this also follows immediately from the definitions.

(d). Let $F \in \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$. Suppose that we have found $(M, \mathrm{Fil}_* M) \in \mathrm{Ob}(\mathrm{Fil}({}_R\mathbf{Mod}))$ and $\alpha : \iota(M, \mathrm{Fil}_* M) \to F$ such that $\alpha(n) : \mathrm{Fil}_n M \to F(n)$ is surjective for every $n \in \mathbb{Z}$. We define $(N, \mathrm{Fil}_* N)$ by $N = \mathrm{Ker}(M \to \varinjlim F)$ and $\mathrm{Fil}_n N = \mathrm{Ker}(\mathrm{Fil}_n M \to F(n))$. This is clearly a filtered R-module. If $x \in N$, there exists $n \in \mathbb{Z}$ such that $x \in \mathrm{Fil}_n M$; as the image of x in $\varinjlim F$ is 0, there exists $m \geq n$ such that the image of x in $F(m)$ is 0, and then $x \in \mathrm{Fil}_m N$. So $N = \bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n N$, and it si clear from the way cokernels are calculated in $\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ that F is the cokernel of the morphism $\iota(N, \mathrm{Fil}_* N) \to \iota(M, \mathrm{Fil}_* M)$.

So, to answer the question, it suffices to find $(M, \mathrm{Fil}_* M)$ satufying the conditions of the previous paragraph. Let $M = \bigoplus_{n \in \mathbb{Z}} F(n)$ and, for every $m \in \mathbb{Z}$, let $\mathrm{Fil}_m M = \bigoplus_{n \leq m} F(n)$. Then $(M, \mathrm{Fil}_* M)$ is an object of $\mathrm{Fil}({}_R\mathbf{Mod})$. Let $\alpha : \iota(M, \mathrm{Fil}_* M) \to F$ be the morphism of functors such that $\alpha(m) : \bigoplus_{n \leq m} F(n) \to F(m)$ is given on the factor $F(n)$ by $F(u_{nm}) : F(n) \to F(m)$, where u_{nm} is the unique morphism from n to m in $\mathbb{Z}$; this clearly defines a morphism of functors.

$\square$

A.3.5 Admissible "topology" on $\mathbb{Q}$

If you have not seen sheaves (on a topological space) in a while, you might want to go read about them a bit, otherwise (b) will be very hard, and (f) won't be as shocking as it should be. Also, if the construction of sheafification that you learned used stalks, you should go and read a construction that uses open covers instead; see for example Section III.1 .

If $a, b \in \mathbb{R}$, we write

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

and

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\}.$$

Consider the space $\mathbb{Q}$ with its usual topology. An *open rational interval* is an open subset of $\mathbb{Q}$ of the form $\mathbb{Q}\cap\,]a, b[$ with $a, b \in \mathbb{Q}$. An *closed rational interval* is a closed subset of $\mathbb{Q}$ of the form $\mathbb{Q} \cap [a, b]$, with $a, b \in \mathbb{Q}$.

We say that an open subset U of $\mathbb{Q}$ is *admissible* if we can write U as a union $\bigcup_{i\in I} A_i$ of open rational intervals such that, for every closed rational interval $B = \mathbb{Q} \cap [a, b] \subset U$, there exists a finite subset J of I and closed rational intervals $B_j \subset A_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$.

If U is an admissible open subset of $\mathbb{Q}$ and $U = \bigcup_{i\in I} U_i$ is an open cover of U, we say that this cover is *admissible* if, for every closed rational interval $B = \mathbb{Q} \cap [a, b] \subset U$, there exist a finite subset J of I and closed rational intervals $B_j \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$.

Let Open_a be the poset of admissible open subsets of $\mathbb{Q}$ (ordered by inclusion), and let $\mathrm{PSh}_a = \mathrm{Func}(\mathrm{Open}_a, \mathbf{Ab})$. This is called the category of presheaves of abelian groups on the admissible topology of $\mathbb{Q}$. If $F : \mathrm{Open}_a^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf and $U \subset V$ are admissible open subsets of $\mathbb{Q}$, we denote the map $F(V) \to F(U)$ by $s \longmapsto s_{|U}$.

We say that a presheaf $F : \mathrm{Open}_a^{\mathrm{op}} \to \mathbf{Ab}$ is a *sheaf* if, for every admissible open subset U of $\mathbb{Q}$ and for every admissible cover $(U_i)_{i\in I}$ of U, the following two conditions hold :

(1) the map $F(U) \to \prod_{i\in I} F(U_i)$, $s \longmapsto (s_{|U_i})$ is injective;

(2) if $(s_i) \in \prod_{i\in I} F(U_i)$ is such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all $i, j \in I$, then there exists $s \in F(U)$ such that $s_i = s_{|U_i}$ for every $i \in I$.

The full subcategory Sh_a of PSh_a whose objects are sheaves is called the category of sheaves of abelian groups on the admissible topology of $\mathbb{Q}$. [11]

(a). Let U be an open subset of $\mathbb{Q}$, and let $V(U)$ be the union of all the open subsets V of $\mathbb{R}$ such that $V \cap \mathbb{Q} = U$. Show that $V(U)$ is the union of all the intervals $[a, b]$, for $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a, b] \subset U$.

(b). Show that every open set in $\mathbb{Q}$ is admissible. [12]

[11] Of course, we could also define presheaves and sheaves with values in $\mathbf{Set}$.

[12] There is a similar notion of admissible open subset in $\mathbb{Q}^n$, where intervals are replaced by products of intervals,

(c). Give an open cover of an open subset of $\mathbb{Q}$ that is not an admissible open cover.

(d). Show that the inclusion functor $\mathrm{Sh}_a \to \mathrm{PSh}_a$ has a left adjoint $F \longmapsto F^{sh}$. (The sheafification functor.)

(e). Show that Sh_a is an abelian category.

(f). Show that the inclusion $\mathrm{Sh}_a \to \mathrm{PSh}_a$ is left exact but not exact, and that the sheafification functor $\mathrm{PSh}_a \to \mathrm{Sh}_a$ is exact.

For every $x \in \mathbb{Q}$ and every presheaf $F \in \mathrm{Ob}(\mathrm{PSh}_a)$, we define the *stalk* of F at x to be $F_x = \varinjlim_{U \ni x} F(U)$, that is, the colimit of the functor $\phi : \mathrm{Open}_a(\mathbb{Q}, x)^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathrm{Open}_a(\mathbb{Q}, x)$ is the full subcategory of Open_a of admissible open subsets containing x and ϕ is the restriction of F.

(g). For every $x \in \mathbb{Q}$, show that the functor $\mathrm{Sh}_a \to \mathbf{Ab}$, $F \longmapsto F_x$ is exact.

(h). Let PSh (resp. Sh) be the usual category of presheaves (resp. sheaves) of abelian groups on $\mathbb{R}$. Show that the functor $\mathrm{Sh} \to \mathrm{PSh}_a$ sending a sheaf F on $\mathbb{R}$ to the presheaf $U \longmapsto F(V(U))$ on $\mathbb{Q}$ is fully faithful, that its essential image is Sh_a, and that it is exact as a functor from Sh_a to Sh.

(i). Find a nonzero object F of Sh_a such that $F_x = 0$ for every $x \in \mathbb{Q}$.

Solution.

(a). The set $V(U)$ is obviously an open subset of $\mathbb{R}$, we have $V(U) \cap \mathbb{Q} = U$, and $V(U)$ is maximal among open subsets of $\mathbb{R}$ satisfying this condition.

Let $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a, b] \subset U$. Then $(]a, b[\cup V(U)) \cap \mathbb{Q} = U$, so $]a, b[\subset V(U)$ by the maximality of $V(U)$; as $a, b \in U \subset V$, we get that $[a, b] \subset V(U)$.

Conversely, let $x \in V(U)$. As $V(U)$ is open in $\mathbb{R}$ and $\mathbb{Q}$ is dense, there exist $a, b \in \mathbb{Q}$ such that $a < x < b$ and $[a, b] \subset V(U)$. Then $\mathbb{Q} \cap [a, b] \subset \mathbb{Q} \cap V(U) \subset U$.

(b). Let U be an open subset of $\mathbb{Q}$, and let $V = V(U)$. Let $((a_i, b_i))_{i \in I}$ be the family of all couples $(a_i, b_i) \in \mathbb{Q}$ such that $a_i < b_i$ and that the open interval $]a_i, b_i[$ is contained in $V(U)$. For every $i \in I$, let $A_i =]a_i, b_i[\cap \mathbb{Q}$; this is an open rational interval. We have $U = \bigcup_{i \in I} A_i$, and we claim that this is an admissible cover of U, which implies that U is admissible. Let B be a closed rational interval, and let $a, b \in \mathbb{Q}$ such that $B = \mathbb{Q} \cap [a, b]$. Then $[a, b] \subset V$ by question (a). As $[a, b]$ is compact, there exists a finite subset J of I such that $[a, b] \subset \bigcup_{j \in J}]a_j, b_j[$. By the shrinking lemma [13] (and the finiteness of J), there exists $\varepsilon > 0$ such that $[a, b] \subset \bigcup_{j \in J} [a_j + \varepsilon, b_j - \varepsilon]$, so $B \subset \bigcup_{j \in J} B_j$, where $B_j = \mathbb{Q} \cap [a_j + \varepsilon, b_j - \varepsilon]$.

(c). Let $U =]0, 2[\cap \mathbb{Q}$. Let $(x_n)_{n \in \mathbb{N}}$ be an increasing sequence of rational numbers that

and this result does not hold for $n \geq 2$.

[13] See Theorem 15.10 of [16].

converges to $\sqrt{2}$. For every $n \in \mathbb{N}$, let $U_n = \mathbb{Q} \cap (]0, x_n[\cup]2/x_n, 2[)$; note that $x_n < \sqrt{2} < 2/x_n$. Then $U = \bigcup_{n\in\mathbb{N}} U_n$. We claim that this is not an admissible cover. Let $a, b \in \mathbb{Q}$ such that $0 < a < 1/\sqrt{2} < b < 1$, and let $B = \mathbb{Q} \cap [a, b]$. If there existed a finite subset M of $\mathbb{N}$ such that $B \subset \bigcup_{n\in M} U_n$, then, as the family U_n is increasing, there would exist $N \in \mathbb{N}$ such that $B \subset U_N$, which is absurd because $x_{N+1} \in B \setminus U_{N+1}$.

(d). The same construction as in Section III.1 gives an additive functor $\mathscr{F} \longmapsto F^{\mathrm{sh}} = \mathscr{F}^{++}$ from PSh_a to Sh_a and a morphism of functors $\iota : \mathrm{id}_{\mathrm{PSh}_a} \to (\cdot)^{\mathrm{sh}}$ such that, if $\mathscr{F}$ is a sheaf, then $\iota(\mathscr{F})$ is an isomorphism. Indeed, we never used the fact that we have a topology in this construction. We only used the fact that we have a notion of open subsets and a notion of covers of open subsets, such that :

(1) any two covers of an open subset admit a common refinement;

(2) if we have a cover $(U_i)_{i\in I}$ of U and we take a cover of each U_i, then the union of these gives a cover of U;

(3) if we have a cover of U and we intersect it with an open subset V of U, then we get a cover of V.

Let's check these properties.

(1) Let $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be two admissible covers of U. We claim that $(U_i \cap V_j)_{i\in I, j\in J}$ is an admissible cover of U. Let $B \subset$ a rational closed interval. There exist finite subsets $I' \subset I$ and $J' \subset J$ and closed rational intervals $A_i \subset U_i$ and $B_j \subset V_j$, for $i \in I'$ and $j \in J'$, such that $B \subset \bigcup_{i\in I'} A_i$ and $B \subset \bigcup_{j\in J'} B_j$. For every $(i, j) \in I' \times J'$, $C_{ij} = A_i \cap B_j \subset U_i \cap V_j$ is a closed rational interval, and we have $B \subset \bigcup_{(i,j)\in I'\times J'} C_{ij}$.

(2) Let $(U_i)_{i\in I}$ be an admissible open cover of U. For every $i \in I$, let $(U_{ij})_{j\in J_i}$ be an admissible open cover of U_i. Let $B = \mathbb{Q} \cap [a, b] \subset U$ be a closed rational interval. Let $I' \subset I$ be a finite subset and $B_i \subset U_i$ be closed rational intervals, for $i \in I'$, such that $B \subset \bigcup_{i\in I'} B_i$. For every $i \in I'$, let $J'_i \subset J_i$ be a finite subset and $B_{ij} \subset U_{ij}$ be closed rational intervals, for $j \in J'_i$, such that $B_i \subset \bigcup_{j\in J'_i} B_{ij}$. Then $B \subset \bigcup_{i\in I'} \bigcup_{j\in J'_i} B_{ij}$, and the set $\bigcup_{i\in I'} J'_i$ is still finite.

(3) Let $(U_i)_{i\in I}$ be an admissible cover of U, let $V \subset U$ be another open set of $\mathbb{Q}$, and let $(V_i)_{i\in I} = (V \cap U_i)_{i\in I}$. Let $B = \mathbb{Q} \cap [a, b]$ be a closed rational interval such that $B \subset V$. Then there exist a finite subset J of I and closed rational intervals $B_j = \mathbb{Q} \cap [a_j, b_j] \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$. After replacing each B_j by its intersection with $[a, b]$, we may assume that $a_j \geq a$ and $b_j \leq b$ for every $j \in J$. Then $B_j \subset B \subset V$, so $B_j \subset V_j$, and we still have $B \subset \bigcup_{j\in J} B_j$.

The fact that $\iota(\mathscr{F})$ is an isomorphism for $\mathscr{F}$ a sheaf means that $\iota(G) : G \to G \circ F \circ G$ is an isomorphism of functors; as G is fully faithful, $\iota(G)^{-1} : G \circ F \circ G \to G$ comes from a unique isomorphism of functors $\varepsilon : F \circ G \to \mathrm{id}_{\mathrm{Sh}_a}$. By Lemma I.4.5 , this ε induces a functorial morphism

$$\alpha : \mathrm{Hom}_{\mathrm{PSh}_a}(\cdot, G(\cdot)) \to \mathrm{Hom}_{\mathrm{Sh}_a}(F(\cdot), \cdot)$$

sending $f : \mathscr{F} \to \mathscr{F}'$ (with $\mathscr{F}$ a presheaf and $\mathscr{F}'$ a sheaf) to $\alpha(\mathscr{F}, \mathscr{F}')(f) = \iota(\mathscr{F}')^{-1} \circ f^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$. As ι is a morphism of functors, we have a commutative square

$$\begin{tikzcd} \mathscr{F}^{\mathrm{sh}} \arrow[r, "f^{\mathrm{sh}}"] & \mathscr{F}'^{\mathrm{sh}} \\ \mathscr{F} \arrow[u, "\iota(\mathscr{F})"] \arrow[r, "f"'] & \mathscr{F}' \arrow[u, "\iota(\mathscr{F}')"'] \end{tikzcd}$$

hence $\alpha(\mathscr{F}, \mathscr{F}')(f) \circ \iota(\mathscr{F}) = f$, and, by the analogue of the uniqueness statement of Proposition III.1.10(vi), $\alpha(\mathscr{F}, \mathscr{F}')(f)$ is the unique morphism from $\mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$ having that property. This implies that $\alpha(\mathscr{F}, \mathscr{F}')(f)$ determines f (so that $\alpha(\mathscr{F}, \mathscr{F}')$ is injective), but also that, if $g : \mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'$ is any morphism of sheaves such that $f = g \circ \iota(\mathscr{F})$, then $g = \alpha(\mathscr{F}, \mathscr{F}')(f)$; the last part gives a construction of a map $\beta : \mathrm{Hom}_{\mathrm{Sh}_a}(\mathscr{F}^{\mathrm{sh}}, \mathscr{F}') \to \mathrm{Hom}_{\mathrm{PSh}_a}(\mathscr{F}, \mathscr{F}')$ such that $\alpha(\mathscr{F}, \mathscr{F}') \circ \beta$ is the identity, so $\alpha(\mathscr{F}, \mathscr{F}')$ is surjective.

(e). We will actually show that the category of sheaves has all small limits and colimits, and that the inclusion functor $\mathrm{Sh}_a \to \mathrm{PSh}_a$ commutes with limits.

If $\mathscr{F}$ is a presheaf, then $\mathscr{F}$ is a sheaf if and only if the sequence

$$0 \to \mathscr{F}(U) \overset{u(\mathscr{F},\mathscr{U})}{\to} \prod_{i\in I} \mathscr{F}(U_i) \overset{v(\mathscr{F},\mathscr{U})}{\to} \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$$

is exact, for every open subset U of $\mathbb{Q}$ and every admissible open cover $\mathscr{U} = (U_i)_{i\in I}$ of U, where $u(\mathscr{F}, \mathscr{U})$ sends $s \in \mathscr{F}(U)$ to $(s_{|U_i})_{i\in I}$ and $v(\mathscr{F}, \mathscr{U})$ sends $(s_i) \in \prod_{i\in I} \mathscr{F}(U_i)$ to $(s_{i|U_i\cap U_j} - s_{j|U_i\cap U_j})_{i,j\in I}$. This sequence is functorial in $\mathscr{F}$, and functorial and contravariant in $\mathscr{U}$. Also, the functors appearing in the sequence commute with limits in $\mathscr{F}$, because of the way limits are formed in categories of presheaves and because direct products (being limits) commute with limits. So, if we have a functor $\alpha : \mathscr{I} \to \mathrm{Sh}_a$ with $\mathscr{I}$ a small category, the presheaf $\varprojlim(G \circ \alpha)$ is also a sheaf; as sheaves form a full subcategory of PSh_a, this limit satisfies the universal property of the limit in the category Sh_a, so it is (the image by G of) the limit of α. Or, in other terms : to form a limit in the category of sheaves, it suffices to take the limit in the category of sheaves.

Now we show that α also has a colimit. In fact, we show that the sheafification of $\varinjlim(G\circ\alpha)$ is the colimit of α. Indeeed, for every sheaf $\mathscr{G}$, we have isomorphisms, functorial in $\mathscr{G}$:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sh}_a}(F(\varinjlim(G \circ \alpha)), \mathscr{G}) &\simeq \mathrm{Hom}_{\mathrm{PSh}_a}(\varinjlim(G \circ \alpha), G(\mathscr{G})) \\ &\simeq \varprojlim_{i\in \mathrm{Ob}(\mathscr{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathrm{PSh}_a}(G(\alpha(i)), G(\mathscr{G})) \\ &= \varprojlim_{i\in \mathrm{Ob}(\mathscr{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathrm{Sh}_a}(\alpha(i), \mathscr{G}), \end{aligned}$$

which is the universal property of the colimit.

As Sh_a is an additive subcatgeory of PSh_a, the fact that it has all limits and colimits shows that every morphism of Sh_a has a kernel and a cokernel; we also showed that the kernel of a morphism of Sh_a is its kernel in PSh_a, and that its cokernel is the sheafification of its cokernel in PSh_a.

Now let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in Sh_a. We want to show that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. By definition, $\mathrm{Im}(f)$ is the kernel of $p : \mathscr{G} \to \mathrm{Coker}(f)$; so, for every open subset U of $\mathbb{Q}$, an element $s \in \mathscr{G}(U)$ is in $(\mathrm{Im}\, f)(U)$ if and only if there exists an admissible open cover $(U_i)_{i\in I}$ of U such that, for every $i \in I$, we have $p(s_{|U_i}) = 0$, that is, $s_{|U_i} \in \mathrm{Im}(\mathscr{F}(U_i) \to \mathscr{G}(U_i))$. In other words, $\mathrm{Im}(f)$ is the sheafification of the separated presheaf $\mathscr{C} : U \longmapsto \mathrm{Im}(\mathscr{F}(U) \to \mathscr{G}(U))$. On the other hand, $\mathrm{Coim}(f)$ is the sheafification of the presheaf $\mathscr{I} : U \longmapsto \mathscr{F}(U)/(\ker f)(U)$. The canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is induced by the morphism $\mathscr{I} \to \mathscr{C}$ sending an element s of $\mathscr{F}(U)/(\mathrm{Im}\, f)(U)$ to $f(s) \in \mathscr{G}(U)$, which is an isomorphism; so $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is also an isomorphism.

(f). We saw that $G : \mathrm{Sh}_a \to \mathrm{PSh}_a$ commutes with limits, so it is left exact. To show that G is not exact, it suffices to find a surjective morphism $u : \mathscr{F} \to \mathscr{G}$ in Sh_a such that $\mathscr{F}(\mathbb{Q}) \to \mathscr{G}(\mathbb{Q})$ is not surjective. Let $\mathscr{F}$ be the constant sheaf with value $\mathbb{Z}$ on $\mathbb{Q}$, that is, the sheafification of the constant presheaf $\mathscr{F}_0 : U \longmapsto \mathbb{Z}$. Let $\mathscr{G}$ be the sheaf sending an open subset U of $\mathbb{Q}$ to $\mathbb{Z}^{U\cap\{0,1\}}$ (with the obvious restriction maps), with the (usual) convention that $\mathbb{Z}^{\varnothing} = 0$. For each open sbuset U of $\mathbb{Q}$, we have the diagonal map $\mathscr{F}_0(U) \to \mathscr{G}(U)$. This gives a morphism of presheaves $\mathscr{F}_0 \to \mathscr{G}$, so we get a morphism of sheaves $\mathscr{F} \to \mathscr{G}$. This morphism is surjective in Sh_a because, if U is an open subset of $\mathbb{Q}$ such that $\{0,1\} \subset U$ and $s \in \mathscr{G}(U)$, then we can find an admissible open cover (U_0, U_1) of U such that $U_i \cap \{0,1\} = \{i\}$ for $i = 0, 1$, and then $s_{|U_i} \in \mathrm{Im}(\mathscr{F}(U_i) \to \mathscr{G}(U_i)) = \mathscr{G}(U_i)$ for $i = 0, 1$. However, the morphism $\mathscr{F}(\mathbb{Q}) = \mathbb{Z} \to \mathscr{G}(\mathbb{Q}) = \mathbb{Z}^2$ is the diagonal morphism, which is not surjective. (To calculate $\mathscr{F}(\mathbb{Q})$, it is easiest to use question (i); the sheaf $\mathscr{F}$ is then identitied to the constant sheaf with values $\mathbb{Z}$ on $\mathbb{R}$, and its global sections are $\mathbb{Z}$ because $\mathbb{R}$ is connected.)

It remains to show that the sheafification functor is exact. By the construction of colimits in Sh_a, we already know that it is right exact, so it suffices to show that it preserves injective morphisms. Let $\mathscr{F} \to \mathscr{F}'$ is an injective morphism of presheaves. If U is an open subset of $\mathbb{Q}$, then $\check{\mathrm{H}}^0(\mathscr{U}, \mathscr{F}) \to \check{\mathrm{H}}^0(\mathscr{U}, \mathscr{F}')$ is injective for every admissible open cover $\mathscr{U}$ of U (by definitions of these groups), so $\mathscr{F}^+(U) \to \mathscr{F}'^+(U)$ is injective because filtrant colimits are exact in $\mathbf{Ab}$. Applying this reasoning twice, we see that $\mathscr{F}^{\mathrm{sh}}(U) \to \mathscr{F}'^{\mathrm{sh}}(U)$ is injective for every U. As kernels in Sh_a are calculated by taking kernels in PSh_a, this means that $\mathscr{F}^{\mathrm{sh}} \to \mathscr{F}'^{\mathrm{sh}}$ is injective.

(g). We can define the stalks of a presheaf (with the same formula). If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is a short exact sequence of presheaves, then the complex $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U) \to 0$ is exact for every open subset U of $\mathbb{Q}$, so, as stalks are defined by filtrant colimits and filtrant colimits are exact in $\mathbf{Ab}$, the complex $0 \to \mathscr{F}_{1,x} \to \mathscr{F}_{2,x} \to \mathscr{F}_{3,x} \to 0$ is exact for every $x \in \mathbb{Q}$. As the inclusion $\mathrm{Sh}_a \subset \mathrm{PSh}_a$ is left exact, we conclude that the functor $\mathrm{Sh}_a \to \mathbf{Ab}$, $\mathscr{F} \longmapsto \mathscr{F}_x$ is left exact for every $x \in \mathbb{Q}$. To show that it is exact, it is therefore enough to show that it sends surjections to surjections. Let $f : \mathscr{F} \to \mathscr{G}$ be a surjective morphism, let $x \in \mathbb{Q}$, and let $s_x \in \mathscr{G}_x$. Choose an open suset $U \ni x$ of $\mathbb{Q}$ and a section $s \in \mathscr{G}(U)$ representing s_x. As we saw in the solution of (d), the surjectivity of f means that there exists an admissible open cover $(U_i)_{i\in I}$ of U and sections $t_i \in \mathscr{F}(U_i)$ such that $f(t_i) = s_{|U_i}$ for every $i \in I$. Let $i_0 \in I$ such that $x \in U_{i_0}$, and let $t_x \in \mathscr{F}_x$ be the image of t_{i_0}. Then $f_x : \mathscr{F}_x \to \mathscr{G}_x$ sends t_x to s_x.

(h). We prove the following facts :

(A) Let U and U' be open subsets of $\mathbb{Q}$. We claim that $V(U \cap U') = V(U) \cap V(U')$. Indeed, the set $V(U) \cap V(U')$ is an open subset of $\mathbb{R}$ such that $\mathbb{Q} \cap V(U) \cap V(U') = U \cap U'$, so $V(U) \cap V(U') \subset V(U \cap U')$ by maximality of $V(U \cap U')$. Conversely, if $a, b \in \mathbb{Q}$ are such that $B := \mathbb{Q} \cap [a,b] \subset U \cap U'$, then $[a,b] \subset V(U)$ and $[a,b] \subset V(U')$, so $[a,b] \subset V(U) \cap V(U')$; this shows that $V(U \cap U') \subset V(U) \cap V(U')$.

(B) Let U be an open subset of $\mathbb{Q}$ and let $(U_i)_{i\in I}$ be an admissible open cover of U. We claim that $V(U) = \bigcup_{i\in I} V(U_i)$. Indeed, $V' := \bigcup_{i\in I} V(U_i)$ is an open subset of $\mathbb{R}$ such that $\mathbb{Q} \cap V' = U$, so $V' \subset V(U)$. Conversely, let $a, b \in \mathbb{Q}$ such that $B := \mathbb{Q} \cap [a,b] \subset U$; by the admissibility conditions, there exists a finite subset J of I and rational closed intervals $B_j = \mathbb{Q} \cap [a_j, b_j] \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$. This implies that $[a,b] = \bigcup_{j\in J} [a_j, b_j]$; moreover, for every $j \in J$, the fact that $B_j \subset U_j$ implies that $[a_j, b_j] \subset V(U_j)$; so we finally get that $[a,b] \subset V'$, as desired.

(C) Let U be an open subset of $\mathbb{Q}$, and let $(V_i)_{i\in I}$ be an open cover of $V(U)$. We claim that, after replacing $(V_i)_{i\in I}$ by a refinement, the open cover $(U \cap V_i)_{i\in I}$ of U is admissible; also, if all the V_i are open intervals, then no refinement is necessary. Indeed, after replacing $(V_i)_{i\in I}$ by a refinement, we can assume that all the V_i are open intervals in $\mathbb{R}$. Let $B = \mathbb{Q} \cap [a,b]$ be a closed rational interval contained in U. Then $[a,b] \subset V(U)$. As $[a,b]$ is compact, there exists a finite subset J of I such that $[a,b] \subset \bigcup_{j\in J} V_j$. By the shrinking lemma, [14] there exist closed intervals with rational end points $[a_j, b_j] \subset V_j$ such that $[a,b] \subset \bigcup_{j\in J}]a_j, b_j[$. If $B_j = \mathbb{Q} \cap [a_j, b_j]$ for every $j \in J$, we have $B_j \subset U \cap V_j$ and $B \subset \bigcup_{j\in J} B_j$. So the open cover $(U \cap V_i)_{i\in I}$ of U is admissible.

(D) If A is an open inteval of $\mathbb{R}$, then $V(A \cap Q) = A$. Indeed, it is clear that $A \subset V(A \cap \mathbb{Q})$. Conversely, write $A =]x, y[$, and let $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a,b] \subset A \cap Q$; then $x < a$ and $b < y$, so $[a,b] \subset A$. Hence $A \supset V(A \cap \mathbb{Q})$.

Now let $\mathscr{F}$ be a sheaf on $\mathbb{R}$ for the usual topology, and let $\Phi(\mathscr{F})$ be the presheaf

[14] See Theorem 15.10 of [16], again.

$U \longmapsto \mathscr{F}(V(U))$ on $\mathbb{Q}$. For an admissible open cover $(U_i)_{i\in I}$ of an open subset U of $\mathbb{Q}$, the sequence

$$0 \to \Phi(\mathscr{F})(U) \to \prod_{i\in I} \Phi(\mathscr{F})(U_i) \to \prod_{i,j\in I} \Phi(\mathscr{F})(U_i \cap U_j)$$

(where the first map sends $s \in \Phi(\mathscr{F})(U)$ to $(s_{|U_i})_{i\in I}$ and the second map sends $(s_i) \in \prod_{i\in I} \Phi(\mathscr{F})(U_i)$ to the family $(s_{i|U_i\cap U_j} - s_{j|U_i\cap U_j})_{i,j\in I}$) is exact, because it is equal to the sequence

$$0 \to \mathscr{F}(V(U)) \to \prod_{i\in I} \mathscr{F}(V(U_i)) \to \prod_{i,j\in I} V(U_i) \cap V(U_j)$$

by (A), and because $(V(U_i))_{i\in I}$ is an open cover of $V(U)$ by (B). So $\Phi(\mathscr{F})$ is a sheaf for the admissible topology on $\mathbb{Q}$.

The functor $\Phi : \mathrm{Sh} \to \mathrm{Sh}_a$ is clearly additive and left exact. We show that it is faithful. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of Sh such that $\Phi(f) = 0$. Then, for every open interval A of $\mathbb{R}$, we have $\mathscr{F}(A) = \Phi(\mathscr{F})(A \cap \mathbb{Q})$ and $\mathscr{G}(A) = \Phi(\mathscr{G})(A \cap \mathbb{Q})$ by (D), so the morphism $f(A) : \mathscr{F}(A) \to \mathscr{G}(A)$ is zero. As open intervals form a basis of the topology of $\mathbb{R}$, this implies that $f = 0$.

We show that Φ is full. Let $\mathscr{F}$, $\mathscr{G}$ be sheaves on $\mathbb{R}$, and let $g : \Phi(\mathscr{F}) \to \Phi(\mathscr{G})$ be a morphism of sheaves on $\mathbb{Q}$. For every open interval A of $\mathbb{R}$, we define $f(A) : \mathscr{F}(A) = \Phi(\mathscr{F})(A \cap \mathbb{Q}) \to \mathscr{G}(A) = \Phi(\mathscr{G})(A \cap \mathbb{Q})$ to be $g(A \cap \mathbb{Q})$ (we are using (D) again). If $A \subset A'$ are open intervals of $\mathbb{R}$, the diagram

$$\begin{array}{ccc} \mathscr{F}(A') & \xrightarrow{F(A')} & \mathscr{G}(A') \\ \downarrow & & \downarrow \\ \mathscr{F}(A) & \xrightarrow[f(A)]{} & \mathscr{G}(A) \end{array}$$

(where the vertical arrows are restriction maps) is commutative because g is a morphism of presheaves. As open intervals form a basis of the topology of $\mathbb{R}$, there is a unique morphism of sheaves $f : \mathscr{F} \to \mathscr{G}$ that is equal to $f(A)$ on sections over any open interval A. It is clear that $\Phi(f) = g$.

We show that the essential image of Φ is Sh_a. Let $\mathscr{F}_0$ be a sheaf on $\mathbb{Q}$ for the admissible topology. We want to define a sheaf $\mathscr{F}$ on $\mathbb{R}$ such that $\Phi(\mathscr{F}) \simeq \mathscr{F}_0$. As open intervals form a base of the topology of $\mathbb{R}$, it suffices to define $\mathscr{F}$ on open intervals (and to check the sheaf condition for covers of an open interval by open intervals). If A is an open interval of $\mathbb{R}$, we set $\mathscr{F}(A) = \mathscr{F}_0(A \cap \mathbb{Q})$; by (D), we then have $\mathscr{F}_0(A \cap \mathbb{Q}) = \mathscr{F}(V(A \cap \mathbb{Q}))$. The sheaf condition for $\mathscr{F}$ follows from the sheaf condition from $\mathscr{F}_0$ and from (C), and the fact that $\Phi(\mathscr{F}) = \mathscr{F}_0$ is obvious.

Finally, we have shown that Φ is an equivalence of categories from Sh to Sh_a. In particular, it commutes with all limits and colimits that exist in these categories, so it is exact.

(i). Let $\mathscr{F}$ be the skryscaper sheaf on $\mathbb{R}$ supported at $\sqrt{2}$ and with value $\mathbb{Z}$. In other words, if V is an open subset of $\mathbb{R}$, we have $\mathscr{F}(V) = 0$ if $\sqrt{2} \notin V$ and $\mathscr{F}(V) = \mathbb{Z}$ if $\sqrt{2} \in V$; the restriction morphisms are either 0 or $\mathrm{id}_{\mathbb{Z}}$. Let $\mathscr{F}_0 = \Phi(\mathscr{F})$; then $\mathscr{F}_0$ is the sheaf on $\mathbb{Q}$ given by $\mathscr{F}(U) = 0$ if $\sqrt{2} \notin V(U)$, and $\mathscr{F}(U) = \mathbb{Z}$ if $\sqrt{2} \in V(U)$. If $x \in \mathbb{Q}$, then there exists an open neighborhood U of x in $\mathbb{Q}$ such that $\sqrt{2} \notin V(U)$ (for example an open rational interval), so $\mathscr{F}_x = 0$.

$\square$

A.3.6 Canonical topology on an abelian category

Let $\mathscr{A}$ be an abelian category. Let $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$ be the category of presheaves of abelian groups on $\mathscr{A}$. We say that a presheaf $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is a *sheaf* (in the canonical topology) if, for every epimorphism $f : X \to Y$ in $\mathscr{A}$, the following sequence of abelian groups is exact :

$$0 \longrightarrow F(Y) \xrightarrow{F(g)} F(X) \xrightarrow{F(p_1)-F(p_2)} F(X \times_Y X)\ ,$$

where $p_1, p_2 : X \times_Y X \to X$ are the two projections. We denote by Sh the full subcategory of PSh whose objects are the sheaves.[15]

(a). If $f : X \to Y$ is an epimorphism in $\mathscr{A}$, show that it is the cokernel of the morphism $p_1 - p_2 : X \times_Y X \to X$, where $p_1, p_2 : X \times_Y X \to X$ are the two projections as before.

(b). Let $f : X \to Y$ be an epimorphism in $\mathscr{A}$ and $g : Z \to Y$ be a morphism. Consider the second projection $p_Z : X \times_Y Z \to Z$. Show that p_Z is an epimorphism.

(c). Show that every representable presheaf on $\mathscr{A}$ is a sheaf.

(d). Show that the inclusion functor $\mathrm{Sh} \to \mathrm{PSh}$ has a left adjoint $F \longmapsto F^{sh}$. (The sheafification functor.)

(e). Show that Sh is an abelian category.

(f). Show that the inclusion $\mathrm{Sh} \to \mathrm{PSh}$ is left exact but not exact, and that the sheafification functor $\mathrm{PSh} \to \mathrm{Sh}$ is exact.

Solution.

[15] Again, we could define sheaves of sets.

(a). We have a cartesian square

$$\begin{tikzcd} X\times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"'] \arrow[ddr, "g"] & \\ X \arrow[r, "f"] \arrow[drr, "g"'] & Y \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

As f is surjective, Proposition II.2.1.15 implies that this square is also cocartesian, that is, Y is the coproduct $X \sqcup_{X\times_Y X} X$. We now show that $f = \mathrm{Coker}(p_1, p_2)$. We have $f\circ p_1 = f\circ p_2$ by definition of the fiber product. Let $g : X \to Z$ be a morphism such that $g\circ p_1 = g\circ p_2$. By the universal property of the coproduct, there exists a unique morphism $h : Y \to Z$ such that $g = h\circ f$. This is also the universal property of $\mathrm{Coker}(p_1, p_2)$.

(b). This is Corollary II.2.1.16(ii) .

(c). If $f : X \to Y$ is surjective, then, by (a), the sequence $X\times_Y X \overset{p_1-p_2}{\to} X \overset{f}{\to} Y \to 0$ is exact. So every left exact functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is a sheaf, and in particular every representable functor.

In fact, every sheaf $\mathscr{F} : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ that is an additive functor is automatically a left exact functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$. Indeed, let $0 \to Z \overset{g}{\to} X \overset{f}{\to} Y \to 0$ be an exact sequence. As the sequence $X \times_Y X \overset{p_1-p_2}{\to} X \overset{f}{\to} Y \to 0$ is exact, we have $Z \overset{\sim}{\to} \mathrm{Im}(g) = \mathrm{Ker}(f) = \mathrm{Im}(p_1-p_2)$, so there exists a unique morphism $h : X\times_Y X \to Z$ such that $g\circ h = p_1 - p_2$. Applying $\mathscr{F}$, we get a commutative diagram where the top row is exact

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}(Y) \arrow[r, "\mathscr{F}(f)"] & \mathscr{F}(X) \arrow[r, "\mathscr{F}(p_1-p_2)"] \arrow[dr, "\mathscr{F}(g)"'] & \mathscr{F}(X\times_Y X) \\ & & & \mathscr{F}(Z) \arrow[u, "\mathscr{F}(h)"'] \end{tikzcd}$$

We have $\mathscr{F}(g)\circ\mathscr{F}(f) = 0$ because $f\circ = 0$, so $\mathrm{Ker}\,\mathscr{F}(g) \supset \mathrm{Im}\,\mathscr{F}(f)$. On the other hand, $\mathrm{Im}\,\mathscr{F}(f) = \mathrm{Ker}\,\mathscr{F}(p_1 - p_2)$ because $\mathscr{F}$ is a sheaf, and $\mathrm{Ker}\,\mathscr{F}(g) \subset \mathrm{Ker}\,\mathscr{F}(p_1 - p_2)$ because $\mathscr{F}(p_1 - p_2) = \mathscr{F}(h)\circ\mathscr{F}(g)$. So $\mathrm{Ker}\,\mathscr{F}(g) = \mathrm{Im}\,\mathscr{F}(f)$, and the sequence $\mathscr{F}(Z) \to \mathscr{F}(X) \to \mathscr{F}(Y) \to 0$ is exact.

(d). As in problem A.3.5, it suffices to construct a functor $\mathrm{PSh} \to \mathrm{Sh}$, $\mathscr{F} \longmapsto \mathscr{F}^{\mathrm{sh}}$ and a morphism of functors $\iota(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ such that $\iota(\mathscr{F})$ is an isomorphism for $\mathscr{F}$ a sheaf and that, if $f : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves and $\mathscr{G}$ is a sheaf, then there exists a unique morphism of sheaves $f' : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\iota(\mathscr{F})"'] & \mathscr{G} \arrow[d, "\iota(\mathscr{G})"] \\ \mathscr{F}^{\mathrm{sh}} \arrow[ur, "f'"'] \arrow[r, "f^{\mathrm{sh}}"'] & \mathscr{G}^{\mathrm{sh}} \end{tikzcd}$$

The rest of the proof is the same as is question A.3.5(c).

The construction of the sheafification functor follows the same lines as the construction of Section III.1, except that we have to use the correct notion of cover. Let X be an object of $\mathscr{A}$. The category of covering families of X is the category $\mathscr{I}_X$ whose objects are surjective morphisms $Y \to X$ and whose morphisms are commutative diagrams
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[dl] \\ X & \end{tikzcd}$$
. Let $\mathscr{F}$ be a presheaf on $\mathscr{A}$. If X is an object of $\mathscr{A}$ and $f : Y \to X$ is a surjective morphism, we set
$$\check{\mathrm{H}}^0(Y \to X, \mathscr{F}) = \mathrm{Ker}(\mathscr{F}(p_1 - p_2) : \mathscr{F}(Y) \to \mathscr{F}(Y \times_X Y)),$$
where $p_1, p_2 : Y \times_X Y \to Y$ are as before the two projections. As $\mathscr{F}(p_1 - p_2) \circ \mathscr{F}(f) = 0$, we have a morphism $\mathscr{F}(X) \to \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$ induced by $\mathscr{F}(f)$. Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two surjective morphisms, and suppose that there exists $g : Y_2 \to Y_1$ such that $f_1 \circ g = f_2$ (that is, g is a morphism in $\mathscr{I}_X$). If $q_1, q_2 : Y_2 \times_X Y_2 \to Y_2$ are the two projections, then $f_1 \circ g \circ q_1 = f_2 \circ q_1 = f_2 \circ q_2 = f_1 \circ g \circ q_2$, so $(g \circ q_1, g \circ q_2)$ defines a morphism $g' : Y_2 \times_X Y_2 \to Y_1 \times_X Y_1$ such that the compositions of g' with the projections $p_1, p_2 : Y_1 \times_X Y_1 \to Y_1$ are equal to $g \circ q_1$ and $g \circ q_2$. In particular, we have $g \circ (q_1 - q_2) = (p_1 - p_2) \circ g'$, hence $\mathscr{F}(q_1 - a_2) \circ \mathscr{F}(g) = \mathscr{F}(g') \circ \mathscr{F}(p_1 - p_2)$. So morphism $\mathscr{F}(g) : \mathscr{F}(Y_1) \to \mathscr{F}(Y_2)$ sends $\check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$ to $\check{\mathrm{H}}^0(Y_2 \to X, \mathscr{F})$. As in Section III.1 , we can show that this morphism does not depend on g. Indeed, let $h : Y_2 \to Y_1$ be another morphism such that $f_1 \circ h = f_2$. Then (g, h) defines a morphism $k : Y_2 \to Y_1 \times_X Y_1$ such that $p_1 \circ k = g$ and $p_2 \circ k = h$. If $s \in \check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$, then
$$\mathscr{F}(g)(s) = \mathscr{F}(k)(\mathscr{F}(p_1)(s)) = \mathscr{F}(k)(\mathscr{F}(p_2)(s)) = \mathscr{F}(h)(s),$$
because $\mathscr{F}(p_1)(s) = \mathscr{F}(p_2)(s)$ by definition of $\check{\mathrm{H}}^0(Y_1 \to X, \mathscr{F})$.

In summary, we have made $\check{\mathrm{H}}^0(\cdot, \mathscr{F})$ into a functor $(\mathscr{I}_X^0)^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathscr{I}_X^0$ is the category that we get from $\mathscr{I}_X$ by contracting all the nonempty Hom sets to singletons. We denote by $\mathscr{F}^+(X)$ the colimit of this functor. We have a canonical morphism $\mathscr{F}(X) \to \mathscr{F}^+(X)$, given by the morphisms $\mathscr{F}(X) \to \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$.

If X' is another object of $\mathscr{A}$ and $u : X' \to X$ is a morphism, then we get a functor $\mathscr{I}_X \to \mathscr{I}_{X'}$ by sending a surjection $Y \to X$ to $Y \times_X X' \to X'$ (which is a surjection by (b)). This allows us to define a morphism $\mathscr{F}^+(X) \to \mathscr{F}(X')$ as in the notes, and so $\mathscr{F}^+$ is a presheaf. It is easy to see that the morphisms $\mathscr{F}(X) \to \mathscr{F}^+(X)$ define a morphism of presheaves $\iota_0(\mathscr{F}) : \mathscr{F} \to \mathscr{F}^+$. It is also easy to see that $\mathscr{F} \longmapsto \mathscr{F}^+$ is a functor, and that the $\iota_0(\mathscr{F})$ define a morphism of functors.

We set $\mathscr{F}^{\mathrm{sh}} = \mathscr{F}^{++}$, with the morphism $\iota(F) : \mathscr{F} \to \mathscr{F}^{++}$ given by $\iota(F) = \iota_0(\mathscr{F}^+) \circ \iota_0(\mathscr{F}^+)$. If $\mathscr{F}$ is a sheaf, then $\mathscr{F}(X) \xrightarrow{\sim} \check{\mathrm{H}}^0(Y \to X, \mathscr{F})$ for every surjective morphism $Y \to X$, so $\iota(\mathscr{F})$ is an isomorphism.

The proof of Proposition III.1.10 now goes through, provided that we can prove that

$(\mathscr{I}_X^0)^{\mathrm{op}}$ is filtrant. If $f : Y \to X$ and $f' : Y' \to X$ are surjective maps, then the two projections $p_1 : Y \times_X Y' \to Y$ and $p_2 : Y \times_X Y' \to Y'$ are surjective by (b), and $f \circ p_1 = f' \circ p_2$, so we have morphisms $(Y \to X) \to (Y \times_X Y' \to X)$ and $(Y' \to X) \to (Y \times_X Y' \to X)$ in $(\mathscr{I}_X^0)^{\mathrm{op}}$, which suffices because the Homs of $(\mathscr{I}_X^0)^{\mathrm{op}}$ are empty or singletons.

(e) and (f) The proof of questions (d) and (e) of problem A.3.5 applies (provided we replace admissible open covers by surjective morphisms), except for the counterexample showing that the inclusion $\mathrm{Sh} \subset \mathrm{PSh}$ does not preserve surjections. Anticipating a bit on problem A.4.3, we can make the following counterexample : Let $A \to B$ be a surjective morphism in $\mathscr{A}$. Then the induced morphism $\mathrm{Hom}_{\mathscr{A}}(\cdot, A) \to \mathrm{Hom}_{\mathscr{A}}(\cdot, B)$ is surjective in Sh. (See the solution of that problem.) But it is not true in general that $\mathrm{Hom}_{\mathscr{A}}(C, A) \to \mathrm{Hom}_{\mathscr{A}}(C, B)$ is surjective for every object C of $\mathscr{A}$ and every choice of surjective $A \to B$, unless $\mathscr{A}$ is a semisimple abelian category. Indeed, if $\mathscr{A}$ is not semisimple, then we can find an exact sequence $0 \to A' \to B \to A \to 0$ that is not split, and than $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$ does not come from an element of $\mathrm{Hom}_{\mathscr{A}}(A, B)$.

□

A.4 Problem set 4

A.4.1 Cartesian and cocartesian squares

(a). Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in an abelian category $\mathscr{A}$. Consider the morphisms $u = \begin{pmatrix} f \\ g \end{pmatrix} : A \to B \oplus C$ and $v = \begin{pmatrix} h & -k \end{pmatrix} : B \oplus C \to D$.

Prove that the following statements are equivalent :

(i) The canonical morphism $A \to B \times_D C$ is an epimorphism.

(ii) The canonical morphism $B \sqcup_A C \to D$ is a monomorphism.

(iii) The complex $A \xrightarrow{u} B \oplus C \xrightarrow{v} D$ is exact.

(b). Let $A \xrightarrow{g} B \xrightarrow{f} C$ be morphisms in $\mathscr{A}$. Show that $g^{-1}(\mathrm{Ker}\, f) = \mathrm{Ker}(f \circ g)$.

(c). Keep the notation of the previous question, and suppose that g is surjective. Show that $g(\mathrm{Ker}(f \circ g)) = \mathrm{Ker}\, f$.

(d). Keep the notation and assumptions of the previous question. If $u : D \to B$ is a morphism such that $f \circ u = 0$, show that there exists a commutative diagram

$$\begin{tikzcd} D' \arrow[r, "g'"] \arrow[d, "u'"'] & D \arrow[d, "u"] & \\ C \arrow[r, "g"'] & B \arrow[r, "f"'] & A \end{tikzcd}$$

such that g' is surjective and $f \circ g \circ u' = 0$.

Solution.

(a). We claim that the canonical morphism $B \times_D C \to B \times C = B \oplus C$ identifies $B \times_D C$ to $\operatorname{Ker} v$. Indeed, we have, for every $E \in \operatorname{Ob}(\mathscr{A})$,

$$\begin{aligned} \operatorname{Hom}_{\mathscr{A}}(E, B \times_D C) &= \{(w_1, w_2) \in \operatorname{Hom}_{\mathscr{A}}(E, B) \times \operatorname{Hom}_{\mathscr{A}}(E, C) \mid h \circ w_1 = k \circ w_2\} \\ &= \{w \in \operatorname{Hom}_{\mathscr{A}}(E, B \times C) \mid v \circ w = 0\} \\ &= \operatorname{Hom}(E, \operatorname{Ker} v). \end{aligned}$$

So (i) is equivalent to the fact that the morphism $A \to \operatorname{Ker} v$ induced by u is an epimorphism, which implies that the canonical morphism $\operatorname{Im}(u) \to \operatorname{Ker}(v)$ is an epimorphism, hence an isomorphism (because it is automatically injective), which is (iii).

We prove that (ii) and (iii) are equivalent. Let $\iota : B \oplus C \to B \oplus C$ be the morphism with matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\iota \circ \iota = \operatorname{id}_{B \oplus C}$, and in particular ι is an isomorphism. Let $u' = \iota \circ u$ and $v' = v \circ \iota$. Then $v' \circ u' = v \circ u = 0$, and we have a commutative square

$$\begin{tikzcd} \operatorname{Im}(u) \arrow[r] \arrow[d, "\iota"', "\sim"] & \operatorname{Ker}(v) \arrow[d, "\iota"', "\sim"] \\ \operatorname{Im}(u') \arrow[r] & \operatorname{Ker}(v') \end{tikzcd}$$

so (iii) is equivalent to the condition that the complex $(*)\ A \xrightarrow{u'} B \oplus C \xrightarrow{v'} D$ be exact. We claim that the canonical morphism $B \oplus C \to B \sqcup_A C$ identifies $B \sqcup_A C$ to $\operatorname{Coker}(u')$. Indeed, for every $E \in \operatorname{Ob}(\mathscr{A})$,

$$\begin{aligned} \operatorname{Hom}_{\mathscr{A}}(\operatorname{Coker} u', E) &= \{w \in \operatorname{Hom}_{\mathscr{A}}(B \oplus C, E) \mid w \circ u' = 0\} \\ &= \{(w_1, w_2) \in \operatorname{Hom}_{\mathscr{A}}(B, E) \times \operatorname{Hom}_{\mathscr{A}}(C, E) \mid w_1 \circ f = w_2 \circ g\} \\ &= \operatorname{Hom}_{\mathscr{A}}(B \sqcup_A C, E). \end{aligned}$$

So (ii) is equivalent to the injectivity of the morphism $\operatorname{Coker}(u') \to D$ induced by v'; if $p : B \oplus C \to \operatorname{Coker}(u')$ is the canonical surjection, this means that (ii) is equivalent to the fact that, for every object E of $\mathscr{A}$, we have

$$\begin{aligned}\{f \in \operatorname{Hom}_{\mathscr{A}}(E, B \oplus C) \mid v' \circ f = 0\} &= \{f \in \operatorname{Hom}_{\mathscr{A}}(E, \mathrm{B} \oplus C) \mid p \circ f = 0\} \\ &= \{f \in \operatorname{Hom}_{\mathscr{A}}(E, \operatorname{Im}(u')\},\end{aligned}$$

where the last equality is because $\operatorname{Im}(u') = \operatorname{Ker}(p)$. This shows that condition (ii) is equivalent to $\operatorname{Ker}(v') = \operatorname{Im}(u')$, that is, to the exactness of the complex (*).

(b). By definition of $g^{-1}(\operatorname{Ker} f)$, we have a commutative diagram, where the square is cartesian :

$$\begin{tikzcd} g^{-1}(\operatorname{Ker} f) \arrow[r, "g'"] \arrow[d, "j"'] & \operatorname{Ker} f \arrow[d, "i"] & \\ A \arrow[r, "g"'] & B \arrow[r, "f"'] & C \end{tikzcd}$$

In particular, $f \circ g \circ j = f \circ i \circ g' = 0$, so $g^{-1}(\operatorname{Ker} f) \subset \operatorname{Ker}(f \circ g)$. Conversely, let C be an object of $\mathscr{A}$ and $h : C \to A$ a morphism such that $f \circ g \circ h = 0$. Then we have a unique morphism $h' : C \to \operatorname{Ker} f$ such that $i \circ h' = g \circ h$, and this in turns defines a unique morphism $k : C \to g^{-1}(\operatorname{Ker} f)$ such that $j \circ k = h$ and $g' \circ k = h'$. Applying this to the inclusion $\operatorname{Ker}(f \circ g) \to A$, we see that this inclusion factors through $g^{-1}(\operatorname{Ker} f) \to A$, that is, $\operatorname{Ker}(f \circ g) \subset g^{-1}(\operatorname{Ker} f)$.

(c). By (b), it suffices to show that $g(g^{-1}(\operatorname{Ker} f))$. In fact, this is true for any subobject of B, so let $B' \subset B$. We want to show that the morphism $g^{-1}(B') \to B'$ induced by g is surjective. By definition of $g^{-1}(B')$, we have a cartesian square

$$\begin{tikzcd} g^{-1}(B') \arrow[r] \arrow[d] & B' \arrow[d] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

and then the conclusion follows from Corollary II.2.1.16.

(d). Let $B' = \operatorname{Im}(u) \subset B$. The morphism u factors as $D \xrightarrow{v} B' \xrightarrow{i} B$ with v surjective and i injective, and we take $D' = g^{-1}(B') \times_{B'} D$. We have a commutative diagram where both squares are cartesian

$$\begin{tikzcd} D' \arrow[r, "g'"] \arrow[d, "a"'] & D \arrow[d, "v"] \\ g^{-1}(B') \arrow[r] \arrow[d, "b"'] & B' \arrow[d, "i"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

and we take $u' = b \circ a$. The morphism g' is surjective by Corollary II.2.1.16, and $f \circ g \circ u' = f \circ i \circ v \circ g' = f \circ u \circ g' = 0$.

$\square$

A.4.2 A random fact

Let $\mathscr{A}$ be an abelian category and $f : B \to A$ be a morphism of $\mathscr{A}$. Show that, for every object C of $\mathscr{A}$, the morphism $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \to \mathrm{Hom}_{\mathscr{A}}(B, C)$ (induced by $B \to \mathrm{Im}(f)$) induces an isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C)).$$

Solution. The statement is saying two things :

(1) The map $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(f), C) \to \mathrm{Hom}_{\mathscr{A}}(B, C)$ is injective; this follows from the fact that $B \to \mathrm{Im}(f)$ is surjective.

(2) A morphism $g : B \to C$ factors through the quotient $\mathrm{Im}(f)$ of B if and only if $g(\mathrm{Ker}\, f) = 0$, or, in other words, the morphism $B \to \mathrm{Im}(f)$ is the cokernel of the morphism $\mathrm{Ker}(f) \to B$. This follows from the fact that $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$. (Remember that $\mathrm{Coim}(f)$ is by definition the cokernel of the morphism $\mathrm{Ker}(f) \to B$.)

$\square$

A.4.3 More sheaves on an abelian category

We use the notation of problem PS3.6 : We fix an abelian category, and we denote by Sh the category of sheaves of abelian groups on $\mathscr{A}$ for the canonical topology. It is a full subcategory of the catgeory of presheaves $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$. Both Sh and PSh are abelian categories, and the forgetful functor $\mathrm{Sh} \to \mathrm{PSh}$ is left exact but not exact; this functor admits a left adjoint $F \longmapsto F^{\mathrm{sh}}$, which is exact.

(a). Let $f : A \to B$ be a surjective morphism in $\mathscr{A}$. Show that, for every morphism $u : C \to B$, there exists a commutative square

$$\begin{tikzcd} C' \arrow[r, "f'"] \arrow[d] & C \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with $f' : C' \to C$ surjective.

(b). Show that the Yoneda embedding $h_{\mathscr{A}} : \mathscr{A} \to \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$, factors as $\mathscr{A} \xrightarrow{h} \mathrm{Sh} \xrightarrow{\mathrm{For}} \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, where For is the forgetful functor and h is a fully faithful left exact additive functor.

(c). Show that the functor $h : \mathscr{A} \to \mathrm{Sh}$ is exact.

Solution.

(a). We take $C' = A \times_B C$ and $f' : C' \to C$ equal to the second projection. The surjectivity of f' follows from Corollary II.2.1.16.

(b). For every $A \in \mathrm{Ob}(\mathscr{A})$, the functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ factors through the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$, so we can see $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ as an object of PSh; also, if $f : A \to B$ is a morphism, then $f^* : \mathrm{Hom}_{\mathscr{A}}(\cdot, B) \to \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is a morphism of presheaves of abelian groups (and not just of presheaves of sets), because composition is bilinear. So the Yoneda embedding factors as $\mathscr{A} \xrightarrow{h'} \mathrm{PSh} \xrightarrow{\mathrm{For}} \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Set})$, where For is the forgetful functor. The functor h' is additive and left exact because $\mathrm{Hom}_{\mathscr{A}}(\cdot, \cdot)$ is additive and left exact in both variables (and in particular the second). Also, for every $A \in \mathrm{Ob}(\mathscr{A})$, the representable presheaf $\mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is a sheaf for the canonical topology by problem A.3.6(c), so we get the factorization of the statement. Finally, the functor h is left exact because the sheafification functor $\mathrm{PSh} \to \mathrm{Sh}$ is exact and isomorphic to the identity functor on Sh, so any complex of sheaves $0 \to F_1 \to F_2 \to F_3$ that is exact in PSh is also exact in Sh.

(c). By question (b) and Lemma II.2.3.2 , it suffices to show that h sends surjections to surjections. Let $f : A \to B$ be a surjective morphism, and let $C \in \mathrm{Ob}(\mathscr{A})$. Let $u : C \to B$ be an element of $h_B(C)$. Choose a commutative diagram as in question (a). Then $f' : C' \to C$ is a covering family for the canonical topology of $\mathscr{A}$, and the morphism $u' : C' \to A$ gives an element of $h_A(C')$ whose image by $f^* : h_A(C') \to h_B(C')$ is $f \circ u' = f'^*(u)$. This shows that $f^* : h_B \to h_A$ is surjective in the category Sh.

□

A.4.4 Other embedding theorems

If we weaken the assumptions in Morita's theorem, we can still get interesting results. There are many variants, we will prove two here.

Let $\mathscr{A}$ be an abelian category, Q an object of $\mathscr{A}$ and $R = \mathrm{End}_{\mathscr{A}}(Q)$. As explained in the paragraph before Theorem II.3.1.6 , we can see the functor $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$ as an additive left exact functor from $\mathscr{A}$ to $\mathbf{Mod}_R$.

Note that we are *not* assuming that Q is projective for now.

We assume that $\mathscr{A}$ admits all small colimits. From now on, we assume that $\mathscr{A}$ admits all small colimits. (You can mostly ignore the smallness condition. It basically means that you can take colimits indexed by all sets that are built out of sets like $\mathrm{Hom}_{\mathscr{A}}(A, B)$. The rigorous way to say it is that $\mathscr{A}$ is a $\mathscr{U}$-category, with $\mathscr{U}$ a universe, and that it admits limits indexed by $\mathscr{U}$-small categories.)

(a). Show that, for every right R-module M, the functor $\mathscr{A} \to \mathbf{Set}$, $A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A))$ is representable. We denote a pair representing this functor by $(M \otimes_R Q, \eta(M))$.

(b). Show that the functor $G = \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) : \mathscr{A} \to \mathbf{Mod}_R$ admits a left adjoint F.

(c). If M is a free right R-module, show that $\eta(M) : M \to G(F(M))$ is injective, and that it is bijective if M is also finitely generated.

(d). Let $\mathscr{A}' \subset \mathscr{A}$ be a full subcategory of $\mathscr{A}$ that is stable by taking finite limits and finite colimits.

(i) Show that $\mathscr{A}'$ is an abelian category and that the inclusion functor $\mathscr{A}' \to \mathscr{A}$ is exact.

From now on, we assume that the category $\mathscr{A}'$ is small. Suppose that Q is a generator of $\mathscr{A}$. For every object A of $\mathscr{A}$, consider the surjective morphism $q_A : \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q \to A$ of Proposition II.3.1.3(i)(e) . Let $P = \bigoplus_{A \in \mathrm{Ob}(\mathscr{A}')} \bigoplus_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} Q$; for every $A \in \mathrm{Ob}(\mathscr{A}')$, we have a surjective morphism $p_A : P \to A$, which is given by q_A on the summand of P indexed by A and by 0 on the other summands. Let $S = \mathrm{End}_{\mathscr{A}}(P)$, and consider the functor $G' = \mathrm{Hom}_{\mathscr{A}}(P, \cdot) : \mathscr{A} \to \mathbf{Mod}_S$.

(ii) Show that G' is faithful, and that it is exact if Q is projective.

(iii) If Q is projective, show that the restriction of G' to $\mathscr{A}'$ is fully faithful.

From now on, we also assume that small filtrant colimits are exact in $\mathscr{A}$ and that Q is a generator of $\mathscr{A}$. [16] We do not assume that Q is projective.

(e). The goal of this question is to show that G is fully faithful. Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ whose objects are finite direct sums of copies of P, and $\mathscr{D}$ the full subcategory of $\mathbf{Mod}_R$ whose objects are finitely generated free R-modules. We denote by $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ the functor $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)_{|\mathscr{C}}$, and by $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ the functor $M \longmapsto \mathrm{Hom}_R(\cdot, M)_{|\mathscr{D}}$.

(i) Show that G induces an equivalence of categories $\mathscr{C} \to \mathscr{D}$.

(ii) Show that $h' : \mathbf{Mod}_R \to \mathrm{PSh}(\mathscr{D})$ is fully faithful.

(iii) Assuming that $h : \mathscr{A} \to \mathrm{PSh}(\mathscr{C})$ is fully faithful, show that $G : \mathscr{A} \to \mathbf{Mod}_R$ is fully faithful.

(iv) Show that h is left exact and faithful, and that, for any morphism f of $\mathscr{A}$, if $h(f)$ is surjective, then f is surjective.

(v) For any object B of $\mathscr{C}$, any morphism $f : B \to A$ in $\mathscr{A}$ and any object C of $\mathscr{A}$, show

[16] In other words, $\mathscr{A}$ is a Grothendieck abelian category.

that the map

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}\, f, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(B) \to h(A)), h(C))$$

is an isomorphism. (Hint : problem A.4.2.)

Let A be an object of $\mathscr{A}$. Denote by $\mathscr{C}/A$ the category of pairs (B, f), where $B \in \mathrm{Ob}(\mathscr{C})$ and $f : B \to A$ is a morphism of $\mathscr{A}$; a morphism $u : (B, f) \to (B', f')$ is a morphism $u : B \to B'$ such that $f = f' \circ u$.

Let I be the set of finite subsets of $\mathrm{Ob}(\mathscr{C}/A)$, ordered by inclusion; the corresponding category is clearly filtrant. Define a functor $\xi : I \to \mathscr{C}$ by sending a finite set $J = \{(B_1, f_1), \dots, (B_n, f_n)\}$ to $B_1 \oplus \dots \oplus B_n$; note that $\xi(J)$ comes with a morphism to A, given by $\begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}$.

(vi) Show that the canonical morphism $\varinjlim_{J \in I} h(\xi(J)) \to h(A)$ is an epimorphism.

(vii) Show that the canonical morphism $\varinjlim_{J \in I} \mathrm{Im}(h(\xi(J)) \to h(A)) \to h(A)$ is an isomorphism.

(viii) Show that the canonical morphism $\varinjlim_{J \in I} \xi(J) \to A$ is an epimorphism.

(ix) Show that the canonical morphism $\varinjlim_{J \in I} \mathrm{Im}(\xi(J) \to A) \to A$ is an isomorphism.

(x) For C another object of $\mathscr{A}$, show that $\mathrm{Hom}_{\mathscr{A}}(A, C) \xrightarrow{h} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C))$ is bijective.

(f). The goal of this question is to show that F is exact.

(i) Show that it suffices to prove that F preserves injections.

Let $f : M \to N$ be a morphism in $\mathbf{Mod}_R$.

(ii) Suppose that M is finitely generated free and that N is free. Show that the composition of $\eta(\mathrm{Ker}\, f) : \mathrm{Ker}\, f \to G(F(\mathrm{Ker}\, f))$ and of the canonical morphism $G(F(\mathrm{Ker}\, f)) \to \mathrm{Ker}(G(F(f)))$ is an isomorphism.
<u>Hint</u> : Use the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta(M)"'] & N \arrow[d, "\eta(N)"] \\ G(F(M)) \arrow[r, "G(F(f))"'] & G(F(N)) \end{tikzcd}$$

(iii) Suppose that M is finitely generated, that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : question 1(c).)

(iv) Suppose that N is free and that f is injective. Show that $F(f)$ is injective. (Hint : M is the union of its finitely generated submodules.)

(v) Suppose that f is injective. Show that we can find a commutative diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r] & N' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

such that f' is injective and N' is free.

(vi) Suppose that f is injective. Applying F to the diagram of (v) and using PS4.1(a), show that $F(f)$ is injective.

In conclusion, here are our embedding results so far :

(1) If $\mathscr{A}$ admits small colimits and a projective generator, we have shown that every small full abelian subcategory $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}' \subset \mathscr{A}$ is exact admits a fully faithful exact functor into a category of modules over some ring.

(2) If $\mathscr{A}$ is a Grothendieck abelian category (it admits small colimits, small filtrant colimits are exact, and $\mathscr{A}$ has a generator), then we have shown that $\mathscr{A}$ admits a fully faithful left exact functor into a category of modules over a ring, with an exact left adjoint. This is known as the Gabriel-Popescu embedding theorem.

(3) We also have Morita's theorem (Theorem II.3.1.6) : If $\mathscr{A}$ admits small colimits and has a projective generator P such that the functor $\mathrm{Hom}_{\mathscr{A}}(P, \cdot)$ commutes with small direct sums, then $\mathscr{A}$ is equivalent to a category of modules over a ring.

Solution.

(a). This is similar to what happens in the proof of Theorem II.3.1.6 , with a few changes to reflect the fact that $\eta(M)$ is not an isomorphism anymore. We denote by $\Phi_M : \mathscr{A} \to \mathbf{Set}$ the functor $A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A))$.

(1) If $M = R$, then the functor $\Phi_M : A \longmapsto \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, A)) \simeq \mathrm{Hom}_{\mathscr{A}}(Q, A)$ is representable by Q, and the morphism $\eta(M) \in \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathscr{A}}(Q, Q)) = \mathrm{Hom}_R(R, R)$ is the identity of R.

(2) If $M = R^{(X)}$ with X a set, then we have isomorphisms of functors

$$\begin{aligned} \Phi_M = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)) \simeq \prod_X \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathscr{A}}(Q, \cdot)) \simeq \prod_X \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) \\ \simeq \mathrm{Hom}_{\mathscr{A}}(Q^{(X)}, \cdot), \end{aligned}$$

so the functor Φ_M is representable by $Q^{(X)}$, and the morphism $\eta(M) \in \mathrm{Hom}_R(R^{(X)}, \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)}))$ is the canonical morphism $R^{(X)} = \mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$ of Subsection I.5.4.2 (which might not be an isomrophism).

(3) In general, we chose an exact sequence $R^{(X)} \xrightarrow{u} R^{(Y)} \to M \to 0$, with X and Y sets. This induces morphisms of functors $\Phi_M \to \Phi_{R^{(Y)}} \xrightarrow{u^*} \Phi_{R^{(X)}}$, and the second of these comes from a morphism $f : Q^{(X)} \to Q^{(Y)}$ between the objects representing $\Phi_{R^{(X)}}$ and $\Phi_{R^{(Y)}}$ such that the following diagram commutes :

(*)
$$\begin{tikzcd} R^{(X)} \arrow[r, "\eta(R^{(X)})"] \arrow[d, "u"'] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)}) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(Q,f)"] \\ R^{(Y)} \arrow[r, "\eta(R^{(Y)})"] & \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(Y)}) \end{tikzcd}$$

Let $B = \mathrm{Coker}\, f$. By Subsection I.5.4.2, there is a canonical morphism $M = \mathrm{Coker}(G(f)) \to G(\mathrm{Coker}\, f) = G(B)B$, which might not be an isomorphism. This induces a morphism of functors $\mathrm{Hom}_{\mathscr{A}}(B, \cdot) \to \mathrm{Hom}_R(G(B), G(\cdot)) \to \mathrm{Hom}_R(M, G(\cdot)) = \Phi_M$. To show that this morphism is an isomorphism, we use, as in the proof of Theorem II.3.1.6, that we have a commutative diagram with exact columns for every $A \in \mathrm{Ob}(\mathscr{A})$:

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \Phi_M(A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{A}}(B, A) \arrow[d] \\ \Phi_{R^{(Y)}}(A) \arrow[r, "\sim"] \arrow[d, "u^*(A)"'] & \mathrm{Hom}_{\mathscr{A}}(Q^{(Y)}, A) \arrow[d, "\mathrm{Hom}_{\mathscr{A}}(f,A)"] \\ \Phi_{R^{(X)}}(A) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathscr{A}}(Q^{(X)}, A) \end{tikzcd}$$

(The fact that the columns are exact only uses the left exact of the Hom functors.) We get the morphism $\eta(M) : M \to \mathrm{Hom}_{\mathscr{A}}(Q, B)$ by taking the morphism between the cokernels of the vertical maps in the commutative square (*).

(b). This follows from (a) and from Proposition I.4.7; in fact, we have $F(M) = M \otimes_R Q$. Also, by the proof of that proposition, the morphisms $\eta(M) : M \to \mathrm{Hom}_{\mathscr{A}}(Q, M \otimes_R Q) = G(F(M))$ define a morphism of functors $\mathrm{id}_{\mathbf{Mod}_R} \to G \circ F$, which is the unit of the adjunction.

(c). If $M = R^{(X)}$ with X a set, we saw in the solution of (a) that $M \otimes_X Q = Q^{(X)}$ and that $\eta(M) : R^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$ is the canonical morphism $\mathrm{Hom}_{\mathscr{A}}(Q, Q)^{(X)} \to \mathrm{Hom}_{\mathscr{A}}(Q, Q^{(X)})$. If X is finite, this morphism is an isomorphism because $\mathrm{Hom}_{\mathscr{A}}(Q, \cdot)$, being an additive functor, commutes with finite coproducts. In general, we claim that $\eta(M)$ is injective. Let $Q^X = \prod_X Q$; we have a family of morphisms $(q_x : Q^{(X)} \to Q)_{x \in X}$, such that the composition of q_x with the morphism $Q \to Q^{(X)}$

corresponding to $y \in X$ is id_Q if $y = x$, and 0 if $y \neq x$. This gives a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{A}}(Q,Q)^{(X)} \arrow[r, "\eta(M)"] \arrow[d, "(1)"'] & \mathrm{Hom}_{\mathscr{A}}(Q,Q^{(X)}) \arrow[d] \\ \mathrm{Hom}_{\mathscr{A}}(Q,Q)^{X} \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{A}}(Q,Q^{X}) \end{tikzcd}$$

where the map (1) is the inclusion of the direct sum into the direct product (in the category of abelian groups), hence an injection. So $\eta(M)$ is injective.

(d). (i) The category $\mathscr{A}'$ is clearly preadditive. It is additive, because finite products in $\mathscr{A}$ of objects of $\mathscr{A}'$ are in $\mathscr{A}'$ by hypothesis, and they are finite products in $\mathscr{A}'$ by the fullness of $\mathscr{A}'$. For the same reason, every morphism in $\mathscr{A}'$ has a kernel and a cokernel, which are its kernel and its cokernel in $\mathscr{A}$. If f is a morphism in $\mathscr{A}'$, then the canonical morphism from its coimage to its image in $\mathscr{A}'$ is the same asthe canonical morphism from its image to its coimage in $\mathscr{A}$, so it is an isomorphism. This shows that $\mathscr{A}'$ is an abelian. We have seen in the construction of finite products, kernels and cokernels in $\mathscr{A}'$ that the inclusion functor from $\mathscr{A}'$ to $\mathscr{A}$ commutes with these, so it commutes with finite limits and colimits, so it is exact.

(ii) By the universal property of the direct sum, the functor G' is isomorphism to $\prod_{A \in \mathrm{Ob}(\mathscr{A})} \prod_{\mathrm{Hom}_{\mathscr{A}}(Q,A)} \mathrm{Hom}_{\mathscr{A}}(Q,\cdot)$. As Q is a generator, the functor $\mathrm{Hom}_{\mathscr{A}}(Q,\cdot)$ is faithful; so G' is also faithful.

If Q is projective, then, by Lemma II.2.4.3, P is also projective, and then the functor $\mathrm{Hom}_{\mathscr{A}}(P,\cdot)$ is exact.

(iii) Let A and B be objects of $\mathscr{A}'$, and let $u : G'(A) \to G'(B)$ be a morphism of right S-modules. We want to show that there exists $g \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ such that $G'(g) = u$. By construction of P, we have surjective morphisms $p_A : P \to A$ and $p_B : P \to B$. As G' is exact, we get a diagram with exact rows

$$\begin{tikzcd} S \arrow[r, "G'(p_A)"] \arrow[d, dashed, "v"] & \mathrm{Hom}_{\mathscr{A}}(P,A) \arrow[r] \arrow[d, "u"] & 0 \\ S \arrow[r, "G'(p_B)"] & \mathrm{Hom}_{\mathscr{A}}(P,B) \arrow[r] & 0 \end{tikzcd}$$

As S is a projective in $\mathbf{Mod}_S$, there exists a morphism $v : S \to S$ making the diagram. This morphism is of the form $g \longmapsto f \circ g$, with $f = v(1) \in S = \mathrm{Hom}_{\mathscr{A}}(P,P)$. Consider the diagram with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(p_A) \arrow[r, "i"] & P \arrow[r, "p_A"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, dashed, "g"] & 0 \\ & & P \arrow[r, "p_B"'] & B \arrow[r] & 0 \end{tikzcd}$$

To show that there exists a morphism $g : A \to B$ making this diagram commute, it suffices to show that $p_B \circ f \circ i = 0$. As G' is faithful, it suffices to show that $G'(p_B) \circ G'(f) \circ G'(i) = 0$; as $G'(p_B) \circ G'(f) = G'(p_B) \circ u = v \circ G'(p_A)$, we have $G'(p_B) \circ G'(f) \circ G'(i) = v \circ G'(p_A \circ i) = 0$.

To finish the proof, it suffices to prove that $G'(g) = u$. We know that $G'(g) \circ G'(p_A) = G'(p_B) \circ v = u \circ G'(p_A)$, so the equality $G'(g) = u$ follows from the fact that $G'(p_A)$ is surjective.

(e). The idea of this seemingly strange procedure is that we are showing that teh subcategory $\mathscr{C}$ (resp. $\mathscr{D}$), that contains a generator, "generates" $\mathscr{A}$ (resp. $\mathbf{Mod}_R$) in some precise sense (this notion is called being *strictly generating*, see Definition 5.3.1 of [8]); so the equivalence $\mathscr{C} \to \mathscr{D}$ of (i) will extend to a fully faithful functor $\mathscr{A} \to \mathbf{Mod}_R$. The proof of this fact is a specialization to our case of the proof of Theorem 5.3.6 of [8].

(i) If X is a finite set, then the canonical morphism $R^{(X)} = \mathrm{Hom}_{\mathscr{A}}(Q,Q)^{(X)} \to G(Q^{(X)}) = \mathrm{Hom}_{\mathscr{A}}(Q,Q^{(X)})$ is an isomorphism. Just as in the second paragraph of the proof of Theorem II.3.1.6, we deduce that, if X and Y are finite sets, then the map $G : \mathrm{Hom}_{\mathscr{A}}(Q^{(X)},Q^{(Y)}) \to \mathrm{Hom}_R(R^{(X)},R^{(Y)})$ is bijective. (As Y is finite, we only use the fact that additive functors commute with finite direct sums, and so we don't need Q to have the extra property of that theorem.)

We have just shown that the restriction of the functor G to $\mathscr{C}$ is fully faithful, and that its essential image is $\mathscr{D}$. So G induces an equivalence of categories from $\mathscr{C}$ to $\mathscr{D}$ by Corollary I.2.3.9.

(ii) Let $H : \mathrm{PSh}(\mathscr{D}) \to \mathbf{Mod}_R$ be the functor sending a presheaf $\mathscr{F}$ to $\mathscr{F}(R)$ (and a morphism $u : \mathscr{F} \to \mathscr{G}$ of preasheaves to $u(R) : \mathscr{F}(R) \to \mathscr{G}(R)$).

For every right R-module M, we have a canonical isomorphism $H(h'(M)) = \mathrm{Hom}_R(R,M) \xrightarrow{\sim} M$, $u \longmapsto u(1)$. This defines an isomorphism of functors $H \circ h' \xrightarrow{\sim} \mathrm{id}_{\mathbf{Mod}_R}$. Let M and N be right R-modules. Then we get a sequence of morphisms of abelian groups

$$\mathrm{Hom}_R(M,N) \xrightarrow{h'} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(h'(M),h'(N)) \xrightarrow{H} \mathrm{Hom}_R(H(h'(M)),H(h'(N)) \simeq \mathrm{Hom}_R(M,N),$$

whose composition is equal to $\mathrm{id}_{\mathrm{Hom}_R(M,N)}$. So the first map is injective. Also, as the functors $\mathrm{Hom}_{\mathscr{A}}(\cdot,M)$ and $\mathrm{Hom}_{\mathscr{A}}(\cdot,N)$ are additive, the presheaves $h'(M)$ $h'(N)$ commute with finite direct sums, so they are determined by their sections on R; this shows that the second map in the sequence above is also injective; as it is surjective, it must be bijective, and this implies that $h' : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(h'(M),h'(N))$ is bijective.

(iii) We have a diagram of categories and functors

$$\begin{tikzcd} \mathscr{A} \arrow[r, "G"] \arrow[d, "h"'] & \mathbf{Mod}_R \arrow[d, "h'"] \\ \mathrm{PSh}(\mathscr{C}) & \mathrm{PSh}(\mathscr{D}) \arrow[l, "\Phi"'] \end{tikzcd}$$

where Φ is the equivalence of categories induced by the equivalence $\mathscr{C} \to \mathscr{D}$ of (i). This diagram is not necessarily commutative, but we have an isomorphism of functors $\Phi \circ h \circ G \simeq h'$. We already know that Φ and h' are fully faithful, so, if h is fully faithful, we can conclude that G is fully faithful.

(iv) The functor h is left exact because $A \longmapsto \mathrm{Hom}_{\mathscr{A}}(\cdot, A)$ is.

Let $f : A \to B$ be a morphism in $\mathscr{A}$ such that $h(f) = 0$. Then the R-linear map $h(A)(Q) = \mathrm{Hom}_{\mathscr{A}}(Q, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(Q, B) = h(B)(Q)$ is 0; in other words, we have $G(f) = 0$. As G is faithful (by Proposition II.3.1.3, we get that $f = 0$. So h is faithful.

Let $f : A \to B$ be a morphism in $\mathscr{A}$ such that $h(f)$ is surjective. Let $g_1, g_2 : B \to C$ be two morphisms such that $g_1 \circ f = g_2 \circ f$. Then $h(g_1) \circ h(f) = h(g_2) \circ h(f)$, so $h(g_1) = h(g_2)$ by the surjectivity of $h(f)$; as h is faithul, this implies that $g_1 = g_2$. So f is an epimorphism.

(v) By problem A.4.2 and the left exactness of h, we have isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}\, f, C) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C))$$

and

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(B) \to h(A)), h(C)) \\ &\qquad \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Ker}(h(f)), h(C))) \\ &\qquad \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}(f)), h(C))). \end{aligned}$$

By these isormophisms, the map that we are trying to understand corresponds to the map

$$\begin{aligned} u : &\mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C)) \\ &\qquad \to \mathrm{Ker}(\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C)) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}(f)), h(C))) \end{aligned}$$

induced by h. As $h : \mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(B), h(C))$ is an isomorphism (by Yoneda's lemma, applied to the representable presheaf $h(B)$ on $\mathscr{C}$) and the map $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}\, f, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(\mathrm{Ker}\, f), h(C))$ is injective (because h is faithful), the map u is bijective.

(vi) Let C be an object of $\mathscr{C}$. Then applying the morphism $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ to C gives the map
$$\varinjlim_{J\in I} \mathrm{Hom}_{\mathscr{A}}(C, \xi(J)) \to \mathrm{Hom}_{\mathscr{A}}(C, A).$$
Saying that this is surjective means that every morphism $f : C \to A$ factors as $C \to \xi(J) \to A$ for $J \in I$, which is true : just take $J = \{(C, f)\}$ and $C \to \xi(J)$ equal to id_C. So $\varinjlim_{J\in I} h(\xi(J))(C) \to h(A)(C)$ is surjective for every $C \in \mathrm{Ob}(\mathscr{C})$, which implies that $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ is an epimorphism.

(vii) As I is filtrant, we have $\mathrm{Im}(\varinjlim_{J\in I} h(\xi(J)) \to h(A)) = \varinjlim_{J\in I} \mathrm{Im}(h(\xi(J)) \to h(A))$. So the result follows immediately from (vi).

(viii) Note that $\varinjlim_{J\in I} h(\xi(J)) \to h(A)$ factors as $\varinjlim_{J\in I} h(\xi(J)) \to h(\varinjlim_{J\in I} \xi(J)) \to h(A)$, where the first morphism is that of Subsection I.5.4.2 and the second is the image by h of the canonical morphism $\varinjlim_{J\in I} \xi(J) \to A$. By (vi), the second morphism is an epimorphism, so, by (iv), the morphism $\varinjlim_{J\in I} \xi(J) \to A$ is an epimorphism.

(ix) As in (vii), this follows immediately from (viii) and from the fact that I is filtrant.

(x) The map $h : \mathrm{Hom}_{\mathscr{A}}(A, C) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C))$ is equal to the composition
$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}}(A, C) &\overset{\sim}{\to} \mathrm{Hom}_{\mathscr{A}}(\varinjlim_{J\in I} \mathrm{Im}(\xi(J) \to A), C) \text{ by (ix)} \\
&\simeq \varprojlim_{J\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathscr{A}}(\mathrm{Im}(\xi(J) \to A), C) \\
&\overset{\sim}{\to} \varprojlim_{J\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathrm{Im}(h(\xi(J)) \to h(A)), h(C)) \text{ by (v)} \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\varinjlim_{J\in I} \mathrm{Im}(h(\xi(J)) \to h(A)), h(C)) \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h(A), h(C)) \text{ by (vii).}
\end{aligned}$$

(f). (i) We already know that F is right exact, because it is a left adjoint (Proposition II.2.3.3.) So the statement follows from Lemma II.2.3.2.

(ii) We have a commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] \arrow[d, "\eta(\mathrm{Ker}\, f)"'] & M \arrow[r, "f"] \arrow[dd, "\eta(M)"'] & N \arrow[dd, "\eta(N)"'] \\
 & G(F(\mathrm{Ker}\, f)) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ker}(G(F(f))) \arrow[r] & G(F(M)) \arrow[r, "G(F(f))"'] & G(F(N))
\end{tikzcd}$$

where the unmarked vertical one is the canonical morphism. So the result follows from a diagram chase in $\mathbf{Mod}_R$.

(iii) As M is finitely generated, there exists a surjective R-linear map $g : M' \to M$, with M' free of finite type. As F is right exact, the morphism $F(g)$ is also surjective. So, by question A.4.1(c), we have $\mathrm{Ker}(F(f)) = F(g)(\mathrm{Ker}(F(f \circ g)))$. Hence, to prove that $\mathrm{Ker}(F(f)) = 0$, it suffices to show that the composition $\mathrm{Ker}(F(f \circ g)) \to F(M') \stackrel{F(g)}{\to} F(M)$ is 0. As G is conservative, it suffices to prove this after applying G, and as G is left exact, it suffices to prove that the composition $\mathrm{Ker}(G(F(f \circ g))) \to G(F(M')) \stackrel{G(F(g))}{\to} G(F(M))$ is 0. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f \circ g) \arrow[r, "u"] \arrow[d, "(1)"'] & M' \arrow[r, "g"] \arrow[d, "\eta(M')"] & M \arrow[d, "\eta(M)"] \\
\mathrm{Ker}(G(F(f \circ g))) \arrow[r, "v"'] & G(F(M')) \arrow[r, "G(F(g))"'] & G(F(M))
\end{tikzcd}$$

We know that $g \circ u = 0$ because $0 = \mathrm{Ker}\, f = g(\mathrm{Ker}(f \circ g))$ by question A.4.1(c), so $\eta(M) \circ g \circ u = 0$. As the morphism (1) is surjective by (ii), this implies that $G(F(g)) \circ v = 0$, as desired.

(iv) Let I be the set of all the finitely generated submodules of M; for $i \in I$, we denote the corresponding submodule by M_i. Then I is filtrant, and $M = \varinjlim_{i \in I} M_i$. As F is a left adjoint, the canonical morphism $\varinjlim_{i \in I} F(M_i) \to F(M)$ is an isomorphism by Proposition I.5.4.3 , and $F(f)$ corresponds to $\varinjlim_{i \in I} F(f_{|M_i})$ by this isomorphism. For each $i \in I$, the morphism $F(f_{|M_i})$ is injective by (iii). As filtrant colimits are exact in $\mathbf{Mod}_R$ by Corollary II.2.3.4, this implies that $F(f)$ is also injective.

(v) Let $g : N' \to N$ be a surjective R-linear map with N' free, let $M' = N' \times_N M$ and let $f' : M' \to M$ and $g' : N' \to N$ be the two projections. The morphism g' is surjective by Corollary II.2.1.16. As f is injective, so is f' (it is true and easy to prove that in any abelian category the pullback of an injective morphism is injective, but in the category of R-modules it is immediate). Let $i : K = \mathrm{Ker}\, g \to N'$ and $i' : \mathrm{Ker}(g') \to M'$ be the canonical injections. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(g') \arrow[r, "i'"] \arrow[d, dashed, "u"'] & M' \arrow[r, "g'"] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & K \arrow[r, "i"'] & N' \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

As $g \circ f' \circ i' = f \circ g \circ i' = 0$, there exists a unique morphism $u : \mathrm{Ker}(g') \to K$ such that $i \circ u = f' \circ i'$. We want to show that u is an isomorphism. As f' is injective, the map $f' \circ i'$ is injective, hence u is also injective. To prove that u is surjective, we can do a bit of diagram chasing : Let $x \in K$. Then $g(i(x)) = 0 = f(0)$, so $y = (i(x), 0) \in N' \times M$ is actually in M', and we have $f'(y) = i(x)$ and $g'(y) = 0$. In particular, there exists $z \in \mathrm{Ker}(g')$ such that $y = i'(z)$. As $i(u(z)) = f'(i'(z)) = i(x)$ and i is injective, we get that $x = u(z)$.

(vi) Applying F to the diagram of (v) gives a commutative diagram with exact rows

$$\begin{tikzcd} F(K) \arrow[r] \arrow[d, equal] & F(M') \arrow[r] \arrow[d, "F(f')"'] \arrow[dr, phantom, "(*)"] & F(M) \arrow[r] \arrow[d, "F(f)"'] & 0 \\ F(K) \arrow[r] & F(N') \arrow[r] & F(N) \arrow[r] & 0 \end{tikzcd}$$

By (iv), the map $F(f')$ is injective. Also, as F is a left adjoint, it commutes with colimits, so the square (*) is cocartesian. By question PS4.1(a), the morphism $F(M') \to F(N') \times_{F(N)} F(M)$ is surjective. As $F(f')$ is injective, this morphism is also injective, so it is an isomorphism; in other words, the square (*) is cartesian. By Corollary II.2.1.16, the morphism $F(f)$ is injective.

$\square$

A.5 Problem set 5

A.5.1 Free presheaves

Let $\mathscr{C}$ be a category and R be a ring.

(a). Show that the forgetful functor $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C})$ has a left adjoint $\mathscr{F} \longmapsto R^{(\mathscr{F})}$.

(b). If X is an object of $\mathscr{C}$ and $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is the corresponding representabe presheaf, we write $R^{(X)}$ for $R^{(h_X)}$. Show that there is an isomorphism of additive functors from $\mathrm{PSh}(\mathscr{C}, R)$ to ${}_R\mathbf{Mod}$ (where $\mathscr{F}$ is the variable):

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(X)}, \mathscr{F}) \simeq \mathscr{F}(X).$$

(c). Suppose that $\mathscr{C}$ is equipped with a Grothendiech pretopology. If $\mathscr{F}$ is a sheaf for this pretopology, is $R^{(\mathscr{F})}$ always a sheaf ?

Solution.

(a). If $\mathscr{F}$ is a presheaf on $\mathscr{C}$, we define a presheaf $R^{(\mathscr{F})}$ by setting, for every $X \in \mathrm{Ob}(\mathscr{C})$, $R^{(\mathscr{F})}(X) = R^{(\mathscr{F}(X))}$; if $f : X \to Y$ is a morphism of $\mathscr{C}$, then we take for $R^{(\mathscr{F})}(f)$ the only R-linear extension of $\mathscr{F}(f)$. The presheaf $R^{(\mathscr{F})}$ is an object of $\mathrm{PSh}(\mathscr{C}, R)$, and its construction is clearly functorial in $\mathscr{F}$.

Now we show that the functor $\mathscr{F} \longmapsto R^{(\mathscr{F})}$ is left adjoint to the forgetful functor. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a presheaf of R-modules. If $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then we define a morphism of presheaves $\alpha(u) : R^{(\mathscr{F})} \to \mathscr{G}$ by taking, for every $X \in \mathrm{Ob}(\mathscr{C})$, the morphism $\alpha(u)(X) : R^{(\mathscr{F}(X))} \to \mathscr{G}(X)$ to be the unique R-linear extension of $u(X) : \mathscr{F}(X) \to \mathscr{G}(X)$. By the universal property of the free R-module on a set, the map $\alpha : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(\mathscr{F})}, \mathscr{G})$ is bijective, and it is easy to check that it defines a morphism of functors on $\mathrm{PSh}(\mathscr{C})^{\mathrm{op}} \times \mathrm{PSh}(\mathscr{C}, R)$.

(b). We have an isomorphism of functors $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(X)}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathscr{F})$ given by question (a), and an isomorphism of functors $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(X)$ given by the Yoneda lemma.

(c). No. Let X be a topological space, let S be a singleton, and let $\mathscr{F}$ be the presheaf on X sending every open subset U of X to S. Then $R^{(\mathscr{F})}(U) = R$ for every open subset U of X, but a sheaf of R-modules on a topological space must take the value $\{0\}$ on $\varnothing$, so $R^{(\mathscr{F})}$ is not a sheaf (unless R is the zero ring).

□

A.5.2 Constant presheaves and sheaves

Let $(\mathscr{C}, \mathscr{T})$ be a site. The *constant presheaf* on $\mathscr{C}$ with value S is the functor $\underline{S}_{psh} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending any object to S and any morphism to id_S. The *constant sheaf* on $\mathscr{C}_{\mathscr{T}}$ with value S is the sheafification of the constant presheaf on $\mathscr{C}$ with value S; we will denote it by $\underline{S}$.

(a). if $\mathscr{X} = (X_i \to X)_{i \in I}$ is a covering family, calculate $\check{\mathrm{H}}^0(\mathscr{X}, \underline{S}_{psh})$.

(b). Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of the topological space $[0, 1]$, with the usual topology. Show that $(\underline{S}_{psh})^+$ is a sheaf if and only if $\mathrm{card}(S) \leq 1$.

(c). Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of a locally connected topological space X. Show that, for every open subset U of X, we have $\underline{S}(U) = S^{\pi_0(U)}$.

Solution.

(a). Suppose that $I = \varnothing$. Then $\prod_{i \in I} \mathscr{F}(X_i)$ and $\prod_{i,j \in I} \mathscr{F}(X_i \times_X X_j)$ are both isomorphic to the terminal object of $\mathbf{Set}$, i.e. to a singleton, so $\check{\mathrm{H}}^0(\mathscr{X}, \underline{S}_{psh})$ is a singleton.

Suppose that $I \neq \varnothing$. Then $\prod_{i \in I} \mathscr{F}(X_i) = S^I$. Also, for all $i, j \in I$, the maps $S = \mathscr{F}(X_i) \to \mathscr{F}(X_i \times_X X_j) = S$ and $S = \mathscr{F}(X_j) \to \mathscr{F}(X_i \times_X X_j) = S$ induced by the two projections are id_S. Let $s = (s_i)_{i \in I} \in S^I$. Then $s \in \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ if and only if, for all $i, j \in I$, the images of s by the projections from S^I to its ith and jth factor are equal, that is, if and only if $s_i = s_j$ for all $i, j \in I$. So the diagonal embedding $S \subset S^I$ induces a bijection $S \xrightarrow{\sim} \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$.

(b). Let $\mathscr{F} = (\underline{S}_{psh})^+$, and let's pretend that we have not read the next question yet.

Suppose that $\mathrm{card}(S) \leq 1$. If S is a singleton, then, for every open cover $\mathscr{U} = (U_i)_{i \in I}$

of an open subset U of $[0,1]$, the canonical map $\mathscr{F}(U) \to \prod_{i\in I}\mathscr{F}(U_i)$ and the two maps $\prod_{i\in I}\mathscr{F}(U_i) \to \prod_{i,j}\mathscr{F}(U_i\cap U_j)$ are isomorphisms, so $\mathscr{F}(U) \xrightarrow{\sim} \check{\mathrm{H}}^0(\mathscr{U},\mathscr{F})$. If S is empty, this stays true as long as U and all the U_i are nonempty; as every open cover of a nonempty open set can be refined by an open cover that has only nonempty elements, we deduce again that $\mathscr{F}$ is a sheaf.

Suppose that $\mathscr{F}$ is a sheaf. Let $U_1 =]1/4,1/2[$, $U_2 =]1/2,3/4[$ and $U = U_1\cup U_2$; we denote by $\mathscr{U}$ the open cover (U_1,U_2) of U. As $U_1\cap U_2\varnothing$, question (a) implies that $\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F}) = S\times S$, and that the canonical map $S = \mathscr{F}(U)\to\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F})\to S\times S$ is the diagonal embedding. This is not bijection if $\mathrm{card}(S)\geq 2$, so we must have $\mathrm{card}(S)\leq 1$.

(c). Write $\mathscr{F} = (\underline{S}_{psh})^+$; by (a), the set $\mathscr{F}(\varnothing)$ is a singleton and we have $\mathscr{F}(V) = S$ for every nonempty open subset V of X.

Let$U$ be an open subset of X. If U is empty, we already know that $\underline{S}(U)$ is a singleton, hence isomorphic to $S^{\varnothing}$. Suppose that U is not empty. As U is locally connected, all its connected components are open (as well as closed), so we have $U = \coprod_{C\in\pi_0(U)} C$ as a topological space. Using the open cover $\{C\in\pi_0(U)\}$ of U, we see that the map $\underline{S}(U)\to\prod_{C\in\pi_0(U)}\underline{S}(C)$ must be bijective. So it suffices to show that, if U is connected and nonempty, then the canonical map $S = \mathscr{F}^+(U)\to\underline{S}(U)$ is bijective.

Let U be a nonempty connected subset of X, and let $\mathscr{U} = (U_i)_{i\in I}$ be an open cover of U. After replacing $\mathscr{U}$ by a refinement, we may assume that all the U_i are nonempty. For every $i\in I$, we denote by $I(i)$ the set of $j\in I$ such that there exists a sequence $i_0 = i, i_1,\dots,i_n = j$ of elements of I such that $U_{i_{r-1}}\cap U_{i_r}\neq\varnothing$ for every $r\in\{1,\dots,n\}$, and we set $V_i = \bigcup_{j\in I(i)} U_j$. Then the sets $I(i)$ form a partition of I. If we choose a subset K of I such that K intersects each $I(i)$ in a singleton, then $V_i\cap V_j = \varnothing$ if $i,j\in K$ and $i\neq j$, and $U = \bigcup_{i\in K} V_i$, so $U = \coprod_{i\in K} V_i$; but U is connected, so K has only on element. Now let $s = (s_i)_{i\in I}\in\prod_{i\in I}\mathscr{F}(U_i) = S^I$. If $i,j\in I$, the two images of s in $\mathscr{F}(U_i\cap U_j)$ are $s_{i|U_i\cap U_j}$ and $s_{j|U_i\cap U_j}$, so teh equality of these two images is an empty condition if $U_i\cap U_j = \varnothing$, and it equivalent to the condition that $s_i = s_j$ if $U_i\cap U_j\neq\varnothing$. But we have just shown that, for all $i,j\in I$, there exists a sequence $i_0 = i, i_1,\dots,i_n = j$ of elements of I such that $U_{i_{r-1}}\cap U_{i_r}\neq\varnothing$ for every $r\in\{1,\dots,n\}$, and we set $V_i = \bigcup_{j\in I(i)} U_j$. So $s\in\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F})$ if and only if $s_i = s_j$ for all $i,j\in I$; in other words, the map $S = \mathscr{F}(U)\to\check{\mathrm{H}}^0(\mathscr{U},\mathscr{F})$ is bijective. So we conclude that $S = \mathscr{F}(U)\xrightarrow{\sim}\check{\mathrm{H}}^0(U,\mathscr{F}) = \underline{S}(U)$.

□

A.5.3 Points

Let $(\mathscr{C},\mathscr{T})$ be a site. We are interested in the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ whose objects are functors $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})\to\mathbf{Set}$ that commutes with all small colimits and with finite limits, and whose morphisms are isomorphisms between such functors. [17]

A reference for many of the results of this problem is MacLane and Moerdijk, *Sheaves in geometry and logic* ([10]), especially Sections VII.5 and VII.6.

(a). Let $\mathscr{C}$ be an arbitrary category. Let $A : \mathscr{C} \to \mathbf{Set}$ be a functor. We denote by $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ the functor $\mathbf{Set} \to \mathrm{PSh}(\mathscr{C})$ sending a set S to the presheaf $X \longmapsto \mathrm{Hom}_{\mathbf{Set}}(A(X), S)$.

(i) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ commutes with all limits.

(ii) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ admits a left adjoint, which we will denote by $(\cdot) \otimes_{\mathscr{C}} A$, and that $((\cdot) \otimes_{\mathscr{C}} A) \circ h_{\mathscr{C}}$ is isomorphic to A. (Hint: First try to construct the adjoint on representable presheaves, and remember problem A.2.2(a).)

We say that the functor $A : \mathscr{C} \to \mathbf{Set}$ is *flat* if the functor $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ commutes with finite limits.

(iii) If A is flat, show that it commutes with all finite limits that exist in $\mathscr{C}$.

(iv) Suppose that $\mathscr{C}$ has all finite limits and that A commutes with finite limits. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$. If X, Y are objects of $\mathscr{C}/\mathscr{F}$, $x \in A(X)$ and $y \in A(Y)$, show that x and y represent the same element of $\mathscr{F} \otimes_{\mathscr{C}} A$ if and only if there exists an object Z of $\mathscr{C}$, morphisms $Z \to X$ and $Z \to Y$, and an element $z \in A(Z)$ whose images in $A(X)$ and $A(Y)$ are x and y respectively.

(v) If $\mathscr{C}$ has all finite limits, show that A is flat if and only if it commutes with finite limits. (Hint : To show that a functor commutes with finite limits, it suffices to show that it sends the final object to the final object and commutes with fibered products. You can admit this easy fact.)

(vi) Suppose that $\mathscr{C}$ has all finite limits. If $\mathscr{T}$ is the trivial pretopology on $\mathscr{C}$ (so that $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) = \mathrm{PSh}(\mathscr{C})$), show that $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ is equivalent to the category of flat functors $\mathscr{C} \to \mathbf{Set}$ (with morphisms being isomorphisms between these functors).

(b). Let $(\mathscr{C}, \mathscr{T})$ be a site. A flat functor $A : \mathscr{C} \to \mathbf{Set}$ is called *continuous* if, for every covering family $(X_i \to X)_{i \in I}$ in $\mathscr{C}$, the map $\coprod_{i \in I} A(X_i) \to A(X)$ is surjective.

For every $X \in \mathrm{Ob}(\mathscr{C})$, we denote by X^{sh} the sheafification of the representable presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$. This defines a functor $\mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$, that commutes with finite limits.

(i) Let $(f_i : X_i \to X)_{i \in I}$ be a covering family. We consider the morphisms

$$\coprod_{i,j \in I} (X_i \times_X X_j)^{\mathrm{sh}} \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{i \in I} X_i^{\mathrm{sh}} \xrightarrow{h} X^{\mathrm{sh}}$$

[17] The idea of this definition is that we are abstracting the formal properties of stalk functors on the category of sheaves on a topological space.

where $h = \coprod_{i\in I} f_i^{\mathrm{sh}}$ and f (resp. g) is equal on $(X_i \times_X X_j)^{\mathrm{sh}}$ to the image by $(.)^{\mathrm{sh}} : \mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ of the first (resp. second) projection $X_i \times_X X_j \to X_i$ (resp. $X_i \times_X X_j \to X_j$).

Show that h is the cokernel of (f, g) in the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$.

(ii) Let $A : \mathscr{C} \to \mathbf{Set}$ be a flat functor, and suppose that $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ factors as $\mathrm{PSh}(\mathscr{C}) \overset{(\cdot)^{\mathrm{sh}}}{\to} \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \overset{x_A}{\to} \mathbf{Set}$. Show that x_A is an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$.

(iii) If $A : \mathscr{C} \to \mathbf{Set}$ satisfies the hypothesis of the previous question, show that A is continuous. [18]

(c). Let $(C, \leq)$ be a preordered set. We see C as a category by taking $\mathrm{Hom}_C(a, b)$ to be a singleton if $a \leq b$, and empty otherwise.

(i) Let $(a_i)_{i\in I}$ be a family of objects of C. Give a description of $\coprod_{i\in I} a_i$ and $\prod_{i\in I} a_i$ in (pre)ordered set terms.

(ii) Give a similar translation of the property "C has all finite limits".

From now on, se suppose that C has all finite limits, and we fix a flat functor $A : C \to \mathbf{Set}$.

(iii) Show that $\mathrm{card}(A(a)) \leq 1$ for every $a \in C$.

(iv) Show that the set $I_A = \{a \in C \mid A(a) \neq \varnothing\}$ is a nonempty upper order ideal. (That is, if $a \in I_A$ and $a \leq b$, then $b \in I_C$.)

(v) If $\mathscr{T}$ is any Grothendieck pretopology on C, show that the points of $C_{\mathscr{T}}$ don't have any nontrivial automorphisms.

(vi) Suppose that any family $(a_i)_{i\in I}$ of elements of C has a least upper bound $\sup(a_i, i \in I)$. We say that a family of morphisms $(a_i \to a)_{i\in I}$ in $\mathscr{C}$ is covering if $a = \sup(a_i, i \in I)$. Suppose that this defines a pretopology on C. If A is continuous, show that I_A is a completely prime upper order ideal, that is, if $\sup(a_i, i \in I) \in I_A$, then at least one of the a_i is in I_A.

(d). Let X be a topological space, let $\mathscr{C} = \mathrm{Open}(X)$, and let $\mathscr{T}$ be the usual topology on X. Remember that a nonempty closed subset Z of X is called *irreducible* if, whenever $Z \subset Y_1 \cup Y_2$ with Y_1, Y_2 closed subsets of X, we have $Z \subset Y_1$ or $Z \subset Y_2$.

(i) Show that a nonempty closed subset Z of X is irreducible if, for every open subset U of Z, the set $Z \cap U$ is either empty or dense in Z.

(ii) Let Z be an irreducible closed subset of X, and let $\mathscr{V}_Z$ be the set of open subsets U of X such that $Z \cap U \neq \varnothing$. For every sheaf $\mathscr{F}$ on X, we set

$$\mathscr{F}_Z = \varinjlim_{U \in \mathrm{Ob}(\mathscr{V}_Z^{\mathrm{op}})} \mathscr{F}(U).$$

[18] In fact, the converse is true : points of $\mathscr{C}_{\mathscr{T}}$ correspond to flat continuous functors $\mathscr{C} \to \mathbf{Set}$.

Show that this defines a point of $\mathscr{C}_{\mathscr{T}}$.

(iii) Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be a point, and let $A : \mathscr{C} \to \mathbf{Set}$ be the corresponding flat continuous functor. Let

$$Z = X - \bigcup_{U \in \mathrm{Ob}(\mathscr{C}),\ A(U)=\varnothing} U.$$

Show that Z is an irreducible closed subset of X, and that x is isomorphic to the functor $\mathscr{F} \longmapsto \mathscr{F}_Z$. [19]

(iv) If x_1 and x_2 are points of $\mathscr{C}_{\mathscr{T}}$ and Z_1 and Z_2 are the corresponding closed irreducible subsets of X, show that there exists a morphism from x_1 to x_2 if and only if $Z_1 \subset Z_2$.

(e). Let $X = [0,1]$ with the Lebesgue measure. We take $\mathscr{C}$ to be the category whose objects are Lebesgue-measurable subsets E of $[0,1]$, and such that $\mathrm{Hom}_{\mathscr{C}}(E, E')$ is a singleton if $E' - E$ has measure 0, and the empty set otherwise. We put the Grothendieck pretopology on $\mathscr{C}$ whose covering families are countable families $(E_n \to E)_{n\in\mathbb{N}}$ such that $E - \bigcup_{n\in\mathbb{N}} E_n$ has measure 0. (You can admit that this is a pretopology; it is not very hard.)

(i) Show that the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ is not empty.

(ii) Show that the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ has no objects (that is, $\mathscr{C}_{\mathscr{T}}$ has no points).

Solution.

(a). (i) Let $\alpha : \mathscr{I} \to \mathbf{Set}$ be a functor, with $\mathscr{I}$ a small category. We want to show that the canonical morphism

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)(\varprojlim \alpha) \to \varprojlim(\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot) \circ \alpha)$$

is an isomorphism in $\mathrm{PSh}(\mathscr{C})$. For every $X \in \mathrm{Ob}(\mathscr{C})$, if we evaluate this morphism at X, we get the canonical morphism

$$\mathrm{Hom}_{\mathbf{Set}}(A(X), \varprojlim \alpha) \to \varprojlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathbf{Set}}(A(X), \alpha(i))$$

(where we use Proposition I.5.3.1 to calculate the right-hand side), which is an isomorphism by definition of the limit.

(ii) By Proposition I.4.7 , it suffices to show that, for every presheaf $\mathscr{F}$ on $\mathscr{C}$, the functor $\mathbf{Set} \to \mathbf{Set}$, $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A, S))$ is representable.

Suppose first that $\mathscr{F} = h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a representable presheaf. By the Yoneda lemma, for every set S, the map

[19] So we have shown that points of $\mathscr{C}_{\mathscr{T}}$ correspond to closed irreducible subsets of X. If X is sober, that is, if every closed irreducible subset has a unique generic point, then points of $\mathscr{C}_{\mathscr{T}}$ correspond to points of X, but this is not true in general.

$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A,S)) \rightarrow \mathrm{Hom}_{\mathscr{C}}(A,S)(X) = \mathrm{Hom}_{\mathbf{Set}}(A(X),S)$ sending $u : h_X \rightarrow \mathrm{Hom}_{\mathscr{C}}(A,S)$ to $u(X)(\mathrm{id}_X)$ is bijective. An easy verification shows that this map defines an isomorphisms of functors. So the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S))$ is represented by the set $A(X)$. Also, if $f : X \rightarrow Y$ is a morphism of $\mathscr{C}$ and $h_f : h_X \rightarrow h_Y$ is its image by the Yoneda embedding, then we have a commutative diagram, for every set S:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_Y, \mathrm{Hom}_{\mathscr{C}}(A,S)) \arrow[r, "\sim"] \arrow[d, "(\cdot)\circ h_f"'] & \mathrm{Hom}_{\mathbf{Set}}(A(Y),S) \arrow[d, "(\cdot)\circ A(f)"] \\
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A,S)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{Set}}(A(X),S)
\end{tikzcd}$$

Indeed, let $u \in \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_Y, \mathrm{Hom}_{\mathscr{C}}(A,S))$. Then its image in $\mathrm{Hom}_{\mathbf{Set}}(A(X),S)$ by the upper right path of the diagram is $u(Y)(\mathrm{id}_Y) \circ A(f)$, and its image by the left bottom path of the diagram is $(u \circ h_f)(X)(\mathrm{id}_X) = u(X)(f) = u(X)(\mathrm{id}_Y \circ f)$. But these two are equal because, as u is a morphism of presheaves, we have a commutative diagram:

$$\begin{tikzcd}
h_Y(X) \arrow[r, equal] & \mathrm{Hom}_{\mathscr{C}}(Y,X) \arrow[r, "u(X)"] & \mathrm{Hom}_{\mathrm{S}}(A(X),S) \\
h_Y(Y) \arrow[r, equal] & \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[r, "u(Y)"'] \arrow[u, "(\cdot)\circ f"'] & \mathrm{Hom}_{\mathbf{Set}}(A(Y),S) \arrow[u, "(\cdot)\circ A(f)"']
\end{tikzcd}$$

It remains to show that the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S))$ is representable for an arbitrary presheaf $\mathscr{F}$ on $\mathscr{C}$. As in problem A.2.2, consider the category $\mathscr{C}/\mathscr{F}$ and the functor $G_{\mathscr{F}} : \mathscr{C}/\mathscr{F} \rightarrow \mathscr{C}$. We have shown in question (a) of that problem that there is a canonical isomorphism $\varinjlim(h_{\mathscr{C}} \circ G_{\mathscr{F}}) \xrightarrow{\sim} \mathscr{F}$. Let $\mathscr{F} \otimes_{\mathscr{C}} A = \varinjlim(A \circ G_{\mathscr{F}}) = \varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)$. Then we have, for every set S,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S)) &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\varinjlim(h_{\mathscr{C}} \circ G_{\mathscr{F}}), \mathrm{Hom}_{\mathscr{C}}(A,S)) \\
&\xrightarrow{\sim} \varprojlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A,S)) \\
&\xrightarrow{\sim} \varprojlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} \mathrm{Hom}_{\mathbf{Set}}(A(X),S)) \\
&\simeq \mathrm{Hom}_{\mathbf{Set}}(\mathscr{F} \otimes_{\mathscr{C}} A, S).
\end{aligned}$$

These isomorphisms are all easily seen to be functorial in S, so the set $\mathscr{F} \otimes_{\mathscr{C}} A$ represents the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S))$.

(iii) We know that $((\cdot) \otimes_{\mathscr{C}} A) \circ h_{\mathscr{C}} \simeq A$ by (ii), and that $h_{\mathscr{C}}$ commutes with all limits that exist in $\mathscr{C}$ by definition of limits, so, if $(\cdot) \otimes_{\mathscr{C}} A$ commutes with finite limits, so does A.

(iv) By Theorem I.5.2.1, we have $\mathscr{F} \otimes_{\mathscr{C}} A = \coprod_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)/\sim$, where $\sim$ is the equivalence relation generated by the relation R defined by: if $X, Y \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})$ and $x \in A(X)$, $y \in A(Y)$, then xRy if there exists a morphism $f : X \to Y$ in $\mathscr{C}/\mathscr{F}$ such that $A(f)(x) = y$.

Let R' be the relation on $\coprod_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)$ defined in the question. We clearly have $xRy \Rightarrow xR'y \Rightarrow x \sim y$ (with the same notation as in the previous paragraph), so it suffices to show that R' is an equivalence relation. It is clearly reflexive and symmetric. We show that it is transitive. Let X_1, X_2, X_3 be objects of $\mathscr{C}/\mathscr{F}$ and $x_1 \in A(X_1)$, $x_2 \in A(X_2)$, $x_3 \in A(X_3)$ such that $x_1 R' x_2$ and $x_2 R' x_3$. This means that we have $Y_1, Y_2 \in \mathrm{Ob}(\mathscr{C})$, morphisms $f_1 : Y_1 \to X_1$, $f_2 : Y_1 \to X_2$, $g_1 : Y_2 \to X_2$, $g_2 : Y_2 \to X_3$ in $\mathscr{C}/\mathscr{F}$ and elements $y_1 \in A(Y_1)$ and $y_2 \in A(Y_2)$ such that $A(f_1)(y_1) = x_1$, $A(f_2)(y_1) = x_2$, $A(g_1)(y_2) = x_2$ and $A(g_2)(y_2) = x_3$. Let $Z = Y_1 \times_{X_2} Y_2$, let $p : Z \to Y_1$ and $q : Z \to Y_2$, and let $z = (y_1, y_2) \in A(Z) = A(Y_1) \times_{A(X_2)} A(Y_2)$.

$$\begin{tikzcd}
 & & Z \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
 & Y_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & Y_2 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
X_1 & & X_2 & & X_3
\end{tikzcd}$$

Then $A(f_1 \circ p)(z) = x_1$ and $A(g_2 \circ q) = x_3$, so $x_1 R' x_3$.

(v) Suppose that A commutes with finite limits. We want to show that it is flat.

Let $*$ be the final object of $\mathscr{C}$ (i.e. the limit of the unique functor $\varnothing \to \mathscr{C}$). Then the final object of $\mathrm{PSh}(\mathscr{C})$ is the presheaf h_*, which is also isomorphic to the constant presheaf with value a fixed singleton. The functor $G_{h_*} : \mathscr{C}/h_* \to \mathscr{C}$ is an isomorphism of categories, so, to show that $h_* \otimes_{\mathscr{C}} A$ is a final object of $\mathbf{Set}$, we need to show that $S := \varinjlim_{X \in \mathrm{Ob}(\mathscr{C})} A(X)$ is a singleton. We have a morphism $A(*) \to S$ and $A(*)$ is a final object in $\mathbf{Set}$, i.e. a singleton, so S is not empty. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$, $x \in A(X)$ and $y \in A(Y)$. Then (x, y) is an element of $A(X \times Y) \simeq A(X) \times A(Y)$, and, if $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the two projections, then $A(p_1)(x, y) = x$ and $A(p_2)(x, y) = y$. So $x \in A(X)$, $(x, y) \in A(X \times Y)$ and $y \in A(Y)$ define the same element of S. This shows that $\mathrm{card}(S) \leq 1$, hence that S is a singleton because S is not empty.

We now show that the functor $(\cdot) \otimes_{\mathscr{C}} A$ commutes with fiber products. Let $\mathscr{F} \to \mathscr{H}$ and $\mathscr{G} \to \mathscr{H}$ be morphisms in $\mathrm{PSh}(\mathscr{C})$, let $E = \mathscr{F} \otimes_{\mathscr{C}} A$, $E' = \mathscr{G} \otimes_{\mathscr{C}} A$, $E'' = \mathscr{H} \otimes_{\mathscr{C}} A$ and $F = (\mathscr{F} \times_{\mathscr{H}} \mathscr{G}) \otimes_{\mathscr{C}} A$. Applying the functor $(\cdot) \otimes_{\mathscr{C}} A$ to the commutative diagram

$$\begin{tikzcd} \mathscr{F} \times_{\mathscr{H}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{F} \arrow[r] & \mathscr{H} \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E' \arrow[d, "q"] \\ E \arrow[r, "p"'] & E'' \end{tikzcd}$$

and we want to show that this induces an isomorphism from F to the fiber product $E \times_{E''} E'$. So let S be another set, and let $u : S \to E$, $v : S \to E'$ be maps such that $p \circ u = q \circ v$. We want to show that these maps factor uniquely through a map $w : S \to F$. Let $s \in S$. To make the notation less cumbersome, we will use the Yoneda embedding to identify $\mathscr{C}$ to a full subcategory of $\mathrm{PSh}(\mathscr{C})$, so we write X instead of h_X if $X \in \mathrm{Ob}(\mathscr{C})$. Choose an object $X \to \mathscr{F}$ of $\mathscr{C}/\mathscr{F}$, an object $Y \to \mathscr{G}$ of $\mathscr{C}/\mathscr{G}$ and elements $x \in A(X)$ and $y \in A(Y)$ such that x represents $u(s)$ and y represents $v(s)$. The fact that $p(u(s)) = q(v(s)$ means that there exists an object Z of $\mathscr{C}$, a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[r] & \mathscr{G} \arrow[dr] & \\ & X \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{H} \end{tikzcd}$$

in $\mathrm{PSh}(\mathscr{C})$ and $z \in A(Z)$ such that the images of z in $A(X)$ and $A(Y)$ are x and y. The diagram we just wrote gives a morphism $Z \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ in $\mathrm{PSh}(\mathscr{C})$, so we get an object of $\mathscr{C}/(\mathscr{F} \times_{\mathscr{H}} \mathscr{G})$, and, if $w : S \to F$ existed, we would necessarily have that $w(s)$ is the element of F represented by z. This proves the uniqueness of w. To prove its existence, we need to show that other choices of representatives of $u(s)$ and $v(s)$ would give the same element of F. So suppose that we have another commutative diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[dr] & Y' \arrow[r] & \mathscr{G} \arrow[dr] & \\ & X' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{H} \end{tikzcd}$$

and an element $z' \in A(Z')$ such that the image x' of z' in $A(X')$ is a representative of $u(s)$ and the image y' of z' in $A(Y')$ is a representative of $v(s)$. We must show that z and z' represent the same element of F. As x and x' represent the same element $u(s)$ of E, there exists an object X'' of $\mathscr{C}$, morphisms $X'' \to X$ and $X'' \to X'$ and an element $x'' \in A(X'')$ whose images in $A(X)$ and $A(X')$ are x and x' respectively. Similarly, we get $Y'' \to Y$, $Y'' \to Y'$ and $y'' \in A(Y'')$. Now replacing X by X'', Z by $X'' \times_X Z$, the morphism $Z \to X$ by the first projection $X'' \times_X Z \to X''$, the morphism $Z \to Y$ by the composition of the second projection $X'' \times_X Z \to Z$ and of $Z \to Y$, $x \in A(X)$ by $x'' \in A(X'')$ and $z \in A(Z)$ by $(x'', z) \in A(X'' \times_X Z) = A(X'') \times_{A(X)} A(Z)$, we may assume that there is a morphism $X \to X'$ such that the image of x in $A(X')$ is x'. Playing the same game with $Y'' \to Y$ (that is, replacing Z with $Y'' \times_Y Z$ etc), we may also assume that $Y'' = Y$ and $y'' = y$. We now have w commutative diagram

$$\begin{tikzcd}
Z' \arrow[rr, bend left] \arrow[drr] \arrow[ddrr] & Y \arrow[r] & Y' \arrow[r] & \mathscr{G} \arrow[dr] & \\
 & & \mathscr{F} \times_{\mathscr{H}} \mathscr{G} \arrow[ur] \arrow[dr] & & \mathscr{H} \\
Z \arrow[uur] \arrow[urr] \arrow[r] & X \arrow[r] & X' \arrow[r] & \mathscr{F} \arrow[ur] &
\end{tikzcd}$$

and element $z \in A(Z)$, $z' \in A(z')$ such that the images of z in $A(X)$ and $A(Y)$ are x and y respectively, that the images of z' in $A(X')$ and $A(Y')$ are x' and y' respectively, the image of x in $A(X')$ is x' and the image of y in $A(Y')$ is y'.

$$\begin{tikzcd}
z' \arrow[rr, mapsto, bend left] \arrow[drr, mapsto] & y \arrow[r, mapsto] & y' \\
z \arrow[ur, mapsto] \arrow[r, mapsto] & x \arrow[r, mapsto] & x'
\end{tikzcd}$$

Let $Z'' = Z \times_{X' \times Y'} Z'$, and let $z'' = (z, z') \in A(Z \times_{X' \times Y'} Z') = A(Z) \times_{A(X') \times A(Y')} A(Z')$. To show that z, z' and z'' induce the same element of F (which will finish the proof), it suffices to show that the morphisms $Z'' \to Z \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ and $Z'' \to Z' \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ are equal. But these morphisms become equal after we compose them with the two projections from $\mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ to $\mathscr{F}$ and $\mathscr{G}$, so they are equal by the universal property of the fiber product.

(vi) If $A : \mathscr{C} \to \mathbf{Set}$ is a flat functor, then the functor $x_A = (\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ commutes with all colimits (as a left adjoint) and with finite limits (by flatness of A), so it is an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$. Also, the construction of $(\cdot) \otimes_{\mathscr{C}} A$ in the solution of (ii) is clearly functorial in A.

Conversely, let $x : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ be an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$, and let $A_x = x \circ h_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$. Then A_x is a flat functor because both $h_{\mathscr{C}}$ and x commutes with finite limits, so we get a functor from $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ to the category of flat functors $\mathscr{C} \to \mathbf{Set}$ (with isomorphisms of such functors as morphisms). Moreover, if x is a point, then it commutes with all colimits, so we have a canonical isomorphism for all $\mathscr{F}$:

$$x(\mathscr{F}) = x(\varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} h_X) \xrightarrow{\sim} \varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} x(h_X) = \varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X) = \mathscr{F} \otimes_{\mathscr{C}} A = x_{A_x}(\mathscr{F}),$$

and this gives an isomorphism of functors $x \xrightarrow{\sim} x_{A_x}$.

Finally, if $A : \mathscr{C} \to \mathbf{Set}$ is a flat functor, we have already seen in (ii) that $x_A \circ h_{\mathscr{C}} \simeq A$; in other words, we have $A_{x_A} \simeq A$.

□

A.5.4 G-sets

Let G be a finite group, let $\mathscr{C} = G - \mathbf{Set}$ be the category whose objects are sets with a left action of G and whose morphisms are G-equivariant maps. We consider the pretopology $\mathscr{T}$ on $\mathscr{C}$ for which a family $(f_i : X_i \to X)_{i \in I}$ is covering if and only if $X = \bigcup_{i \in I} f_i(X_i)$. [20]

Let A be G with its actionby left translations. More generally, for every subgroup H of G, we denote by A_H the set G/H with the action of G by left translations.

Useful fact: If $X \to Y$ is a surjective map in $\mathbf{Set}$ or $G - \mathbf{Set}$, then it is the cokernel of the two projections $X \times_Y X \to X$. (You still need to justify thisn if you want to use it.)

(a). Show that every object of $G - \mathbf{Set}$ is a coproduct of objects isomorphic to some A_H.

(b). Calculate $A \times_{A_H} A$ in the category $G - \mathbf{Set}$.

(c). Show that every representable presheaf on $G - \mathbf{Set}$ is a sheaf.

(d). Show the automorphisms of A in $G - \mathbf{Set}$ are exactly the maps $c_g : A \to A$, $a \longmapsto ag$, for $g \in G$.

(e). If $\mathscr{F}$ is a presheaf on $G - \mathbf{Set}$, show the family $(\mathscr{F}(c_g))_{g \in G}$ defines a left action of G on $\mathscr{F}(A)$.

(f). Consider the functor $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to G - \mathbf{Set}$ defined by $\Phi(\mathscr{F}) = \mathscr{F}(A)$ and the functor $\Psi : F - \mathbf{Set} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ fiven by $\Psi(X) = \mathrm{Hom}_{G-\mathbf{Set}}(\cdot, X)$. Show that $\Phi \circ \Psi \simeq \mathrm{id}_{G-\mathbf{Set}}$.

(g). Show that $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})}$. (Hint: For any G-set X, if $|X|$ is the set X with the trivial G-action, then we have a surjective G-equivariant map $A \times |X| \to X$, $(g, x) \longmapsto g \cdot x$, which induces an injection $\mathscr{F}(X) \to \prod_{x \in X} \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathscr{F}(A))$.)

(h). Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be the functor $\mathscr{F} \longmapsto \mathscr{F}(A)$, where we forget the action of G on $\mathscr{F}(A)$ to see $\mathscr{F}(A)$ as a set. Show that every point of $\mathscr{C}_{\mathscr{T}}$ is isomorphic to x. (See the beginning of Problem A.5.33 for the definition of points.) (Suggestion: if y is a point, calculate $y(\Psi(\{1\}))$, then $y(\Psi(A))$, then construct a morphism of functors $\mathrm{Hom}_{G-\mathbf{Set}}(A, \cdot) \to y \circ \Psi$, then show that it is an isomorphism.)

(i). Show that the group of automorphisms of the point x is isomorphic to G.

[20] It is very easy to check that this is a pretopology, you don't need to do it.

Solution.

(a). Let X be a set with an action of G. Then X is the disjoint union of its G-orbits, and a G-orbit $G \cdot x$ is isomorphic to A_H, where H is the stabilizer of x.

(b). Let B be the set G with the trivial action of G. We have a G-equivariant bijection $A \times A \to A \times B$, $(x, y) \longmapsto (x, x^{-1}y)$. (Where $A \times A$ is the direct product in $G - \mathbf{Set}$, so G acts via $g \cdot (x, y) = (gx, gy)$.) If H is a subgroup of G, this bijection sends the G-subset $A \times_{A_H} A = \{(x, y) \in A \times A \mid x^{-1}y \in H\}$ to $A \times H$, where the factor H has the trivial action of G.

(c). This is exactly the content of the "useful fact" from the statement. Let's prove it. Let E be a G-set, and let $(f_i : X_i \to X)_{i \in I}$ be a covering family in $G - \mathbf{Set}$. Let $(u_i : X_i \to E)_{i \in I}$ be a family of G-equivariant maps such that, for all $i, j \in I$, the pullbacks of u_i and u_j to $X_i \times_X X_j$ (by the two projectins) agree. This means that, for every $x \in X$, if $x_i \in f_i^{-1}(x)$ and $x_j \in f_j^{-1}(x)$, then $u_i(x) = u_j(x)$. As $X = \bigcup_{i \in I} f_i(X_i)$, there exists a unique map $u : X \to E$ such that $u \circ f_i = u_i$ for every $i \in I$, and it suffices to check that u is G-equivariant. Let $x \in X$ and $g \in G$; choose $i \in I$ and $x_i \in X_i$ such that $x = f_i(x_i)$; then $g \cdot x = f_i(g \cdot x_i)$, so $u(g \cdot x) = u_i(g \cdot x_i) = g \cdot u(x_i) = g \cdot u(x)$.

(d). It is clear that the c_g are all automorphisms of A in $G - \mathbf{Set}$.

Conversely, let $\varphi : A \to A$ be an automorphism in $G - \mathbf{Set}$, and let $g = \varphi(1)$. Then, for everty $h \in A$, we have $\varphi(h) = \varphi(h \cdot 1) = h \cdot \varphi(1) = hg$. So $\varphi = c_g$.

(e). For every $g \in G$, the map $\mathscr{F}(c_g)$ is an automorphism of $\mathscr{F}(A)$ (in the category $\mathbf{Set}$), and we have $\mathscr{F}(c_1) = \mathrm{id}_{\mathscr{F}(A)}$ because $c_1 = \mathrm{id}_A$. If $g, h \in G$, we have $c_{gh} = c_h \circ c_g$, so $\mathscr{F}(c_{gh}) = \mathscr{F}(c_g) \circ \mathscr{F}(c_h)$. So we do get a left action of G on $\mathscr{F}(A)$.

(f). The functor Φ is well-defined, because, if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves and $g \in G$, then $\alpha(A) \circ \mathscr{F}(c_g) = \mathscr{G}(c_g) \circ \alpha(A)$, so $\alpha(A)$ is a G-equivariant map.

Let X be a G-set. Then we have a map $u(X) : \Phi(\Psi(X)) = \mathrm{Hom}_G(A, X) \to X$ sending $f : A \to X$ to $f(1)$, and this clearly defines a morphism of functors $u : \Phi \circ \Psi \to \mathrm{id}_{G-\mathbf{Set}}$. We show that it is an isomorphism. If $f, f' : A \to X$ are two G-equivariant maps such that $f(1) = f'(1)$, then, for every $g \in G$, we have $f(g) = f(g \cdot 1) = g \cdot f(1) = g \cdot f'(1) = f'(g)$. So $u(X)$ is injective. Let $x \in X$, and define a map $f : A \to X$ by $f(g) = g \cdot x$; then f is G-equivariant, and $u(X)(f) = x$; so $u(X)$ is surjective.

(g). If $\mathscr{F}$ is a sheaf, then $\Psi(\Phi(\mathscr{F})) = \mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$, so we must find an isomorphism of sheaves $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A)) \simeq \mathscr{F}$ that is functorial in $\mathscr{F}$.

Let $\mathscr{F}$ be a sheaf. For every G-set X, let $p_X : A \times |X| \to X$, $(g, x) \longmapsto g \cdot x$ be the G-equivariant surjection of the hint. It is a covering family in G-set, hence induces an injection $\iota(X, \mathscr{F}) : \mathscr{F}(X) \to \mathscr{F}(A \times |X|) = \mathscr{F}(\coprod_{x \in X} A) = \prod_{x \in X} \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathscr{F}(A))$, that is a morphism of functors in X in $\mathscr{F}$. We first check that the image of $\iota(X, \mathscr{F})$ is contained in the set G-equivariant maps. Write $A \times |X| = \coprod_{x \in X} A_x$, with $A_x = A$ for every $x \in X$; we have $p_{X|A_x}(g) = g \cdot x$, for $g \in A$ and $x \in X$. Let $x \in X$ and $g \in G$; we set $y = g \cdot x$. Then we have a commutative diagram in $G - \mathbf{Set}$:

$$\begin{tikzcd} A \arrow[r, "c_g"] \arrow[d, "c_1"'] & A_x \arrow[d, "p_{X|A_x}"] \\ A_y \arrow[r, "p_{X|A_y}"'] & X \end{tikzcd}$$

So, if $e \in \mathscr{F}(X)$ and $u = \iota(X, \mathscr{F})(e) : X \to \mathscr{F}(A)$, then $\mathscr{F}(c_g)(u(x)) = \mathscr{F}(c_1)(u(g \cdot x))$. This shows that u is G-equivariant.

To finish the proof, we must show that $\iota(X, \mathscr{F})$ is surjective for every G-set X and every sheaf $\mathscr{F}$. Fix $\mathscr{F}$.

If $X = A$, then $p_A : A \times |A| \to A$ is the map $(g, h) \longmapsto gh$; if we write as before $A \times |A| = \coprod_{h \in G} A_h$ with $A_h = A$ for every h, then $p_{A|A_h} = c_h$ for every $h \in G$. So $\mathscr{F}(p_A) : \mathscr{F}(A) \to (\mathscr{F}(A))^A = \mathrm{Hom}_{\mathbf{Set}}(A, \mathscr{F}(A))$ is the map sending $e \in \mathscr{F}(A)$ to $A \to \mathscr{F}(A)$, $g \longmapsto \mathscr{F}(c_g)(e)$. It is easy to see that every G-equivariant map $u : A \to \mathscr{F}(A)$ is of this form (take $e = u(1)$). So $\iota(A, \mathscr{F})$ is surjective, hence bijective.

Note that the functors $\mathscr{F}$ and $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$ both send coproducts to products. For the second functor, this is by definition of a coproduct. For the first functor, suppose that $X = \coprod_{i \in I} X_i$. Then the then the family of injections $(X_i \to X)_{i \in I}$ is covering and $X_i \times_X X_j = \varnothing$ for $i \neq j$, so the morphism $\mathscr{F}(X) \to \prod_{i \in I} \mathscr{F}(X_i)$ is bijective. So if X is a disjoint union of copies of A, then $\iota(X, \mathscr{F})$ is a bijection.

Let X be an arbirtrary G-set. We have a surjective G-equivariant map $p_X : A \times |X| = \coprod_{x \in X} A_x \to X$, where $A_x = A$ for every $x \in X$ and $p_{A|A_x}$ sends $g \in A$ to $g \cdot x$. Let $x, y \in X$. Then we have a G-equivariant isomorphism $A_x \times_X A_y = \{(g, h) \in A \times A \mid g \cdot x = h \cdot y\} \overset{\sim}{\to} A \times G_{y,x}, (g, h) \mid (g, g^{-1}h)$, where $G_{x,y}$ is the set $\{g \in G \mid g \cdot y = x\}$ with the trivial action of G; in particular, $A_x \times_X A_y$ is a disjoint union of copies of A. So $P := (A \times |X|) \times_X (A \times |X|)$ also is a disjoint union of copies of A. Let $p_1, p_2 : P \to A \times |X|$ be the two projections. Then we have commutative diagrams

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r, "\mathscr{F}(p_X)"] \arrow[d, "\iota(X,\mathscr{F})"'] & \mathscr{F}(A \times |X|) \arrow[r, "\mathscr{F}(p_i)"] \arrow[d, "\iota(A \times |X|, \mathscr{F})"'] & \mathscr{F}(P) \arrow[d, "\iota(P,\mathscr{F})"'] \\ \mathrm{Hom}_{G-\mathbf{Set}}(X, \mathscr{F}(A)) \arrow[r, "p_X^*"'] & \mathrm{Hom}_{G-\mathbf{Set}}(A \times |X|, \mathscr{F}(A)) \arrow[r, "p_i^*"'] & \mathscr{F}(P) \end{tikzcd}$$

for $i = 1, 2$, the maps $\mathscr{F}(p_X) : \mathscr{F}(X) \to \mathscr{F}(A \times |X|)$ and $p_X^* : \mathrm{Hom}_{G-\mathbf{Set}}(X, \mathscr{F}(A)) \to \mathrm{Hom}_{G-\mathbf{Set}}(A \times |X|, \mathscr{F}(A))$ are the kernels of $(\mathscr{F}(p_1), \mathscr{F}(p_2))$ and (p_1^*, p_2^*) respectively (because $\mathscr{F}$ and $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$ are sheaves), and the maps $\iota(A \times |X|, \mathscr{F})$ and $\iota(P, \mathscr{F})$ are bijective by the previous paragraph, so $\iota(X, \mathscr{F})$ is bijective.

(h). Note that $x \circ \Psi : G - \mathbf{Set} \to \mathbf{Set}$ is the functor $X \longmapsto \mathrm{Hom}_{G-\mathbf{Set}}(A, X)$. For every G-set X, we have a bijection $\mathrm{Hom}_{G-\mathbf{Set}}(A, X) \xrightarrow{\sim} X$ sending $u : A \to X$ to $u(1)$, and this gives an isomorphism from $x \circ \Psi$ to the forgetful functor $G - \mathbf{Set} \to \mathbf{Set}$. As Ψ is an equivalence of categories by (f) and (g), this shows that x commutes with all small limits and colimits, and in particular that it is a point.

Let $y : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be a point, that is, a functor that commutes with all small colimits and all finite limits. The functor $F := y \circ \Psi : G - \mathbf{Set} \to \mathbf{Set}$ has the same property, so it sends the terminal object A_G of $G - \mathbf{Set}$ to a terminal object of $\mathbf{Set}$, i.e. a singleton. For every nonempty G-set X, the unique map $X \to A_G$ identifies A_G to the cokernel of the two projections $X \times X \to X$, so $F(A_G) \to F(X)$ is a kernel morphism, hence injective, and so $F(X)$ is not empty.

We calculate $F(A)$. We have an isomorphism of G-sets $A \times A \xrightarrow{\sim} \coprod_{x \in G} A_x$, where $A_x = A$ for every $x \in G$, sending $(g, h) \in A \times A$ to $g \in A_{g^{-1}h}$. Let $q_1, q_2 : \coprod_{x \in G} A_x \to A$ be the map corresponding to the two projections $p_1, p_2 : A \times A \to A$ by this isomorphism. Then, for every $x \in G$, we have $q_{1|A_x} = \mathrm{id}_A$ and $q_{2|A_x} = c_x$. Applying F and using the fact that F commutes with coproducts and finite products, we get two maps $F(q_1), F(q_2) : \coprod_{x \in G} F(A_x) \to F(A)$, such that $F(q_1)_{|F(A_x)} = \mathrm{id}_{F(A)}$ and $F(q_2)_{|F(A_x)} = F(c_x)$ for every $x \in G$, and such that the induced map $(F(q_1), F(q_2)) : \coprod_{x \in G} F(A_x) \to F(A) \times F(A)$ is bijective. Let $e \in F(A)$ (we know that $F(A) \neq \varnothing$ by the previous paragraph). Then (q_1, q_2) induces a bijection $\coprod_{x \in G} \{e\} \xrightarrow{\sim} \{e\} \times F(A)$, so we get a bijection $\iota : F(A) \xrightarrow{\sim} G = A$, and it is easy to see that $\iota \circ F(c_x) = c_x \circ \iota$ for every $x \in G$.

Now that we have an isomorphism $\iota : F(A) \xrightarrow{\sim} A$, we can construct a morphism of functors α from $x \circ \Psi = \mathrm{Hom}_{G-\mathbf{Set}}(A, \cdot)$ to F by sending $f : A \to X$ to $F(f)(\iota^{-1}(1)) \in F(X)$. We know that $\alpha(A)$ is bijective, so $\alpha(X)$ is bijective if the G-set X is a coproduct of copies of A, because both functors commute with coproducts. As every G-set is the cokernel of two G-equivariant maps between coproducts of copies of A (see the solution of (g)), and as both functors commute with cokernel, $\alpha(X)$ is an isomorphism for every X.

(i). As Ψ is an equivalence of categories, it suffices to calculate the group of automorphisms of $x \circ \Psi$. We can apply the other Yoneda lemma (see for example Corollary I.3.2.8): as $x \circ \Psi$ is a representable functor, every automorphism of this functor comes from an automorphism of the representing object, that is, of A. So, by question (d), every automorphism of $x \circ \Psi$ is of the form $\mathrm{Hom}_{G-\mathbf{Set}}(c_g, \cdot)$. If $g, h \in G$, then $c_{gh} = c_h \circ c_g$, so $\mathrm{Hom}_{G-\mathbf{Set}}(c_{gh}, \cdot) = \mathrm{Hom}_{G-\mathbf{Set}}(c_g, \cdot) \circ \mathrm{Hom}_{G-\mathbf{Set}}(c_h, \cdot)$. So we get an isomorphism $G \xrightarrow{\sim} \mathrm{Aut}(x \circ \Psi)$.

□

A.6 Problem set 6

We make the following useful convention: if $(x_0, \dots, x_n)$ is some list and if $i \in \{0, \dots, n\}$, then $(x_0, \dots, \hat{x}_i, \dots, x_n)$ means $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

Also, if S is a set and $\mathbb{Z}^{(S)}$ is the free $\mathbb{Z}$-module on S, we denote the canonical basis of this free module by $(e_s)_{s \in S}$.

A.6.1 Salamander lemma

Prove the salamander lemma (Theorem IV.2.1.3).

Solution. If we turn the complex of (ii) 90 degree to the left and see it as a complex in the opposite category of $\mathscr{A}$, then we are exactly in the situation of (i). So it suffices to prove (i).

We give names to some morphisms of the complex

$$\begin{tikzcd}
 & \bullet \arrow[d, "\beta"] & & \\
\bullet \arrow[r, "\alpha"] \arrow[dr, "\delta"'] & C \arrow[d, "\iota"] \arrow[dr, "\gamma"] & \bullet \arrow[d, "\pi"] & \\
\bullet \arrow[r, "\zeta"'] & A \arrow[r, "\epsilon"] \arrow[d, "\eta"'] \arrow[dr, "\theta"] & B \arrow[r, "\lambda"] \arrow[d, "\mu"] \arrow[dr, "\nu"] & \bullet \\
 & \bullet & D \arrow[r, "\rho"] \arrow[d, "\sigma"'] & \bullet \\
 & & \bullet &
\end{tikzcd}$$

We check the exactness of the sequence at each object. By the Freyd-Mitchell embedding theorem (Theorem III.3.1), we may assume that $\mathscr{A}$ is a category of left R-modules. (Hence take elements in the objects of $\mathscr{A}$.)

In ${}_{=}A = \operatorname{Ker}\epsilon / \operatorname{Im}\gamma$, the subobject $\operatorname{Im}(1)$ is the image of $\iota(\operatorname{Ker}\gamma) \subset A$, and $\operatorname{Ker}(2) = (\operatorname{Ker}\epsilon \cap (\operatorname{Im}\iota + \operatorname{Im}\zeta))/\operatorname{Im}\zeta$. So $\operatorname{Im}(1) \subset \operatorname{Ker}(2)$. Conversely, take an element of $\operatorname{Ker}(2)$, lift it to $x \in \operatorname{Ker}\epsilon$, and choose $y \in C$ such that $x \in \iota(y) + \operatorname{Im}\zeta$. Then $\gamma(y) \in \epsilon(x) + \epsilon(\operatorname{Im}\zeta) = 0$, so y defines an element of $C_\square = \operatorname{Ker}\gamma/(\operatorname{Im}\alpha + \operatorname{Im}\beta)$, so $y \in \operatorname{Im}(1)$.

In $A_\square = \operatorname{Ker}\theta/(\operatorname{Im}\iota + \operatorname{Im}\zeta)$, the subobject $\operatorname{Im}(2)$ is the image of $\operatorname{Ker}\epsilon \subset A$, and $\operatorname{Ker}(3)$ is the set of elements that have a lift $x \in \operatorname{Ker}\theta$ such that $\epsilon(x) \in \operatorname{Im}(\gamma)$. So we clearly have $\operatorname{Im}(2) \subset \operatorname{Ker}(3)$. Consider an element of $\operatorname{Ker}(3)$, choose a lift $x \in \operatorname{Ker}\theta$ of that element such that $\epsilon(x) = \gamma(y)$, for some $y \in C$. Then $x - \iota(y)$ and x have the same image in $A_\square$, and $\epsilon(x - \iota(y)) = 0$, so the image of x in $A_\square$ is in $\operatorname{Im}(2)$.

In ${}^{\square}B = (\operatorname{Ker}\lambda \cap \operatorname{Ker}\mu)/\operatorname{Im}\gamma$, the subobject $\operatorname{Im}(3)$ is the image of $\epsilon(\operatorname{Ker}\theta) \subset B$, and $\operatorname{Ker}(4)$ is the set of elements of ${}^{\square}B$ that have a lift $x \in (\operatorname{Ker}\lambda \cap \operatorname{Ker}\mu) \cap \operatorname{Im}\epsilon$. So we clearly have $\operatorname{Im}(3) \subset \operatorname{Ker}(4)$. Conversely, consider an element of $\operatorname{Ker}(4)$, and choose a lift $x \in \operatorname{Ker}\lambda \cap \operatorname{Ker}\mu$ of this element such that we can write $x = \epsilon(y)$, with $y \in A$. Then $\theta(y) = \mu(x) = 0$, so $x \in \epsilon(\theta(y))$, and its image in ${}^{\square}B$ is in $\operatorname{Im}(3)$.

In ${}_{=}B = \operatorname{Ker}\lambda/\operatorname{Im}\epsilon$, the subobject $\operatorname{Im}(4)$ is the image of $\operatorname{Ker}\lambda \cap \operatorname{Ker}\mu \subset B$, and $\operatorname{Ker}(5)$ is the set of elements of ${}_{=}B$ that have a lift $x \in \operatorname{Ker}\lambda$ such that $\mu(x) \in \operatorname{Im}(\theta)$. So we clearly have $\operatorname{Im}(4) \subset \operatorname{Ker}(5)$. Conversely, consider an element of $\operatorname{Ker}(5)$, and choose a lift $x \in \operatorname{Ker}\lambda$ of this element such that we can write $\mu(x) = \theta(y)$, with $y \in A$. Then $\lambda(x - \epsilon(y)) = 0$, the elements x and $x - \epsilon(y)$ of $\operatorname{Ker}\lambda$ have the same image in ${}_{=}B$, and $\mu(x - \epsilon(y)) = 0$, so $x - \epsilon(y) \in \operatorname{Ker}\lambda \cap \operatorname{Ker}\mu$, and its image in ${}_{=}B$ is in $\operatorname{Im}(4)$.

$\square$

A.6.2 Some bar resolutions

(a). Let S be a nonempty set. We define a complex of $\mathbb{Z}$-modules $X^\bullet$ by:

- $X^n = 0$ and $d_X^n = 0$ if $n \geq 2$;
- $X^1 = \mathbb{Z}$ and $d_X^1 = 0$;
- $X^0 = \mathbb{Z}^{(S)}$ and $d_X^0 : X^0 \to X^1 = \mathbb{Z}$ sends every e_s to 1;
- if $n \geq 1$, then $X^{-n} = \mathbb{Z}^{(S^{n+1})}$ and $d^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^n)}$ sends $e_{(s_0,\dots,s_n)}$ to $\sum_{i=0}^n (-1)^i e_{(s_0,\dots,\hat{s}_i,\dots,s_n)}$, for all $s_0, \dots, s_n \in S$.

Show that $X^\bullet$ is indeed a complex (i.e. that $d_X^{n+1} \circ d_X^n = 0$ for every $n \in \mathbb{Z}$), and that it is acyclic. (Hint: Fix $s \in S$. If $n \geq -1$, consider the morphism $t^{-n} : X^{-n} \to X^{-n-1}$ sending $e_{(s_0,\dots,s_n)}$ to $e_{(s,s_0,\dots,s_n)}$.)

(b). Let G be a group. For every $n \geq 0$, let $X_n(G) = \mathbb{Z}^{(G^{n+1})}$. By (a), we have an acyclic complex of $\mathbb{Z}$-modules $X^\bullet$, where $X^1 = \mathbb{Z}$, $X^{-n} = X_n(G)$ if $n \geq 0$, $X^n = 0$ if $n \geq 2$, and the differentials are as in (a).

(i) We make G act as $X_n(G)$ by $g \cdot e_{(g_0,\dots,g_n)} = e_{(gg_0,\dots,gg_n)}$, and we make G act trivially on $\mathbb{Z}$. Show that $X^\bullet$ is an acyclic complex of $\mathbb{Z}[G]$-modules.

(ii) Show that $X_n(G)$ is a free $\mathbb{Z}[G]$-module for every $n \geq 0$. [21]

Let I_n be the $\mathbb{Z}$-submodule of $X_n(G)$ generated by the $e_{(g_0,\dots,g_n)}$ such that $g_i = g_{i+1}$ for some $i \in \{0, \dots, n-1\}$.

(iii) Show that I_n is a free $\mathbb{Z}[G]$-submodule of $X_n(G)$ and that $d^{-n}(I_n) \subset I_{n-1}$ for $n \geq 0$,

[21] The complex $X^\bullet$ is called the *unnormalized bar resolution* of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

with $I_{-1} = \{0\}$.

(iv) By the previous question, we get a complex of $\mathbb{Z}[G]$-modules $Y^\bullet$ such that $Y^n = 0$ for ≥ 2, $Y^1 = \mathbb{Z}$, $Y^{-n} = X_n(G)/I_n$ if $n \geq 0$ and d_Y^n is the morphism induced by d_X^n for every $n \in \mathbb{Z}$. Show that $Y^\bullet$ is acyclic. (Hint: Try to imitate the method of (a).) [22]

Solution.

(a). If we set $S^0 = \{()\}$ (the set whose only element is the empty sequence of elements of S), then we can see X^1 as the free $\mathbb{Z}$-module on S^0, with basis element $1 = e_{()}$. In this way, the formula for d^{-n} also works if $n = 0$.

We prove that $X^\bullet$ is a complex. If $n \geq 0$, then $d^{n+1} \circ d^n = 0$ because $d^{n+1} = 0$. We assume that $n \geq 1$ and we calculate $d^{-n+1} \circ \circ d^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^{n-1})}$. Let $s_0, \dots, s_n \in S$. Then

$$\begin{aligned}
d^{-n+1} \circ d^{-n}(e_{(s_0,\dots,s_n)}) &= \sum_{i=0}^{n} (-1)^i d^{-n+1}(e_{s_0,\dots,\hat{s}_i,\dots,s_n}) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} e_{(s_0,\dots,\hat{s}_j,\dots,\hat{s}_i,\dots,s_n)} + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j-1} e_{(s_0,\dots,\hat{s}_i,\dots,\hat{s}_j,\dots,s_n)} \\
&= \sum_{j=0}^{n} \sum_{i=j+1}^{n} (-1)^{i+j} e_{(s_0,\dots,\hat{s}_j,\dots,\hat{s}_i,\dots,s_n)} + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j-1} e_{(s_0,\dots,\hat{s}_i,\dots,\hat{s}_j,\dots,s_n)} \\
&= 0.
\end{aligned}$$

We fix $s \in S$. We define $t^m : X^m \to X^{m-1}$ by $t^m = 0$ for $m \geq 2$, and $t^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^{n+2})}$, $e_{(s_0,\dots,s_n)} \longmapsto e_{(s,s_0,\dots,s_n)}$ if $n \geq -1$. We want to prove that $(t^m)_{m\in\mathbb{Z}}$ is a homotopy between $\mathrm{id}_{X^\bullet}$ and 0. We have to check that $\mathrm{id}_{X^m} = t^{m+1} \circ d^m + d^{m-1} \circ t^m$ for every $m \in \mathbb{Z}$. If $m \geq 2$, then both sides are equal to 0. If $m = 1$, then we want to check that $\mathrm{id}_{\mathbb{Z}} = d^0 \circ t^1$; the right hand side sends $e_{()}$ to $d^0(e_s) = e_{()}$, so we get the desired identity. Suppose that $m \geq 0$, and write $n = -m$. Let $(s_0, \dots, s_n) \in S^{n+1}$. Then $(t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n})(e_{(s_0,\dots,s_n)})$ is equal to

$$\begin{aligned}
&(t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n})(e_{(s_0,\dots,s_n)}) \\
&= \sum_{i=0}^{n} (-1)^i e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} + d^{-n}(e_{(s,s_0,\dots,s_n)}) \\
&= \sum_{i=0}^{n} (-1)^i e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} + e_{(s_0,\dots,s_n)} + \sum_{i=0}^{n} (-1)^{i+1} e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} \\
&= e_{(s_0,\dots,s_n)}.
\end{aligned}$$

So $t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n} = \mathrm{id}_{X^{-n}}$.

[22] The complex $Y^\bullet$ is called the *normalized bar resolution* of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

(b). (i) As the formation of kernels and cokernsl commutes with the forgetful functor from ${}_{\mathbb{Z}[G]}\mathbf{Mod}$ to $\mathbf{Ab}$, and as we know that $X^\bullet$ is an acyclic complex of $\mathbb{Z}$-modules by (a), it suffices to show that $X^\bullet$ is a complex of $\mathbb{Z}[G]$-modules, i.e. that its differentials are $\mathbb{Z}[G]$-linear. But this is clear from the definitions of the differentials and of the action of $\mathbb{Z}[G]$.

(ii) It suffices to find a $\mathbb{Z}[G]$-basis of $X_n(G)$. If $(g_1,\dots,g_n) \in G^n$, then the morphism $\mathbb{Z}[G] \to X_n(G)$, $a \longmapsto a \cdot e_{(1,g_1,g_1g_2\dots,g_1g_2\dots g_n)}$ is injective with image $V_{g_1,\dots,g_n} := \mathrm{Span}(\{e_{(h_0,h_1,\dots,h_n)},\ h_{i-1}^{-1}h_i = g_i \text{ for } 1 \leq i \leq n\}$. As $X_n(G) = \bigoplus_{(g_1,\dots,g_n)\in G^n} V_{(g_1,\dots,g_n)}$ (because these subspaces are generated by mutually disjoint subsets of the canonical basis of $X_n(G)$), we deduce that the family $(e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)})_{(g_1,\dots,g_n)\in G^n}$ is a $\mathbb{Z}[G]$-basis of $X_n(G)$.

(iii) We have found a $\mathbb{Z}[G]$-basis $(e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)})_{(g_1,\dots,g_n)\in G^n}$ of $X_n(G)$ in (ii), and the calculation of $V_{g_1,\dots,g_n} = \mathbb{Z}[G] \cdot e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)}$ in the proof of that question show that $V_{g_1,\dots,g_n}$ is included in I_n if one of the g_i is equal to 1, and that $V_{g_1,\dots,g_n} \cap I_n = \{0\}$ otherwise. So I_n is the $\mathbb{Z}[G]$-submodule of $X_n(G)$ generated by the $e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)}$ such that at least one of the g_i is equal to 1, and in particular it is a free $\mathbb{Z}[G]$-submodule of $X_n(G)$.

We check that $d^{-n}(I_n) \subset I_{n-1}$. Let $(g_0,\dots,g_n) \in G^{n+1}$, and suppose that $d_i = d_{i+1}$ for some $i \in \{0,\dots,n-1\}$. Then

$$\begin{aligned} d^{-n}(e_{(g_0,\dots,g_n)}) &= \sum_{j\in\{0,\dots,n\}-\{i,i+1\}} (-1)^j e_{(g_0,\dots,\hat{g}_j,\dots,g_n)} + (-1)^i e_{(g_0,\dots,g_{i-1},g_i,g_{i+2},\dots,g_n)} \\ &\qquad + (-1)^{i+1} e_{(g_0,\dots,g_{i-1},g_{i+1},g_{i+2},\dots,g_n)} \\ &= \sum_{j\in\{0,\dots,n-\{i,i+1\}} (-1)^j e_{(g_0,\dots,\hat{g}_j,\dots,g_n)}. \end{aligned}$$

The last sum is clearly in I_{n-1}.

(iv) It suffices to show that $Y^\bullet$ is acyclic as a complex of $\mathbb{Z}$-modules. Let $t^m : X^m \to X^{m-1}$ be the morphisms of (a), for example for $s = 1$ (the unit element of G). Then, if $n \geq 0$, $t^{-n} : X_n(G) \to X_{n+1}(G)$ sends I_n to I_{n+1}, so it induces a morphism $\bar{t}_n : Y^{-n} \to Y^{-n+1}$. We also denote by $\bar{t}^1$ the morphism $t^1 : Y^1 = \mathbb{Z} \to Y^0 = X_0(G)$ (note that $I_0 = \{0\}$) and set $\bar{t}^m = 0$ for $m \geq 2$. Then, by (a), the family $(t^m)_{m\in\mathbb{Z}}$ defines a homotopy between $\mathrm{id}_{Y^\bullet}$ and 0.

□

A.6.3 Čech cohomology, part 1

This problem uses problem A.5.1.

Let $\mathscr{C}$ be a category that admits fiber products, and let $\mathscr{X} = (f : X_i \to X)_{i\in I}$ be a family of morphisms of $\mathscr{C}$. If $i_0, \ldots, i_p \in I$, we write $X_{i_0,\ldots,i_p} = X_{i_0} \times_X X_{i_1} \times_X \ldots \times_X X_{i_p}$. For every $p \in \mathbb{Z}$, we define an abelian presheaf $\mathscr{C}_p(\mathscr{X}) \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C}, \mathbb{Z}))$ in the following way:

- if $p < 0$, then $\mathscr{C}_p = 0$;
- if $p \geq 0$, then
$$\mathscr{C}_p(\mathscr{X}) = \bigoplus_{i_0,\ldots,i_p\in I} \mathbb{Z}^{(X_{i_0,\ldots,i_p})}.$$

We also define a morphism of presheaves $d_p : \mathscr{C}_p(\mathscr{X}) \to \mathscr{C}_{p-1}(\mathscr{X})$ in the following way:

- if $p \leq 0$, then $d_p = 0$;
- if $p \geq 1$, then d_p is given on the component $\mathbb{Z}^{(X_{i_0,\ldots,i_p})}$ by the morphism $\mathbb{Z}^{(X_{i_0,\ldots,i_p})} \to \bigoplus_{q=0}^{p} \mathbb{Z}^{(X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p})} \subset \mathscr{C}_{p-1}(\mathscr{X})$ equal to $\sum_{q=0}^{p}(-1)^q \delta^q_{i_0,\ldots,i_p}$, where $\delta^q_{i_0,\ldots,i_p} : \mathbb{Z}^{(X_{i_0,\ldots,i_p})} \to \mathbb{Z}^{(X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p})}$ is the image of the canonical projection $X_{i_0,\ldots,i_p} \to X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p}$ by the functor $\mathscr{C} \overset{h_{\mathscr{C}}}{\to} \mathrm{PSh}(\mathscr{C}) \overset{\mathbb{Z}^{(\cdot)}}{\to} \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$.

(a). Show that $\mathrm{Ker}(d_p) \supset \mathrm{Im}(d_{p+1})$ for every $p \in \mathbb{Z}$ and that this is an equality for $p \neq 0$.

<u>Hint</u>: For every object Y of $\mathscr{C}$, we have
$$\mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0,\ldots,i_p}) = \coprod_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0})_h \times \ldots \times \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_p})_h,$$
where, for every $i \in I$, $\mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h = \{g \in \mathrm{Hom}_{\mathscr{C}}(Y, X_i) \mid f_i \circ g = h\}$. Set $S_h = \coprod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h$ and think of question A.6.22(a).

(b). Let $\varepsilon : \mathscr{C}_0(\mathscr{X}) \to \mathbb{Z}^{(X)}$ be the morphism that is equal on the component $\mathbb{Z}^{(X_i)}$ to the image of $f_i : X_i \to X$ by the functor $\mathscr{C} \overset{h_{\mathscr{C}}}{\to} \mathrm{PSh}(\mathscr{C}) \overset{\mathbb{Z}^{(\cdot)}}{\to} \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$. Show that $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(d_1)$.

For every $p \in \mathbb{Z}$, we define a functor $\check{C}^p(\mathscr{X}, \cdot) : \mathrm{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$ by $\check{C}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(\mathscr{C}^p(\mathscr{X}), \mathscr{F})$, and a morphism of functors $d^p : \check{C}^p(\mathscr{X}, \cdot) \to \check{C}^{p+1}(\mathscr{X}, \cdot)$ by $d^p = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(d_{p+1}, \cdot)$. The family $(\check{C}^p(\mathscr{X}, \mathscr{F}), d^p)_{p\in\mathbb{Z}}$ is called the *Čech complex of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. For every $p \geq 0$, we set $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Ker}(d^p(\mathscr{F}))/\mathrm{Im}(d^{p-1}(\mathscr{F}))$. This is called the *pth Čech cohomology group of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. Note that the definition of $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F})$ is functorial in $\mathscr{F}$, so $\check{\mathrm{H}}^p(\mathscr{X}, \cdot)$ is a functor from $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ to $\mathbf{Ab}$.

(c). Show that, for every abelian presheaf $\mathscr{F}$ and every $p \geq 0$, we have
$$\check{C}^p(\mathscr{X}, \mathscr{F}) = \prod_{i_0,\ldots,i_p} \mathscr{F}(X_{i_0,\ldots,i_p}),$$
and that the definition of $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ given here generalizes that of Definition III.2.2.4.

(d). If $\mathscr{F}$ is an injective object of $\mathrm{PSh}(\mathscr{C},\mathbb{Z})$, show that $\check{\mathrm{H}}^p(\mathscr{X},\mathscr{F})=0$ for every $p\geq 1$.

(e). Suppose that we have a Grothendieck topology $\mathscr{T}$ on $\mathscr{C}$, that $\mathscr{X}$ is a covering family, and that $\mathscr{F}$ is an injective object of $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}},\mathbb{Z})$. Show that $\check{\mathrm{H}}^p(\mathscr{X},\mathscr{F})=0$ for $p\geq 1$ and that $\check{\mathrm{H}}^0(\mathscr{X},\mathscr{F})=\mathscr{F}(X)$.

(f). Let $\mathscr{F}\in\mathrm{Ob}(\mathrm{PSh}(\mathscr{C},\mathbb{Z}))$, let $\mathscr{F}\to\mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathrm{PSh}(\mathscr{C},\mathbb{Z})$. Show that we have canonical isomorphisms

$$\mathrm{H}^n(\check{\mathrm{H}}^0(\mathscr{X},\mathscr{I}^\bullet))\simeq\check{\mathrm{H}}^n(\mathscr{X},\mathscr{F}).$$

[23]

Solution.

(a). If $p=0$, then $d_p=0$, so $\mathrm{Ker}(d_p)\supset\mathrm{Im}(d_{p+1})$. If $p\leq -1$, then $\mathscr{C}_p(\mathscr{X})(Y)=0$, so $\mathrm{Ker}(d_p(Y))=\mathrm{Im}(d_{p+1}(Y))=0$. To treat the other cases, it suffices to prove that, for every $Y\in\mathrm{Ob}(\mathscr{C})$, we have $\mathrm{Ker}(d_p(Y))=\mathrm{Im}(d_{p+1}(Y))$ for $p\geq 1$.

We fix $Y\in\mathrm{Ob}(\mathscr{C})$, and we use the notation of the hint. For every $h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)$, let $S_h=\coprod_{i\in I}\mathrm{Hom}_{\mathscr{C}}(Y,X_i)_h$. Fix $p\geq 0$. The fact that

$$\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_0,\dots,i_p})=\coprod_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)}\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_0})_h\times\ldots\times\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_p})_h$$

for all $i_0,\dots,i_p$ is obvious, so we get

$$\coprod_{(i_0,\dots,i_p)\in I^{p+1}}\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_0,\dots,i_p})=\coprod_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)}S_h^{p+1},$$

and

$$\begin{aligned}\mathscr{C}_p(\mathscr{X})(Y)&=\bigoplus_{(i_0,\dots,i_p)\in I^{p+1}}\mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_0,\dots,i_p}))}\\&=\bigoplus_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)}\mathbb{Z}^{(S_h^{p+1})}.\end{aligned}$$

So $\mathscr{C}_p(\mathscr{X})(Y)$ is the direct sum indexed by $h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)$ of the terms of degree $-p$ of the complex of Problem A.6.2(a) for $S=S_h$, and $d_p:\mathscr{C}_p(\mathscr{X})(Y)\to\mathscr{C}_{p-1}(\mathscr{X})(Y)$ is the direct sum of the differentials of this complex if $p\geq 1$ (this follows immediately from the definition of d_p). As the complex of A.6.2(a) is acyclic, this implies that $\mathrm{Ker}(d_p(Y))=\mathrm{Im}(d_{p+1}(Y))$ if $p\geq 1$.

(b). Let $Y\in\mathrm{Ob}(\mathscr{C})$. We use the same notation as in the solution of (a). Then

$$\mathscr{C}_0(\mathscr{X})(Y)=\bigoplus_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)}\mathbb{Z}^{(S_h)},$$

[23] In other words, $\check{\mathrm{H}}^n(\mathscr{X},\cdot)$ is the nth right derived functor of $\check{\mathrm{H}}^0(\mathscr{X},\cdot)$.

$$\mathbb{Z}^{(X)}(Y) = \mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(Y,X))} = \bigoplus_{h \in \mathrm{Hom}_{\mathscr{C}}(X,Y)} \mathbb{Z},$$

and $\varepsilon(Y)$ is the sum of the morphisms $d_0 : \mathbb{Z}^{(S_h)} \to \mathbb{Z}$ from Problem A.6.2(a). So the result follows again from Problem A.6.2(a).

(c). The $\check{\mathcal{C}}^p(\mathscr{X}, \mathscr{F}) = \prod_{i_0,\ldots,i_p} \mathscr{F}(X_{i_0,\ldots,i_p})$ follows immediately from the definition of $\mathscr{C}_p(\mathscr{X})$, the universal property of the direct sum and question (b) of Problem A.5.1.

In particular, we have $\check{\mathcal{C}}^0(\mathscr{X}, \mathscr{F}) = \prod_{i \in I} \mathscr{F}(X_i)$ and $\check{\mathcal{C}}^1(\mathscr{X}, \mathscr{F}) = \prod_{i,j \in I} \mathscr{F}(X_i \times_X X_j)$, and (by definition of $d_1 : \mathscr{C}_1(\mathscr{X}) \to \mathscr{C}_0(\mathscr{X})$) $d^0 : \check{\mathcal{C}}^0(\mathscr{X}, \mathscr{F}) \to \check{\mathcal{C}}^1(\mathscr{X}, \mathscr{F})$ sends a family $(s_i)_{i \in I}$ to $(p_{i,ij}^* s_i - p_{j,ij}^* s_j)_{i,j \in I}$, where $p_{i,ij} : X_i \times_X X_j \to X_i$ and $p_{j,ij} : X_i \times_X X_j \to X_j$ are the two projections. So $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathrm{Ker}(d^0)$ is equal to the set $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ of Definition III.2.2.4.

(d). If $\mathscr{F}$ is an injective object of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, then the functor $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(\cdot, \mathscr{F})$ is exact, so the statement follows from (a).

(e). The fact that $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathscr{F}(X)$ follows from the end of (c) and from the definition of a sheaf (see Remark III.2.2.5).

The inclusion functor $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \subset \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ is right adjoint to the sheafification functor and the sheafification functor is exact, so Φ sends injective objects of $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z})$ to injective objects of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ by Lemma II.2.4.4. So the fact that $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = 0$ for $p \geq 1$ follows from (e).

(f). Applying the functors $\check{\mathcal{C}}^p(\mathscr{X}, \cdot)$ to the complex $\mathscr{F} \to \mathscr{I}^\bullet$, we get a double complex in $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, whose pth row is $\check{\mathcal{C}}^p(\mathscr{X}, \mathscr{F}) \to \check{\mathcal{C}}^p(\mathscr{X}, \mathscr{I}^\bullet)$, whose (-1)th column is the complex $\check{\mathcal{C}}^\bullet(\mathscr{X}, \mathscr{F})$ and whose nth column is the complex $\check{\mathcal{C}}^\bullet(\mathscr{X}, \mathscr{I}^n)$ for $n \geq 0$ (the other columns are 0).

We consider the double complex, where we write $\check{\mathcal{C}}^p(\cdot)$ and $\check{\mathrm{H}}^0(\cdot)$ for $\check{\mathcal{C}}^p(\mathscr{X}, \cdot)$ and

$\check{\mathrm{H}}^0(\mathscr{X},\cdot)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \check{\mathrm{H}}^0(\mathscr{I}^0) \arrow[r] \arrow[d] & \check{\mathrm{H}}^0(\mathscr{I}^1) \arrow[r] \arrow[d] & \check{\mathrm{H}}^0(\mathscr{I}^2) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \check{\mathcal{C}}^0(\mathscr{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^0(\mathscr{I}^0) \arrow[r] \arrow[d] & \check{\mathcal{C}}^0(\mathscr{I}^1) \arrow[r] \arrow[d] & \check{\mathcal{C}}^0(\mathscr{I}^2) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \check{\mathcal{C}}^1(\mathscr{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^1(\mathscr{I}^0) \arrow[r] \arrow[d] & \check{\mathcal{C}}^1(\mathscr{I}^1) \arrow[r] \arrow[d] & \check{\mathcal{C}}^1(\mathscr{I}^2) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \check{\mathcal{C}}^2(\mathscr{F}) \arrow[r] \arrow[d] & \check{\mathcal{C}}^2(\mathscr{I}^0) \arrow[r] \arrow[d] & \check{\mathcal{C}}^2(\mathscr{I}^1) \arrow[r] \arrow[d] & \check{\mathcal{C}}^2(\mathscr{I}^2) \arrow[r] \arrow[d] & \dots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Every column of this double complex except for the first one is exact by (d). Also, every row except for the first one is exact, because the functor $\mathrm{PSh}(\mathscr{C},\mathbb{Z}) \to \mathbf{Ab}$, $\mathscr{G} \to \mathscr{G}(Y)$ is exact for every object Y of $\mathscr{C}$, and direct products of exact sequences in $\mathbf{Ab}$ are exact. So the $\infty \times \infty$ lemma (Corollary IV.2.2.4) gives a canonical isomorphism between the cohomology of the first row and the cohomology of the first column, which is exactly what the question is asking for.

□

A.6.4 The fpqc topology is subcanonical

Let A be a commutative ring and B be a commutative A-algebra. For every $n \geq 1$, we write $B^{\otimes n}$ for the n-fold tensor product $B \otimes_A B \otimes_A \ldots \otimes_A B$. We consider the following sequence $\mathscr{A}_{B/A}$ of morphisms of A-modules:

$$0 \to A \overset{d^0}{\to} B \overset{d^1}{\to} B^{\otimes 2} \overset{d^2}{\to} B^{\otimes 3} \to \ldots$$

where the morphism $\mathscr{A}^0_{B/A} = A \to \mathscr{A}^1_{B/A} = B$ is the structural morphism and $d^n : \mathscr{A}^n_{B/A} = B^{\otimes n} \to \mathscr{A}^{n+1}_{B/A} = B^{\otimes(n+1)}$ is defined by

$$d^n(b_1 \otimes \ldots \otimes b_n) = \sum_{i=1}^{n+1} (-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n.$$

For example, $d^1(b) = 1 \otimes b - b \otimes 1$ and $d^2(b_1 \otimes b_2) = 1 \otimes b_1 \otimes b_2 - b_1 \otimes 1 \otimes b_2 + b_1 \otimes b_2 \otimes 1$.

(a). Show that $\mathscr{A}_{B/A}$ is a complex. [24]

(b). Suppose that the morphism of A-algebras $A \to B$ has a section, that is, that there exists a morphism of A-algebras $s : B \to A$ such that $s \circ d^0 = \mathrm{id}_A$. Show that $\mathscr{A}_{B/A}$ is homotopic to 0 as a complex of A-modules.

(c). Under the hypothesis of (b), show that $\mathscr{A}_{B/A} \otimes_A M$ is acyclic for every A-module M.

(d). We don't assume that $A \to B$ has a section anymore. Let M be a A-module. Show that we have a canonical isomorphism

$$B \otimes_A (\mathscr{A}_{B/A} \otimes_A M) \xrightarrow{\sim} \mathscr{A}_{B \otimes_A B/B} \otimes_B (M \otimes_A B),$$

where we see $B \otimes_A B$ as a B-algebra via the morphism $b \longmapsto b \otimes 1$.

(e). Suppose that the morphism $A \to B$ is faithfully flat. Show that the complex $\mathscr{A}_{B/A} \otimes_A M$ is acyclic fo every A-module M.
Remark. If $(f_i)_{i \in I}$ is a family of elements generating the unit ideal of A, then $B := \prod_{i \in I} A_{f_i}$ is a faithfully flat A-algebra, and, for any A-module M, the complex $\mathscr{A}_{B/A} \otimes_A M$ is the Čech complex of the quasi-coherent sheaf on $\operatorname{Spec} A$ corresponding to M for the open cover $(D_{f_i})_{i \in I}$. Applying the result of (e), we see that the Čech cohomology of any quasi-coherent sheaf on $\operatorname{Spec} A$ for the open cover $(D_{f_i})_{i \in I}$ is zero in degree ≥ 1.

Let $A - \mathbf{CAlg}$ be the the category of commutative A-algebras, and $\mathscr{C} = (A - \mathbf{CAlg})^{\mathrm{op}}$; to distinguish between objects of $A - \mathbf{CAlg}$ and $\mathscr{C}$, we write $\operatorname{Spec} B$ for the object of $\mathscr{C}$ corresponding to a commutative A-algebra B. We consider the fpqc topology on $\mathscr{C}$; this means that covering families in $\mathscr{C}$ are morphisms $\operatorname{Spec} C \to \operatorname{Spec} B$ such that $B \to C$ is a faithfully flat A-algebra morphism; also, if $B = 0$, then the empty family covers $\operatorname{Spec} B$.

(f). Show that this is a Grothendieck pretopology on $\mathscr{C}$.

(g). Let M be a A-module. We define a presheaf $\mathscr{F}_M$ on $\mathscr{C}$ by $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M$; if $\operatorname{Spec} C \to \operatorname{Spec} B$ is a morphism of $\mathscr{C}$, corresponding to a morphism of A-algebras $u : B \to C$, then $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M \to \mathscr{F}_M(\operatorname{Spec} C) = C \otimes_A M$ sends $b \otimes m$ to $u(b) \otimes m$. Show that $\mathscr{F}_M$ is a sheaf.

(h). Show that every representable presheaf on $\mathscr{C}$ is a sheaf.

Solution.

(a). This is very similar to the beginning of A.6.2(a). If $a \in A$, then $1 \otimes a = a \otimes 1$ in $B \otimes_A B$,

[24] It is called the *Amitsur complex*, hence the notation.

so $d^1 \circ d^0(a) = 0$. Suppose that $n \geq 1$, and let $b_1, \ldots, b_n \in B$. Then

$$\begin{aligned}
d^{n+1} \circ d^n(b_1 \otimes b_n) &= d^{n+1}(\sum_{i=1}^{n+1}(-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n) \\
&= \sum_{i=1}^{n+1}\sum_{j=1}^{i}(-1)^{i+j} b_1 \otimes \ldots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n \\
&\quad + \sum_{i=1}^{n+1}\sum_{j=i}^{n+1}(-1)^{i+j+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_n \\
&= \sum_{j=1}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j} b_1 \otimes \ldots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n \\
&\quad + \sum_{i=1}^{n+1}\sum_{j=i}^{n+1}(-1)^{i+j+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_n \\
&= 0.
\end{aligned}$$

(b). We write $C^n = B^{\otimes n}$ for $n \geq 1$, $C^0 = A$, $C^n = 0$ for $n \leq -1$, and we denote $d^n : C^n \to C^{n+1}$ the morphism defined in the beginning. We define $s^n : C^n \to C^{n-1}$ in the following way:

- if $n \leq 0$, then $s^n = 0$;
- $s^1 = s : B \to A$;
- if $n \geq 2$, then $s^n : B^{\otimes n} \to B^{\otimes(n-1)}$ sends $b_1 \otimes \ldots \otimes b_n$ to $(-1)^{n-1} s(b_n)(b_1 \otimes \ldots \otimes s_{n-1})$ (this is A-linear in each b_i, hence does define a morphism on the tensor product).

We claim that $(s^n)_{n \in \mathbb{Z}}$ is a homotopy between $\mathrm{id}_{C^\bullet}$ and 0. To prove this claim, we have to calculate the morphism $g^n := d^{n-1} \circ s^n + s^{n+1} \circ d^n$ for every $n \in \mathbb{Z}$. If $n \leq -1$, then $g^n = 0 = \mathrm{id}_{C^n}$. If $n = 0$, then $g^n = s \circ d^0 = \mathrm{id}_A$. Suppose that $n \geq 1$. Then, for all $b_1, \ldots, b_n \in B$, we have that $g^n(b_1 \otimes \ldots \otimes b_n)$ is equal to

$$\begin{aligned}
&(-1)^{n-1} s(b_n) d^{n-1}(b_1 \otimes \ldots \otimes b_{n-1}) + s^{n+1}(\sum_{i=1}^{n+1}(-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n) \\
&= s(b_n) \sum_{i=1}^{n}(-1)^{i+n} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_{n-1} \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i+1} s(b_n)(b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_{n-1}) + s(1)(b_1 \otimes \ldots \otimes b_n) \\
&= b_1 \otimes \ldots \otimes b_n.
\end{aligned}$$

So $g^n = \mathrm{id}_{C^n}$.

Note that the homotopy that we just constructed is A-linear, so $\mathscr{A}_{B/A}$ is homotopic to 0 as a complex of A-modules.

(c). As the functor $(\cdot) \otimes_A M : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ is additive and the complex of A-modules $\mathscr{A}_{B/A}$ is homotopic to 0, the complex $\mathscr{A}_{B/A} \otimes_A M$ is also homotopic to 0, and in particular acyclic.

(d). In degree 0, this isomorphism is the isomorphism $B \otimes_A (A \otimes_A M) \simeq B \otimes_B (M \otimes_A B)$ sending $B \otimes (1 \otimes m)$ to $b \otimes (m \otimes 1) = 1 \otimes (m \otimes b)$. If $n \geq 1$, we have morphism $u : B \otimes_A (\mathscr{A}^n_{B/A} \otimes_A M) \to \mathscr{A}^n_{B \otimes_A B/B} \otimes_B M$ and $v : \mathscr{A}^n_{B \otimes_A B/B} \otimes_B M \to B \otimes_A (\mathscr{A}^n_{B/A} \otimes_A M)$ defined by

$$u(b_0 \otimes (b_1 \otimes \ldots \otimes b_n \otimes m)) = ((1 \otimes b_1) \otimes \ldots \otimes (1 \otimes b_n)) \otimes (m \otimes b_0)$$

and

$$v((b'_1 \otimes b_1) \otimes \ldots \otimes (b'_n \otimes b_n) \otimes (m \otimes b_0)) = (b_0 b'_1 \ldots b'_n) \otimes (b_1 \otimes \ldots \otimes b_n \otimes m)$$

if $b_0, b_1, b'_1, \ldots, b_n, b'_n \in B$ and M. It is easy to check that these morphisms are well-defined and inverses of each other.

(e). Note that the structural morphism $B \to B \otimes_A B$, $b \longmapsto b \otimes 1$ has a section $B \otimes_A B \to B$, $b_1 \otimes b_2 \longmapsto b_1 b_2$ which is a morphism of B-algebras. So, by (c) and (d), the complex of B-modules $B \otimes_A (\mathscr{A}_{B/A} \otimes_A M)$ is acyclic. As B is a faithfully flat A-algebra, this implies that the complex of A-modules $\mathscr{A}_{B/A} \otimes_A M$ is acyclic.

(f). We check the axioms of Definition III.2.1.1. Axiom (CF3) is clear, because an isomorphism of rings is faithfully flat. Axiom (CF2) says that the composition os two faithfully flat morphisms of A-algebras is also faithfully flat, which is also true. Axiom (CF1) says that, if $B \to C$ and $B \to D$ are faithfully flat morphisms of A-algebras, then $B \to C \otimes_B D$ is also faithfully flat, which is also true.

(g). The sheaf condition says that:

(1) If $B = 0$, then the sequence $0 \to B \otimes_A M \to 0$ is exact, which is certainly true.

(2) For every faithfully flat A-algebra morphism $B \to C$, the sequence

$$0 \to M' \xrightarrow{f} C \otimes_B M' \xrightarrow{g} (C \otimes_B C) \otimes_B M'$$

is exact, where $M' = B \otimes_A M$, f sends $m \in M'$ to $1 \otimes m \in C \otimes_B M$, and g sends $c \otimes m \in C \otimes_B M$ to $(1 \otimes c) \otimes m - (c \otimes 1) \otimes m$. This exactness follows from question (e).

(h). Let D be a commutative A-algebra. We want to show that the presheaf $\operatorname{Hom}_{\mathscr{C}}(\cdot, \operatorname{Spec} D)$ is a sheaf. If we consider the empty cover of $\operatorname{Spec}(0)$, the sheaf condition says that $\operatorname{Hom}_{\mathscr{C}}(\operatorname{Spec}(0), \operatorname{Spec}(D)) = \operatorname{Hom}_{A-\mathbf{CAlg}}(D, 0)$ should be a singleton, which is true. Let $u : B \to C$ be a faithfully flat morphism of commutative A-algebras. The sheaf condition for the covering family $\operatorname{Spec} C \to \operatorname{Spec} B$ says that:

(1) The map $\operatorname{Hom}_{A-\mathbf{CAlg}}(D, B) \to \operatorname{Hom}_{A-\mathbf{CAlg}}(D, C)$, $v \longmapsto u \circ v$ is injective; this is true because u, being faithfully flat, is injective.

(2) If $f : D \to C$ is a morphism of A-algebras such that $f(c) \otimes 1 = 1 \otimes f(c)$ in $C \otimes_B C$ for every $c \in C$, then there exists a morphism of A-algebras $v : D \to B$ such that $f = u \circ v$.

We prove (2). By (e), the kernel of the morphism $g : C \to C \otimes_B C$, $c \longmapsto 1 \otimes c - c \otimes 1$ is $u(B)$. The condition on f says that $g \circ f = 0$; as u is injective, it implies that we can write $f = u \circ v$, for a uniquely determined A-linear morphism $v : D \to B$. As u is an injective morphism of A-algebras and f is a morphism of A-algebras, the map v is also a morphism of A-algebras.

□

A.6.5 Čech cohomology, part 2

Let X be a topological space.

(a). Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of abelian sheaves on X, and let U be an open subset of X. Suppose that every open cover of U has a refinement $\mathscr{U}$ such that $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$. Show that the sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U) \to 0$$

is exact.

(b). Let $\mathscr{B}$ be a basis of the topology of X, and $\mathbf{Cov}$ be a set of open covers of open subsets of X, such that:

(1) If $(U_i)_{i \in I}$ is in $\mathbf{Cov}$, then $\bigcup_{i \in I} U_i$ and all the $U_{i_0} \cap \ldots \cap U_{i_p}$ are in $\mathscr{B}$, for $p \in \mathbb{N}$ and $i_0, \ldots, i_p \in I$.

(2) If $U \in \mathscr{B}$, then any open cover of U has a refinement in $\mathbf{Cov}$.

Let $\mathscr{I}$ be the full category of injective objects in $\mathrm{Sh}(X, \mathbb{Z})$, and $\mathscr{C}$ be the full category whose objects are abelian sheaves $\mathscr{F}$ such that $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{F}) = 0$ for every $\mathscr{U} \in \mathbf{Cov}$ and every $n \geq 1$. [25]

(i) Show that $\mathscr{C}$ contains $\mathscr{I}$ and is stable by taking cokernels of injective morphisms.

(ii) If $\mathscr{F}$ is an object of $\mathscr{C}$, show that, for every $U \in \mathscr{B}$, we have $\mathrm{H}^1(U, \mathscr{F}) = 0$.

[25] For example, if X is a scheme, we could take $\mathscr{B}$ to be the set of open affine subschemes of X and $\mathscr{U}$ to be the set of open covers of open affine subschemes of X by principal open affines, and then $\mathscr{C}$ would contain all the quasi-coherent sheaves on X.

(iii) Show by induction on n that, for every $n \geq 1$, every $U \in \mathscr{B}$ and every object $\mathscr{F}$ of $\mathscr{C}$, we have $\mathrm{H}^n(U, \mathscr{F}) = 0$.

(iv) Let $\mathscr{F}$ be an object of $\mathscr{C}$ and $\mathscr{X} = (U_i)_{i\in I}$ be an open cover of X such that, for every $p \in \mathbb{N}$ and all $i_0, \ldots, i_p$, we have $U_{i_0} \cap \ldots \cap U_{i_p} \in \mathscr{B}$. Show that the canonical morphism $\check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^n(X, \mathscr{F})$ of Example IV.4.1.14(2) is an isomorphism for every $n \geq 0$. [26]

Solution.

(a). We give names to the morphisms of the exact sequence: $0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$. Let U be an open subset of X. We know that the sequence $0 \to F(U) \to \mathscr{G}(U) \to \mathrm{H}(U)$ is exact, so it suffices to show that $\mathscr{G}(U) \to \mathrm{H}(U)$ is surjective.

Let $s \in \mathscr{F}(U)$. Choose an open cover $\mathscr{U} = (U_i)_{i\in I}$ such that, for every $i \in I$, there exists $s_i \in \mathscr{G}(U_i)$ such that $g(s_i) = s_{|U_i}$. By the hypothesis, after replacing $\mathscr{U}$ by a refinement, we may assume that $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$. For $i, j \in I$, let $s_{ij} = s_{i|U_i\cap U_j} - s_{j|U_i\cap U_j}$. As $g(s_i)$ and $g(s_j)$ are equal on $U_i \cap U_i$, there exists $t_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $f(t_{ij}) = s_{ij}$. If $i, j, k \in I$, then we have

$$\begin{aligned} s_{ij|U_i\cap U_j\cap U_k} - s_{ik|U_i\cap U_j\cap U_k} + s_{jk|U_i\cap U_j\cap U_k} &= s_{i|U_i\cap U_j\cap U_k} - s_{j|U_i\cap U_j\cap U_k} \\ &\quad - (s_{i|U_i\cap U_j\cap U_k} - s_{k|U_i\cap U_j\cap U_k}) + s_{j|U_i\cap U_j\cap U_k} - s_{k|U_i\cap U_j\cap U_k} \\ &= 0, \end{aligned}$$

so the family $(t_{ij})_{(i,j)\in I^2} \in \check{\mathcal{C}}^1(\mathscr{U}, \mathscr{F})$ is in the kernel of d^1. As $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$, there exists $(t_i)_{i\in I} \in \check{\mathcal{C}}^0(\mathscr{U}, \mathscr{F}) = \prod_{i\in I} \mathscr{F}(U_i)$ such that $d^0((t_i)_{I\in I}) = (t_{ij}))$, that is, $t_{ij} = t_{i|U_i\cap U_j} - t_{j|U_i\cap U_j}$. For every $i \in I$, let $s'_i = s_i - f(t_i)$. Then, for $i, j \in I$, we have

$$s'_{i|U_i\cap U_j} - s'_{j|U_i\cap U_j} = s_{ij} - f(t_{ij}) = 0.$$

So there exists $s' \in \mathscr{G}(U)$ such that $s'_{|U_i} = s'_i$ for every $i \in I$. Moreover, we have $g(s')_{|U_i} = g(s'_i) = g(s_i) = s_{|U_i}$ for every $i \in I$, so $g(s') = s$.

(b). (i) Let $\mathscr{F}$ be an object of $\mathscr{I}$. We know that $\check{\mathrm{H}}^p(\mathscr{U}, \mathscr{F}) = 0$ for every covering family family $\mathscr{U}$ of an open subset of X and for every $p \geq 1$ by question (e) of problem A.6.3, so $\mathscr{F}$ is in $\mathscr{C}$.

Now let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of abelian sheaves on X, and suppose that $\mathscr{F}$ and $\mathscr{G}$ are in $\mathscr{C}$. By question (a), the sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is also exact as a sequence of abelian presheaves. Let $\mathscr{U} \in \mathbf{Cov}$ By problem A.6.2, the functors $\check{\mathrm{H}}^n(\mathscr{U}, \cdot)$ are the right derived functors of $\check{\mathrm{H}}^0(\mathscr{U}, \cdot)$ on the category $\mathrm{PSh}(X, \mathbb{Z})$, so we have a long exact sequence

$$\ldots \to \check{\mathrm{H}}^n(\mathscr{U}, \mathscr{G}) \to \check{\mathrm{H}}^n(\mathscr{U}, \mathscr{H}) \to \check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{F}) \to \check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{G}) \to \ldots$$

[26] If X is a scheme, this shows that, for every quasi-coherent sheaf $\mathscr{F}$ on X, the cohomology of $\mathscr{F}$ is isomorphism to its Čech cohmology relative to any open affine cover of X.

If $n \geq 1$, then $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{G}) = 0$ and $\check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{F}) = 0$ by the hypothesis on $\mathscr{F}$ and $\mathscr{G}$, so $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{H}) = 0$. This shows that $\mathscr{H}$ is an object of $\mathscr{C}$.

(ii) Let $\mathscr{F}$ be an object of $\mathscr{C}$, let $f : \mathscr{F} \to \mathscr{G}$ be an injective morphism of abelian sheaves with $\mathscr{G}$ an injective object of $\mathrm{Sh}(X, \mathbb{Z})$, and $\mathscr{H} = \mathrm{Coker}(f)$. Let $U \in \mathscr{B}$. Then we have an exact sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U) \to \mathrm{H}^1(U, \mathscr{F}) \to \mathrm{H}^1(U, \mathscr{G}) \to \dots$$

But $\mathrm{H}^1(U, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective, and the morphism $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective by (i), so $\mathrm{H}^1(U, \mathscr{F}) = 0$.

(iii) We already know that the result holds for $n = 1$ by question (iii). Suppose that it holds for some $n \geq 1$. Let let $\mathscr{F}$ be an object of $\mathscr{C}$. Choose an injective morphism $f : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an object of $\mathscr{I}$, and let $\mathscr{H} = \mathrm{Coker}\, f$. Let $U \in \mathscr{B}$. We have a long exact sequence of cohomology

$$\dots \mathrm{H}^n(U, \mathscr{H}) \to \mathrm{H}^{n+1}(U, \mathscr{F}) \to \mathrm{H}^{n+1}(U, \mathscr{G}) \to \dots$$

By question (ii), the sheaf $\mathscr{H}$ is an object of $\mathscr{C}$, so $\mathrm{H}^n(U, \mathscr{H}) = 0$ by the induction hypothesis. Moreover, as $\mathscr{G}$ is an injective object of $\mathrm{Sh}(X, \mathbb{Z})$, we have $\mathrm{H}^{n+1}(U, \mathscr{G}) = 0$. So $\mathrm{H}^{n+1}(U, \mathscr{F}) = 0$.

(iv) We use the notation of Example IV.4.1.14(2) . By question (iii), for every $p \in \mathbb{N}$, all $i_0, \dots, i_p \in I$, and every $q \geq 1$, we have

$$R^q\Phi(\mathscr{F})(U_{i_0} \cap \dots \cap U_{i_p}) = \mathrm{H}^q(U_{i_0} \cap \dots \cap U_{i_p}, \mathscr{F}) = 0.$$

By definition of Čech cohomology, this implies that, for every $p \in \mathbb{N}$ and every $q \geq 1$, we have $\check{\mathrm{H}}^p(\mathscr{X}, R^q\Phi(\mathscr{F})) = 0$. Let

$$E_2^{pq} = \check{\mathrm{H}}^p(\mathscr{X}, R^q\Phi(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

be the Čech cohomology to cohomology spectral sequence for the open cover $\mathscr{X}$. By the calculation we just did, we have $E_2^{pq} = 0$ if $q \geq 1$, so the spectral sequence degenerates at E_2 and $E_\infty^{pq} = E_2^{pq}$ is zero unless $q = 0$. So for every $p \in \mathbb{N}$, the subobject $E_\infty^{p,0} = E_2^{p,0} = \check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F})$ of $\mathrm{H}^p(X, \mathscr{F})$ is actually equal to $\mathrm{H}^p(X, \mathscr{F})$, which shows that the morphism $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^p(\mathscr{X}, \mathscr{F})$ is an isomorphism.

□

A.7 Problem set 7

A.7.1 Diagram chasing lemmas via spectral sequences

This problem will ask to reprove some of the diagram chasing lemmas using the two spectral sequences of a double complex. This is circular, because of course the diagram chasing lemmas are used to establish the existence of the spectral sequences. The goal is just to get you used to manipulating spectral sequences on simple examples.

(a). **The $\infty \times \infty$ lemma**: Suppose that we have a double complex $X = (X^{n,m}, d_1^{n,m}, d_2^{n,m})$ such that $X^{n,m} = 0$ if $n < 0$ or $m < 0$. Suppose also that the complexes $(X^{\bullet,n}, d_{1,X}^{\bullet,n})$ and $(X^{n,\bullet}, d_{2,X}^{n,\bullet})$ are exact if $n \neq 0$. Using the two spectral sequences of the double complex, prove that we have canonical isomorphisms

$$\mathrm{H}^n(X^{\bullet,0}, d_{1,X}^{\bullet,0}) \simeq \mathrm{H}^n(X^{0,\bullet}, d_{2,X}^{0,\bullet}).$$

(Hint: Both spectral sequences degenerate at the first page.)

(b). **The four lemma**: Consider a commutative diagram with exact rows in $\mathscr{A}$:

(*)
$$\begin{tikzcd} A' \arrow[r] & B' \arrow[r, "g"] & C' \arrow[r] & D' \\ A \arrow[r] \arrow[u, "u"] & B \arrow[r, "f"'] \arrow[u, "v"] & C \arrow[r] \arrow[u, "w"] & D \arrow[u, "t"] \end{tikzcd}$$

Suppose that u is surjective and t is injective. We want to show that $f(\operatorname{Ker} v) = \operatorname{Ker} w$ and that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$.

(i) Show that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$ if and only if the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ induced by g is injective.

We consider the double complex X represented on diagram (*), with the convention that all the objects that don't appear are 0, the object A is in bidegree $(0,0)$, the differential $d_{1,X}$ is horizontal and the differential $d_{2,X}$ is vertical. (So, for example, $X^{3,0} = C$, $X^{1,0} = A'$ and $X^{2,2} = 0$.) Let ${}^I E$ and ${}^{II} E$ the two spectral sequences of this double complex.

(ii) Show that ${}^{II} E$ degenerates at the second page.

(iii) Show that $\mathrm{H}^2(\mathrm{Tot}(X)) = 0$.

(iv) Write the first page of ${}^I E$.

(v) Show that ${}^I E$ degenerates at the second page.

(vi) Show that $f(\operatorname{Ker} v) = \operatorname{Ker} w$ and that $\operatorname{Im} v = g^{-1}(\operatorname{Im} w)$.

(c). **The long exact sequence of cohomology**: We consider a short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$; to simplify the notation, we will assume that

$A^n = B^n = C^n = 0$ for $n < 0$. Consider the following double complex X:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \dots
\end{array}$$

where $X^{0,0} = A$, the differential $d_{1,X}$ (resp. $d_{2,X}$) is represented horizontally (resp. vertically), and $X^{n,m} = 0$ if $n < 0$, $m < 0$ or $m \geq 3$. Let ${}^{I}E$ and ${}^{II}E$ be the two spectral sequences of X.

(i) Show that ${}^{I}E$ degenerates at the first page and that $\mathrm{H}^n(\mathrm{Tot}(X)) = 0$ for every $n \in \mathbb{Z}$.

(ii) Calculate ${}^{II}E_1$.

(iii) Show that ${}^{II}E$ degenerates at the third page.

(iv) Show that ${}^{II}E_2^{00} = {}^{II}E_\infty^{00}$ and that ${}^{II}E_2^{1q} = {}^{II}E_\infty^{1q}$ for every $q \geq 0$.

(v) Show that $d_2^{0q} : {}^{II}E_2^{0q} \to {}^{II}E_2^{2,q-1}$ is an isomorphism for every $q \geq 1$.

(vi) Show that we have a long exact sequence

$$\dots \to \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet) \to \mathrm{H}^n(C^\bullet) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(A^\bullet) \to \mathrm{H}^{n+1}(B^\bullet) \to \dots$$

where δ^n comes from a differential of the spectral sequence ${}^{II}E$.

Solution.

(a). Consider the two spectral sequences ${}^{I}E$ and ${}^{II}E$ of the double complex X. We have ${}^{I}E_1^{p,q} = \mathrm{H}^q(X^{p,\bullet}, d_{2,X}^{p,\bullet})$; as all the columns of the double complex are supposed exact except for $X^{0,\bullet}$, this implies that ${}^{I}E_1^{pq} = 0$ for $p \neq 0$. As ${}^{I}E_r^{pq}$ is a subquotient of ${}^{I}E_q^{pq}$ for $r \geq 1$, we deduce that ${}^{I}E_r^{pq} = 0$ for every $r \geq 1$ and every $p \neq 0$, and in particular $d_r^{pq} : {}^{I}E_r^{pq} \to {}^{I}E_r^{p+r,q-r+1}$ is the zero morphism if $r \geq 1$, because its source or target is 0. So the spectral sequence ${}^{I}E$ degenerates at the first page, and we have ${}^{I}E_\infty^{pq} = {}^{I}E_1^{pq}$. Also, as X is a first quadrant double complex, the spectral sequence ${}^{I}E$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, so we get

$$\mathrm{H}^n(\mathrm{Tot}(X)) = {}^{I}E_\infty^{0,n} = \mathrm{H}^n(X^{0,\bullet}, d_{2,X}^{0,\bullet}).$$

On the other hand, we have ${}^{II}E_1^{pq} = \mathrm{H}^q(X^{\bullet,p}, d_{1,X}^{\bullet,p})$. As all the rows of the double complex are supposed exact except for $X^{\bullet,0}$, this implies that ${}^{II}E_1^{pq} = 0$ if $p \neq 0$. Reasoning as in the first paragraph, we deduce that the spectral sequence ${}^{II}E$ degenerates at the first page, and that we have ${}^{II}E_\infty^{pq} = {}^{II}E_1^{pq}$. As ${}^{II}E$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, this gives canonical isomorphisms

$$\mathrm{H}^n(\mathrm{Tot}(X)) = {}^{II}E_\infty^{0,n} = \mathrm{H}^n(X^{\bullet,0}, d_{1,X}^{\bullet,0}).$$

(b). (i) As $\operatorname{Im} w$ is the kernel of the canonical morphism $C' \to \operatorname{Coker} w$, the subobject $g^{-1}(\operatorname{Im} w)$ of B' is the kernel of the morphism $B' \xrightarrow{g} C' \to \operatorname{Coker} w$, which is also equal to the morphism $B' \to \operatorname{Coker} v \to \operatorname{Coker} w$, where the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ is induced by g. Note also that we always have $\operatorname{Im} v \subset g^{-1}(\operatorname{Im} w)$, because $g \circ v = w \circ f$. So the kernel of the morphism $\operatorname{Coker} v \to \operatorname{Coker} w$ induced by g is $g^{-1}(\operatorname{Im} w)/\operatorname{Im} v$, which gives the result.

(ii) Let us give names to all the morphisms in the diagram:

$$\begin{tikzcd} A' \arrow[r, "b"] & B' \arrow[r, "g"] & C' \arrow[r, "d"] & D' \\ A \arrow[u, "u"] \arrow[r, "a"'] & B \arrow[u, "v"] \arrow[r, "f"'] & C \arrow[u, "w"] \arrow[r, "c"'] & D \arrow[u, "t"'] \end{tikzcd}$$

As the rows are exact, we have ${}^{II}E_1^{0,0} = \operatorname{Ker} a$, ${}^{II}E^{0,3} = \operatorname{Coker} c$, ${}^{II}E_1^{0,1} = {}^{II}E_1^{0,2} = 0 = {}^{II}E_1^{1,1} = {}^{II}E_1^{1,2} = 0$, ${}^{II}E_1^{1,0} = \operatorname{Ker} b$, ${}^{II}E_1^{1,3} = \operatorname{Coker} d$, and the other ${}^{II}E_1^{p,q}$ are all 0. In other words, the first page of ${}^{II}E$ looks like this:

$$\begin{array}{ccc} 0 & 0 & 0 \\ \operatorname{Coker} c & \operatorname{Coker} d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \operatorname{Ker} a & \operatorname{Ker} b & 0 \end{array}$$

In particular, for every $r \geq 1$, we have ${}^{II}E_r^{pq} = 0$ if $(p,q) \not\in \{(0,0),(3,0),(1,0),(1,3)\}$, so, if $r \geq 2$, every d_r^{pq} has its source or target zero. Hence ${}^{II}E$ degenerates at the second page, and ${}^{II}E_\infty = {}^{II}E_2$.

(iii) As X is a first quadrant double complex, the spectral sequences ${}^{I}E$ and ${}^{II}E$ both converge to the cohomology of $\mathrm{Tot}(X)$. Also, by the calculation in question (ii), for all $(p,q) \in \mathbb{Z}$ such that $p+q = 2$, we have ${}^{II}E_\infty^{pq} = {}^{II}E_2^{pq} = 0$. So $\mathrm{H}^2(\mathrm{Tot}(X)) = 0$.

(iv) By definition of ${}^{I}E$, its first page is (where every term that doesn't appear is 0):

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & \operatorname{Coker} v & \operatorname{Coker} w & \operatorname{Coker} t & 0 \\ \operatorname{Ker} u & \operatorname{Ker} v & \operatorname{Ker} w & 0 & 0 \end{array}$$

(v) For every $r \geq 1$, we have ${}^I E_r^{pq} = 0$ unless $p \in \{0, 1\}$. In particular, if $r \geq 2$, then either the source of the target of d_r^{pq} is 0, so $d_r^{pq} = 0$. This shows that ${}^I E$ degenerates at the second page, hence that ${}^I E_\infty = {}^I E_2$.

(vi) As ${}^I E_\infty = {}^I E_2$, there exists a filtration on $\mathrm{H}^2(\mathrm{Tot}(X))$ whose quotients are the $\mathrm{E}_2^{p,2-p}$. But we have seen in questino (iii) that $\mathrm{H}^2(\mathrm{Tot}(X)) = 0$, so ${}^I E_2^{p,2-p} = 0$ for every $p \in \mathbb{Z}$. On the other hand, by question (iv), we have ${}^I E_2^{2,0} = \mathrm{Ker}\, w/f(\mathrm{Ker}\, v)$, ${}^I E_2^{1,1} = \mathrm{Ker}(g : \mathrm{Coker}\, v \to \mathrm{Coker}\, w)$ and ${}^I E_2^{0,2} = 0$. This shows that $\mathrm{Ker}\, w = f(\mathrm{Ker}\, v)$ and that the morphism $\mathrm{Coker}\, v \to \mathrm{Coker}\, w$ induced by g is injective; by question (i), that last fact is equivalent to the fact that $\mathrm{Im}\, v = g^{-1}(\mathrm{Im}\, w)$, so we are done.

(c). (i) As all the columns of the complex are exact, we have ${}^I E_1^{pq} = 0$ for all $p, q \in \mathbb{Z}$, so the spectral sequence ${}^I E$ degenerates at the first page, and we have ${}^I E_\infty = {}^I E_1 = 0$. Also, as X is a first quadrant double complex, the spectral sequence ${}^I E$ converges to $\mathrm{H}^\bullet(\mathrm{Tot}(X))$, so $\mathrm{H}^n(\mathrm{Tot}(X)) = 0$ for every $n \in \mathbb{Z}$.

(ii) Applying the formula for ${}^{II} E_1$, we get that it is equal to:

$$\begin{array}{cccc}
\cdots & \cdots & \cdots & 0 \\
\mathrm{H}^2(A^\bullet) & \mathrm{H}^2(B^\bullet) & \mathrm{H}^2(C^\bullet) & 0 \\
\mathrm{H}^1(A^\bullet) & \mathrm{H}^1(B^\bullet) & \mathrm{H}^1(C^\bullet) & 0 \\
\mathrm{H}^0(A^\bullet) & \mathrm{H}^0(B^\bullet) & \mathrm{H}^0(C^\bullet) & 0
\end{array}$$

In other words, we have ${}^{II} E_1^{pq} = 0$ if $p \notin \{0, 1, 2\}$, ${}^{II} E_1^{0,q} = \mathrm{H}^q(A^\bullet)$, ${}^{II} E_1^{1,q} = \mathrm{H}^q(B^\bullet)$ and ${}^{II} E_1^{2,q} = \mathrm{H}^q(C^\bullet)$.

(iii) By question (ii), we have ${}^{II} E_r^{pq} = 0$ if $r \geq 1$ and $p \notin \{0, 1, 2\}$. So, if $r \geq 3$, then either or target of $d_r^{pq} : {}^{II} E_r^{pq} \to {}^{II} E_e^{p+r,q-r+1}$ is 0, hence $d_r^{pq} = 0$. This shows that ${}^{II} E$ degenerates at the third page.

(iv) For every $r \geq 2$ and every $q \in \mathbb{Z}$, we have ${}^{II} E_r^{1+r,q-r+1} = 0$ and ${}^{II} E_r^{1-r,q+r-1} = 0$, so $d_r^{1,q} : {}^{II} E_r^{1q} \to {}^{II} E_r^{1+r,q-r+1}$ and $d_r^{1-r,q+r-1} : {}^{II} E_r^{1-r,q+r-1} \to {}^{II} E_r^{1,q}$ are both zero, so ${}^{II} E_{r+1}^{1,q} = {}^{II} E_r^{1,q}$. This shows that ${}^{II} E_\infty^{1,q} = {}^{II} E_2^{1,q}$ for every $q \in \mathbb{Z}$.

Also, if $r \geq 2$, we have ${}^{II} E_r^{r,-r+1} = {}^{II} E_2^{-r,r-1} = 0$, so $d_r^{0,0} : {}^{II} E_r^{0,0} \to {}^{II} E_r^{r,-r+1}$ and $d_r^{-r,r-1} : {}^{II} E_r^{-r,r-1} \to {}^{II} E_r^{0,0}$ are both zero, so ${}^{II} E_{r+1}^{0,0} = {}^{II} E_r^{0,0}$. This shows that' ${}^{II} E_\infty^{0,0} = {}^{II} E_2^{0,0}$. (Note that we only used the fact that we have a first quadrant spectral sequence in this paragraph.)

(v) As the spectral sequence ${}^{II} E$ degenerates at the third page and its limit $\mathrm{H}^\bullet(\mathrm{Tot}(X))$ is 0 by question (i), we have ${}^{II} E_3^{pq} = {}^{II} E_\infty^{p,q} = 0$ for all $p, q \in \mathbb{Z}$. As ${}^{II} E_3^{0,q} = \mathrm{Ker}(d_2^{0,q})$ and ${}^{II} E_3^{2,q-1} = \mathrm{Coker}(d_2^{0,q})$, this shows that $d_2^{0,q}$ is an isomorphisms for every $q \in \mathbb{Z}$.

(vi) We have ${}^{II} E_2^{0,0} = {}^{II} E_\infty^{0,0} = 0$ by questions (iv) and (v), so the morphism

$d_1^{0,0} : \mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet)$ is injective. Also, for every $q \geq 0$, we have ${}^{II}E_\infty^{1,q} = {}^{II}E_2^{1,q} = 0$, so the sequence

$$\mathrm{H}^q(A^\bullet) \xrightarrow{d_1^{0,q}} \mathrm{H}^q(B^\bullet) \xrightarrow{d_1^{1,q}} \mathrm{H}^q(C^\bullet)$$

is exact. Finally, we have seen in question (v) that, for every $q \geq 1$, the morphism

$$d_2^{0,q} : {}^{II}E_2^{0,q} = \mathrm{Ker}(d_1^{0,q}) \to {}^{II}E_2^{2,q-1} = \mathrm{Coker}(d_1^{1,q-1})$$

is an isomorphism; inverting it, we get an exact sequence

$${}^{II}E_1^{1,q-1} = \mathrm{H}^{q-1}(B^\bullet) \xrightarrow{d_1^{1,q-1}} {}^{II}E_1^{2,q-1} = \mathrm{H}^{q-1}(C^\bullet) \xrightarrow{\delta^{q-1}} {}^{II}E_1^{0,q} = \mathrm{H}^q(A^\bullet) \xrightarrow{d_1^{0,q}} {}^{II}E_1^{1,q} = \mathrm{H}^q(B_\bullet).$$

Putting all these exact sequences together gives the long exact that we wanted.

□

A.7.2 Group cohomology

(a). **Cohomology of cyclic groups**: If G is a group, $a \in \mathbb{Z}[G]$ and M is a left $\mathbb{Z}[G]$-module, we denote by $a : M \to M$ the $\mathbb{Z}[C_n]$-linear map $x \longmapsto a \cdot x$. For every $n \geq 1$, we denote by C_n the cyclic group of order n and by σ a generator of C_n, and we write $N = 1 + \sigma + \sigma^2 + \ldots + \sigma^{n-1}$. We also write $C_\infty = \mathbb{Z}$ and $\sigma = 1 \in C_\infty$. If $n \in \{1, 2, \ldots\} \cup \{\infty\}$, we have a $\mathbb{Z}[C_n]$-linear map $\epsilon : \mathbb{Z}[C_\infty] \to \mathbb{Z}$ sending each element of C_n to $1 \in \mathbb{Z}$.

(i) If $n \geq 1$, show that:

$$\ldots \to \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{\sigma-1} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{\sigma-1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is an exact sequence.

(ii) If M is a $\mathbb{Z}[C_n]$-module, show that:

$$\mathrm{H}^q(C_n, M) = \begin{cases} M^{C_n} & \text{if } q = 0 \\ M^{C_n}/N \cdot M & \text{if } q \geq 2 \text{ is even} \\ \{x \in M \mid N \cdot x = 0\}/(\sigma - 1) \cdot M & \text{if } q \text{ is odd.} \end{cases}$$

(iii) Show that

$$0 \to \mathbb{Z}[C_\infty] \xrightarrow{\sigma-1} \mathbb{Z}[C_\infty] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is an exact sequence.

(iv) If M is a $\mathbb{Z}[C_\infty]$-module, show that:

$$\mathrm{H}^q(C_\infty, M) = \begin{cases} \{x \in M \mid \sigma \cdot x = x\} & \text{if } q = 0 \\ M/(\sigma - 1) \cdot M & \text{if } q = 1 \\ 0 & \text{if } q \geq 2. \end{cases}$$

(b). Let n be a integer, and let $G = C_n \rtimes C_2$ be the dihedral group of order $2n$, where the nontrivial element of C_2 acts on C_n by multiplication by -1. Then $K = C_n$ is a normal subgroup of G, and $G/K \simeq C_2$.

(i) Show that

$$\mathrm{H}^q(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q \geq 2 \text{ is even} \\ 0 & \text{if } q \text{ is odd,} \end{cases}$$

and show that the nontrivial element of C_2 acts by $(-1)^{q/2}$ on $\mathrm{H}^q(C_n, \mathbb{Z})$ if q is even.

(ii) Calculate $\mathrm{H}^p(C_2, \mathrm{H}^q(C_n, \mathbb{Z}))$ for all $p, q \geq 0$.

(iii) If n is odd, show that

$$\mathrm{H}^m(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2 \mod 4 \\ \mathbb{Z}/2n\mathbb{Z} & \text{if } m > 0 \text{ and } m = 0 \mod 4 \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

(c). Let G be a group, and suppose that G has a normal subgroup K such that $G/K \simeq \mathbb{Z}$. Let M be a $\mathbb{Z}[G]$-module.

(i) Show that the Hochschild-Serre spectral sequence degenerates at E_2.

(ii) We fix a generator σ of G/K and, for every $q \in \mathbb{N}$, we write $\mathrm{H}^q(K, M)^\sigma = \{x \in \mathrm{H}^q(K, M) \mid \sigma(x) = x\}$ and $\mathrm{H}^q(K, M)_\sigma = \mathrm{H}^q(K, M)/(\sigma - 1) \cdot H^q(K, M)$.

Show that $\mathrm{H}^0(G, M) = \mathrm{H}^0(K, M)^\sigma$, and that we have short exact sequences

$$0 \to \mathrm{H}^{m-1}(K, M)_\sigma \to \mathrm{H}^m(G, M) \to \mathrm{H}^m(K, M)^\sigma \to 0$$

for every $m \geq 1$.

(d). Let G be a group.

(i) If K is a central subgroup of G, show that G/K acts trivially on $\mathrm{H}_*(K, \mathbb{Z})$ and on $\mathrm{H}^*(K, \mathbb{Z})$.

Let σ be an element of infinite order in the center of G, and $K = \langle \sigma \rangle$. Let M be a $\mathbb{Z}[G]$-module. We write $M^\sigma = \{x \in M \mid \sigma \cdot x = x\}$ and $M_\sigma = M/(\sigma - 1) \cdot M$.

(ii) Show that the Hochschild-Serre spectral sequence calculating $\mathrm{H}^*(G, M)$ degenerates at E_3.

(iii) Show that $\mathrm{H}^0(G, M) = \mathrm{H}^0(G/K, M^\sigma)$, and that we have a long exact sequence:

$$0 \to \mathrm{H}^1(G/K, M^\sigma) \to \mathrm{H}^1(G, M) \to \mathrm{H}^0(G/K, M_\sigma) \to \mathrm{H}^2(G/K, M^\sigma) \to \mathrm{H}^2(G, M) \to \mathrm{H}^1(G/K, M_\sigma) \to \mathrm{H}^3(G/K, M^\sigma) \to \dots$$

Solution.

(a). (i) Let $x = \sum_{i=0}^{n-1} a_i\sigma^i \in \mathbb{Z}[C_n]$, with $a_0, \dots, a_{n-1} \in \mathbb{Z}$; we also write $a_n = a_0$ and $a_{-1} = a_{n-1}$. We have $\epsilon(x) = a_0 + a_1 + \dots + a_{n-1}$, $(\sigma - 1)(x) = \sum_{i=0}^{n-1}(a_{i+1} - a_i)\sigma^i$ and $N(x) = (a_0 + a_1 + \dots + a_{n-1})\sum_{i=0}^{n-1}\sigma^i$. So

$$\mathrm{Ker}(\sigma - 1) = \{x = \sum_{i=0}^{n-1} a_i\sigma^i \mid a_0 = a_1 = \dots = a_{n-1}\} = \mathrm{Im}(N)$$

and

$$\mathrm{Im}(\sigma - 1) = \{x = \sum_{i=0}^{n-1} a_i\sigma^i \mid a_0 + a_1 + \dots + a_{n-1} = 0\} = \mathrm{Ker}(N) = \mathrm{Ker}(\epsilon).$$

(ii) Question (i) gives a resolution of the trivial $\mathbb{Z}[C_n]$-module $\mathbb{Z}$ by free $\mathbb{Z}[C_n]$-modules, so we can use it to calculate $\mathrm{H}^n(C_n, M) = \mathrm{Ext}^n_{\mathbb{Z}[C_n]}(\mathbb{Z}, M)$ by Theorem IV.3.4.1. So $\mathrm{H}^n(C_m, M)$ is the cohomology of the following complex (concentrated in degree ≥ 0):

$$\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], M) \overset{(\cdot)\circ(\sigma-1)}{\to} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], M) \overset{(\cdot)\circ N}{\to} \mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], M) \overset{(\cdot)\circ(\sigma-1)}{\to} \dots$$

We have an isomorphism $\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], M) \overset{\sim}{\to} M$ sending $u : \mathbb{Z}[C_n] \to M$ to $u(\sigma)$. By isomorphism, the endomorphism1 $(\cdot) \circ (\sigma - 1)$ (resp. $(\cdot) \circ N$) of $\mathrm{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], M)$ corresponds to the action of $\sigma - 1$ (resp. N) on M. Moreover, as σ generates C_n, we have $\mathrm{Ker}(\sigma - 1 : M \to M) = M^{C_n}$. This gives the desired formulas for $\mathrm{H}^q(C_n, M)$.

(iii) Let $x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathbb{Z}[C_\infty]$, with $a_n = 0$ for $|n|$ big enough. Then $\epsilon(x) = \sum_{n\in\mathbb{Z}} a_n$ and $(\sigma - 1)(x) = \sum_{n\in\mathbb{Z}}(a_{n+1} - a_n)\sigma^n$. So

$$\mathrm{Im}(\sigma - 1) = \{x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathbb{Z}[C_\infty] \mid \sum_{n\in\mathbb{Z}} a_n = 0\} = \mathrm{Ker}(\epsilon).$$

On the other hand, we have $x = \sum_{n\in\mathbb{Z}} a_n\sigma^n \in \mathrm{Ker}(\sigma - 1)$ if and only if $a_n = a_{n+1}$ for every $n \in \mathbb{Z}$, i.e. if and only if all the a_n are equal; as we must have $a_n = 0$ for $|n|$ big enough, this forces all the a_n to be 0. So $\mathrm{Ker}(\sigma - 1) = \{0\}$.

(iv) This is exactly the same proof as in question (ii), except that we wrote $\mathrm{Ker}(\sigma - 1 : M \to M)$ as $\{x \in M \mid \sigma \cdot x = x\}$ instead of M^{C_∞}. (These are just two ways of writing the same object.)

(b). (i) We apply the formulas of question (a)(ii). As C_n acts trivially on $\mathbb{Z}$, we have $\mathbb{Z}^{C_n} = \mathbb{Z}$, and N acts on $\mathbb{Z}$ by multiplicatoion by $\sum_{i=0}^{n-1} 1 = n$, so $N \cdot \mathbb{Z} = n\mathbb{Z}$ and $\{x \in \mathbb{Z} \mid N \cdot x = 0\} = \{0\}$. This immediately gives the desired formula for $\mathrm{H}^q(C_n, \mathbb{Z})$.

Let τ be the nontrivial element of C_2. Then, if we make C_n act on it via $(g, x) \longmapsto (\tau g \tau^{-1}) \cdot x$, the resolution of (a)(i) is isomorphic to the following (exact) complex of $\mathbb{Z}[C_n]$-modules:

$$(*) \quad \ldots \to \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{-\sigma+1} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{-\sigma+1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

To calculate the action of τ on $\mathrm{H}^\bullet(C_n, \mathbb{Z})$, we need to extend the action of τ on $\mathbb{Z}$ (which is given by $\mathrm{id}_\mathbb{Z}$) to a morphism between the resolution of (a)(i) and (*). Here is a possibility:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "(-\sigma)^3"'] & \mathbb{Z}[C_n] \arrow[r, "N"] \arrow[d, "(-\sigma)^2"'] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "(-\sigma)^2"'] & \mathbb{Z}[C_n] \arrow[r, "N"] \arrow[d, "-\sigma"'] & \mathbb{Z}[C_n] \arrow[r, "\sigma-1"] \arrow[d, "-\sigma"'] & \mathbb{Z}[C_n] \arrow[r, "\epsilon"] \arrow[d, "1"'] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}_\mathbb{Z}"'] & 0 \\
\ldots \arrow[r] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "N"] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "N"] & \mathbb{Z}[C_n] \arrow[r, "-\sigma+1"] & \mathbb{Z}[C_n] \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

So, on $\mathrm{H}^{2i}(C_n, \mathbb{Z})$, the action of τ is given by $(-\sigma)^i$; as $\mathrm{H}^{2i}(C_n, \mathbb{Z})$ is a quotient $\mathbb{Z}^{C_n} = \mathbb{Z}$, where C_n acts trivially, the action of $(-\sigma)^i$ is given by multiplication by $(-1)^i$.

(ii) We apply the formulas of (a)(ii) for $n = 2$. If q is odd, then $\mathrm{H}^q(C_n, \mathbb{Z}) = 0$, so $\mathrm{H}^p(C_2, \mathrm{H}^q(C_n, \mathbb{Z})) = 0$ for every $p \geq 0$. If $q = 0$, then $\mathrm{H}^q(C_n, \mathbb{Z}) = \mathbb{Z}$ with the trivial action of C_2, so, by question (i),

$$\mathrm{H}^p(C_2, \mathrm{H}^0(C_n, \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \geq 2 \text{ is even} \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Suppose that $q \geq 2$ is even and write $q = 2i$. Then $\mathrm{H}^q(C_n, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ and the nontrivial element τ of C_2 acts by $(-1)^i$ on $\mathrm{H}^q(C_n, \mathbb{Z})$. We use the formula of (a)(ii), and we distinguish four cases:

(1) <u>i is even and n is odd</u>: Then $\mathrm{H}^{2i}(C_n, \mathbb{Z})^{C_2} = \mathrm{H}^{2i}(C_n, \mathbb{Z})$, $\mathrm{H}^{2i}(C_n, \mathbb{Z})/(1 + \tau) \cdot \mathrm{H}^{2i}(C_n, \mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z}) = 0$ and $\{x \in \mathbb{Z}/n\mathbb{Z} \mid 2x = 0\} = \{0\}$. So

$$\mathrm{H}^p(C_2, \mathrm{H}^{2i}(C_n, \mathbb{Z})) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{cases}$$

(2) <u>i is even and n is even</u>: Then $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2} = \mathrm{H}^{2i}(C_n,\mathbb{Z})$, $\mathrm{H}^{2i}(C_n,\mathbb{Z})/(1+\tau)\cdot\mathrm{H}^{2i}(C_n,\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})/2(\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/(n/2)\mathbb{Z}$ and $\{x\in\mathbb{Z}/n\mathbb{Z} \mid 2x=0\} = (n/2)\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$. So

$$\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z})) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } p=0 \\ \mathbb{Z}/(n/2)\mathbb{Z} & \text{if } p\geq 2 \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \text{ is odd.} \end{cases}$$

(3) <u>i is odd and n is odd</u>: Then $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2} = 0$, the element $1+\tau$ of $\mathbb{Z}[C_2]$ acts on $\mathrm{H}^n(C_n,\mathbb{Z})$ by 0 and the element $\tau-1$ acts by multiplication by -2. So $\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z}))=0$ for every $p\geq 0$.

(4) <u>i is odd and n is even</u>: As in case (3), we have $\mathrm{H}^{2i}(C_n,\mathbb{Z})^{C_2} = 0$, the element $1+\tau$ of $\mathbb{Z}[C_2]$ acts on $\mathrm{H}^n(C_n,\mathbb{Z})$ by 0 and the element $\tau-1$ acts by multiplication by -2. So

$$\mathrm{H}^p(C_2,\mathrm{H}^{2i}(C_n,\mathbb{Z})) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \mathbb{Z}/(n/2)\mathbb{Z} & \text{if } p \text{ is odd.} \end{cases}$$

(iii) We use the Hochschild-Serre spectral sequence:

$$E_2^{pq} = \mathrm{H}^p(C_2,\mathrm{H}^q(C_n,\mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(G,\mathbb{Z}).$$

By question (ii) (and the fact that n is odd), we have

$$E_2^{pq} = \begin{cases} \mathbb{Z} & \text{if } p=q=0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q=0 \text{ and } p\geq 2 \text{ is even} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q\geq 1 \text{ is in } 4\mathbb{N} \text{ and } p=0 \\ 0 & \text{otherwise.} \end{cases}$$

So the second page of the spectral sequence looks like this:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbb{Z}/n\mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \dots
\end{array}$$

In particular, for every $r\geq 2$, we have $E_2^{pq}=0$ unless p and q are even; this implies that $d_r : E_r^{pq} \to E_2^{p+r,q-r+1}$ is always 0 (if r is odd, then p and $p+r$ cannot be even at the same time; if r is even, then q and $q-r+1$ cannot be even at the same time). So the spectral sequence degenerates at E_2, and we have $E_\infty^{pq} = E_2^{pq}$.

If m is odd, then $E_\infty^{p,m-p} = 0$ for every $p \in \mathbb{Z}$, so $\mathrm{H}^m(G,\mathbb{Z}) = 0$. If $m = 2 \mod 4$, then the only $E_\infty^{p,m-p}$ that is nonzero is $E_\infty^{m,0} = \mathbb{Z}/2\mathbb{Z}$, so $\mathrm{H}^m(G,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Finally, suppose that $m > 0$ and $m = 0 \mod 4$. Then the only two $E_\infty^{p,m-p}$ that are nonzero are $E_\infty^{0,m} = \mathbb{Z}/n\mathbb{Z}$ and $E_\infty^{m,0} = \mathbb{Z}/2\mathbb{Z}$, so we have an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{H}^m(G,\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

As $\mathrm{H}^m(G,\mathbb{Z})$ is an abelian group and n is odd, this gives an isomorphism

$$\mathrm{H}^m(G,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/2n\mathbb{Z}.$$

(c). (i) We have $E_2^{pq} = \mathrm{H}^p(G/K, \mathrm{H}^q(K,M))$, so, by (a)(iv), we get $E_2^{pq} = 0$ if $p \not\in \{0,1\}$. So $d_r^{pq} = 0$ if $r \geq 2$, and the spectral sequence degenerates at E_2.

(ii) By (a)(iv) again, we have

$$E_\infty^{pq} = E_2^{pq} = \mathrm{H}^p(G/K, \mathrm{H}^q(K,M)) = \begin{cases} \mathrm{H}^q(K,M)^\sigma & \text{if } p=0 \\ \mathrm{H}^q(K,M)_\sigma & \text{if } p=1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $m \in \mathbb{Z}$. Then $\mathrm{H}^m(G,M)$ has a decreasing filtration $\mathrm{Fil}^p\mathrm{H}^m(G,M)$ such that $\mathrm{Fil}^p\mathrm{H}^m(G,M) = 0$ if $p \geq 2$, $\mathrm{Fil}^1\mathrm{H}^m(G,M) = E_\infty^{1,m-1}$, $\mathrm{Fil}^0\mathrm{H}^m(G,M)/\mathrm{Fil}^1\mathrm{H}^m(G,M) = E_\infty^{0,m}$ and $\mathrm{Fil}^p\mathrm{H}^m(G,M) = \mathrm{H}^m(G,M)$ if $p \leq 0$. In other words, we have an exact sequence

$$0 \to E_\infty^{1,m-1} \to \mathrm{H}^m(G,M) \to E_\infty^{0,m} \to 0.$$

Combining this with the formula for E_∞^{pq} gives the result. (If $m=0$, then $E_\infty^{1,m-1} = 0$, so we get $\mathrm{H}^m(G,M) = E_\infty^{0,0} = \mathrm{H}^0(K,M)^\sigma$.)

(d). (i) The action of G on K by conjugation is trivial, and its action on $\mathbb{Z}$ is also trivial, so G acts trivially on $\mathrm{H}_*(G,\mathbb{Z})$ and $\mathrm{H}^*(G,\mathbb{Z})$.

(ii) We have $E_2^{pq} = \mathrm{H}^p(\mathrm{G}/K, \mathrm{H}^q(K,M))$, so, by (a)(iv), $E_2^{pq} = 0$ if $q \not\in \{0,1\}$. In particular, if $r \geq 3$, then the source or target of $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ is 0 for every choice of $(p,q) \in \mathbb{Z}$, so all teh d_r^{pq} are zero. So the spectral sequence degenerates at E_3.

(iii) By question (i), we have $E_\infty^{pq} = E_3^{pq}$, so $E_\infty^{pq} = 0$ if $q \not\in \{0,1\}$,

$$E_\infty^{m,0} = E_3^{m,0} = \mathrm{Coker}(E_2^{m-2,1} \to E_2^{m,0}) = \mathrm{Coker}(\mathrm{H}^{m-2}(G/K, M_\sigma) \to \mathrm{H}^m(G/K, M^\sigma))$$

and

$$E_\infty^{m-1,1} = E_3^{m-1,1} = \mathrm{Ker}(E_2^{m-1,1} \to E_2^{m+1,0}) = \mathrm{Coker}(\mathrm{H}^{m-1}(G/K, M_\sigma) \to \mathrm{H}^{m+1}(G/K, M^\sigma)).$$

Let $m \in \mathbb{Z}$. Then $\mathrm{H}^m(G, M)$ has a decreasing filtration $\mathrm{Fil}^p\mathrm{H}^m(G, M)$ such that $\mathrm{Fil}^p\mathrm{H}^m(G, M) = 0$ if $p \geq m+1$, $\mathrm{Fil}^p\mathrm{H}^m(G, M) = \mathrm{H}^m(G, M)$ if $p \leq m-1$, $\mathrm{Fil}^m\mathrm{H}^m(G, M) = E_\infty^{m,0}$, and $\mathrm{Fil}^{m-1}\mathrm{H}^m(G, M)/\mathrm{Fil}^m\mathrm{H}^m(G, M) = E_\infty^{m-1,1}$. If $m = 0$, then $E_\infty^{m-1,1} = 0$, so we get

$$\mathrm{H}^0(G, M) = E_\infty^{0,0} = \mathrm{H}^0(G/K, M^\sigma).$$

If $m = 1$, we get an exact sequence

$$\mathrm{H}^{m-2}(G/K, M_\sigma) \to \mathrm{H}^m(G/K, M^\sigma) \to \mathrm{H}^m(G, M) \to \mathrm{H}^{m-1}(G/K, M_\sigma) \to \mathrm{H}^{m+1}(G/K, M^\sigma).$$

Putting all these exact sequences together gives the desired long exact sequence.

□

A.7.3 Flabby and soft sheaves

Let X be a topological space. If $\mathscr{F}$ is a sheaf on X and Y is a subset of X, we set

$$\mathscr{F}(Y) = \varinjlim_{Y \subset U \in \mathrm{Open}(X)^{\mathrm{op}}} \mathscr{F}(U).$$

If $Y \subset Y'$, we have a map $\mathscr{F}(Y') \to \mathscr{F}(Y)$ induced by the restriction maps of $\mathscr{F}$.

We say that $\mathscr{F}$ is *flabby* (or *flasque*) if, for every open subset U of X, the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective. We say that $\mathscr{F}$ is *soft* if, for every *closed* subset F, the map $\mathscr{F}(X) \to \mathscr{F}(F)$ is surjective.

Let R be a ring. If M is a left R-module and $x \in X$, we write $S_{x,M}$ for the presheaf on X given by $S_{x,M}(U) = M$ if $x \in U$ and $S_{x,M}(U) = 0$ if $x \notin U$, with the obvious restriction maps (equal to 0 or id_M). It is easy to see that this is a sheaf, and we call it the *skryscraper sheaf* at x with value M.

(a). Show that any flabby sheaf is soft.

(b). Let $d \geq 1$, and let $\mathscr{F}$ be the sheaf $U \longmapsto C^\infty(U, \mathbb{C})$ on $\mathbb{R}^d$. Show that the sheaf $\mathscr{F}$ is soft. [27]

(c). For every $x \in X$, show that the functor ${}_R\mathbf{Mod} \to \mathrm{Sh}(\mathscr{F}, R)$, $M \longmapsto S_{x,M}$ is right adjoint to the functor $\mathscr{F} \longmapsto \mathscr{F}_x$.

(d). If $(M_x)_{x \in X}$ is a family of R-modules, show that $\prod_{x \in X} S_{x,M_x}$ is a flabby sheaf, and that it is an injective sheaf if every M_x is an injective R-module.

[27] More generally, if X is a smooth manifold, then the sheaf Ω_X^k of degree k differential forms on X is soft. As the sequence $0 \to \underline{\mathbb{C}}_X \to \Omega_X^1 \to \Omega_X^2 \to \ldots$ is exact by the Poincaré lemme, this, and the fact that soft sheaves are $\mathrm{H}^0(X, \cdot)$-acyclic, shows that the cohomology of the constant sheaf $\underline{\mathbb{C}}_X$ is isomorphic to the de Rham cohomology of X.

(e). For every sheaf of R-modules $\mathscr{F}$ on X, we set $G(\mathscr{F}) = \prod_{x \in X} S_{x,\mathscr{F}_x}$. Show that the canonical morphism $\mathscr{F} \to G(\mathscr{F})$ (sending any $s \in \mathscr{F}(U)$ to the family $(s_x)_{x \in U}$) is injective. [28]

(f). Show that sheaves of R-modules on X have a functorial resolution by flabby injective sheaves.

(g). Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence in $\mathrm{Sh}(X, R)$, with $\mathscr{F}$ flabby. Show that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

An open cover $(U_i)_{i \in I}$ of X is called *locally finite* if every point of X has a neighborhood that meets only finitely many of the U_i. We say that X is *paracompact* if every open cover of X has a locally finite refinement. We admit the following facts:

(1) A metric space is paracompact.

(2) If X is paracompact and $(U_i)_{i \in I}$ is an open cover of X, then there exists an open cover $(V_i)_{\in I}$ of X such that $\overline{V_i} \subset U_i$ for every $i \in I$. [29]

(h). Suppose that X is a separable metric space. [30] Let $0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$ be a short exact sequence of sheaves of R-modules on X, with $\mathscr{F}$ soft. The goal of this question is to prove that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

(i) Let $s \in \mathscr{H}(X)$. Show that there exists a locally finite open cover $(U_n)_{n \in \mathbb{N}}$ and sections $t_n \in \mathscr{G}(U_n)$ such that $g(t_n) = s_{|U_n}$ for every $n \in \mathbb{N}$.

(ii) Take an open cover $(V_n)_{n \in \mathbb{N}}$ of X such that $F_n := \overline{V_n} \subset U_n$ for every $n \in \mathbb{N}$. Prove by induction on n that, for every $n \geq 0$, there exists a section $a_n \in \mathscr{G}(F_0 \cup \ldots \cup F_n)$ such that $g(a_n) = s_{|F_0 \cup \ldots \cup F_n}$.

(iii) Show that s has a preimage in $\mathscr{G}(X)$.

(i). If $\mathscr{F}$ is a flabby sheaf of R-modules on a topological space X, or a soft sheaf of R-modules on a separable metric space X, show that $\mathrm{H}^n(X, \mathscr{F}) = 0$ for every $n \geq 1$. (Hint: Try to adapt the strategy of Problem A.6.5(b).)

Solution.

(a). Let $\mathscr{F}$ be a flabby sheaf. Let F be a closed subset of X and $s \in \mathscr{F}(F)$. By definition of $\mathscr{F}(F)$, there exists an open subset $U \supset F$ of X and a representative $s' \in \mathscr{F}(U)$ of s. As $\mathscr{F}$ is flabby, there exists $t \in \mathscr{F}(X)$ such that $t_{|X} = s'$, and then $t_{|F} = s$. So $\mathscr{F}$ is soft.

(b). Let F be a closed subset of $\mathbb{R}^d$, and let $s \in \mathscr{F}(F)$. By definition of $\mathscr{F}(F)$, there exists an open subset $V \supset F$ of $\mathbb{R}^d$ and a C^∞ function $f : U \to \mathbb{C}$ representing s.

[28] The "G" is for "Godement", who invented this method of constructing flabby resolutions of sheaves.

[29] This follows from the fact that there exists a partition of unity subordinate to $(U_i)_{i \in I}$, which uses the fact that paracompact spaces are normal and Urysohn's lemma.

[30] It would be enough to assume that X is paracompact.

I claim that there exists a locally finite open cover $(U_i)_{i\in I}$ of $\mathbb{R}^n$ and a subset J of I such that $F \subset \bigcup_{j\in J} U_j \subset U$ and $U_i \cap F = \varnothing$ if $i \in I - J$. Here is a way to prove this claim: For every $x \in F$, choose an open neighborhood U_x of x such that $B_x \subset U$. For every $x \in \mathbb{R}^n - F$, choose an open neighborhood U_x of x such that $U_x \cap F = \varnothing$. As $\mathbb{R}^n$ is paracompact, there exists $I \subset \mathbb{R}^n$ such that $(U_x)_{x\in I}$ is a locally finite open cover of $\mathbb{R}^n$. Let $J = I \cap F$. Then $F \cap (\bigcup_{x\in I-J} U_x) = \varnothing$, so $F \subset \bigcup_{x\in J} U_x \subset U$.

Now choose a C^∞ partition of unity $(\varphi_i)_{i\in I}$ subordinate to the open cover $(U_i)_{i\in I}$, and let $\varphi = \sum_{j\in J} \varphi_j$. Then $\operatorname{supp}(\varphi) \subset \bigcup_{j\ni J} U_j \subset U$ and, if $x \in F$, then $1 = \sum_{i\in I} \varphi_i(x) = \sum_{j\in J} \varphi_j(x) = 1$. Define a function $g : \mathbb{R}^n \to \mathbb{C}$ by $g(x) = \varphi(x) f(x)$ if $x \in U$ and $g(x) = 0$ if $x \notin U$. Then g is C^∞ on U, and $g = 0$ on $\mathbb{R}^n - \operatorname{supp}(\varphi)$. So g is C^∞, i.e. $g \in \mathscr{F}(\mathbb{R}^n)$, and $g_{|F} = s$.

(c). Let $\mathscr{F} \in \operatorname{Ob}(\operatorname{Sh}(X, R))$ and $M \in \operatorname{Ob}({}_R\mathbf{Mod})$. Then we have

$$\operatorname{Hom}_R(\mathscr{F}_x, M) = \operatorname{Hom}_R(\varinjlim_{x\in U} \mathscr{F}(U), M) = \varprojlim_{x\in U} \operatorname{Hom}_R(\mathscr{F}(U), M).$$

On the other hand, a morphism $\mathscr{F} \to S_{x,M}$ is a family $(f_U)_{U\in\operatorname{Open}(X)}$ of morphisms of R-modules $f_U : \mathscr{F}(U) \to S_{x,M}(U)$, with $f_U = 0$ if $x \notin U$ (because then $S_{x,M}(U) = 0$ and $f_U : \mathscr{F}(U) \to M$ if $x \in U$, satisfying the condition that, if $x \in V \subset U$, then, for every $s \in \mathscr{F}(U)$, we have $f_V(s_{|V}) = f_U(s)$. In other words, the family $(f_U)_{U\ni x}$ is an element of $\varprojlim_{x\in U} \operatorname{Hom}_R(\mathscr{F}(U), M) = \operatorname{Hom}_R(\mathscr{F}_x, M)$. This defines an isomorphism $\operatorname{Hom}_{\operatorname{Sh}(X,R)}(\mathscr{F}, S_{x,M}) \simeq \operatorname{Hom}_R(\mathscr{F}_x, M)$, that is clearly functorial in $\mathscr{F}$ and M.

(d). Let $\mathscr{F} = \prod_{x\in X} S_{x,M_x}$. Let U be an open subset of X. Then $\mathscr{F}(U) = \prod_{x\in U} M_x$ and $\mathscr{F}(X) = \prod_{x\in X} M_x$, and the restriction morphism $\mathscr{F}(X) \to \mathscr{F}(U)$ is given by the canonical projection on the factors indexed by $x \in U$, which is clearly surjective. So $\mathscr{F}$ is flabby.

Suppose that M_x is an injective R-module for every $x \in X$. Then, for every $x \in X$, the sheaf $S_{x,M}$ is injective by Lemma II.2.4.4 and question (c). By Lemma II.2.4.3, the sheaf $\prod_{x\in X} S_{x,M_x}$ is also injective.

(e). Denote by $c : \mathscr{F} \to G(\mathscr{F})$ the canonical morphism. Let U be an open subset of X and $s \in \mathscr{F}(U)$ such that $c(s) = 0$. As $c(s) = (s_x)_{x\in U}$, we have $s_x = 0$ for every $x \in U$, so $s = 0$.

(f). As ${}_R\mathbf{Mod}$ is a Grothendieck abelian category, there exists a functor $\Phi : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ and a morphism of functors $\iota : \operatorname{id}_{{}_R\mathbf{Mod}} \to \Phi$ such that, for every $M \in \operatorname{Ob}({}_R\mathbf{Mod})$, the R-module $\Phi(M)$ is injective and $\iota(M) : M \to \Phi(M)$ is an injective morphism. (See Theorem II.3.2.4 .)

For every sheaf of R-modules $\mathscr{F}$ on X, let $G'(\mathscr{F}) = \prod_{x\in X} \Phi(\mathscr{F}_x)$ and let $c' : \mathscr{F} \to G'(\mathscr{F})$ be the composition of $c : \mathscr{F} \to G(\mathscr{F})$ and of $\prod_{x\in X} \iota(\mathscr{F}_x) : G(\mathscr{F}) \to G'(\mathscr{F})$. Then G' is a functor and c' is a morphism of functors. Also, the sheaf $G'(\mathscr{F})$ is always injective and flabby by question (d). The construction of the proof of Lemma IV.3.1.2 gives a functorial resolution of $\mathscr{F}$ by injective and flabby sheaves. [31]

(g). We give names to the morphisms of the sequence:

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0.$$

Let $s \in \mathscr{H}(X)$. We want to show that there exists $t \in \mathscr{G}(X)$ such that $g(t) = s$. We consider the set I of pairs (U, t), where $U \subset X$ is an open subset and $t \in \mathscr{G}(U)$ is such that $g(t) = s_{|U}$. Note that I is not empty, because s admits preimages by g locally on X. We consider the following (partial) order on I: $(U_1, t_1) \leq (U_2, t_2)$ if $U_1 \subset U_2$ and $t_1 = t_{2|U_1}$. Let J be a nonempty totally ordered subset of I; for every $i \in J$, let (U_i, t_i) be the corresponding pair. Let $U = \bigcup_{i \in J} U_i$. If $i, j \in J$, I claim that $t_{i|U_i \cap U_j} = t_{j|U_i \cap U_j}$; indeed, we may assume that $i \leq j$, and then $U_i \subset U_j$ and $t_i = t_{j|U_i}$. So there exists $t \in \mathscr{G}(U)$ such that $t_{|U_i} = t_i$ for every $i \in J$, and then $(U, t) \geq (u_i, t_i)$ for every $i \in J$. By Zorn's lemma, the set I has a maximal element (U, t). I claim that $U = X$, which finishes the proof. Suppose that $U \neq X$. Then there exists an open subset $V \not\subset U$ of X and $t' \in \mathscr{G}(V)$ such that $g(t') = s_{|V}$. In particular, we have $g(t_{U \cap V} - t'_{U \cap V}) = 0$, so there exists $u \in \mathscr{F}(U \cap V)$ such that $f(u =) t_{U \cap V} - t'_{U \cap V}$. As $\mathscr{F}$ is flabby, there exists $v \in \mathscr{F}(X)$ such that $u = v_{|U \cap V}$. Let $t'' = t' + f(v_{|V})$. Then $t'_{|U \cap V} = t'_{|U \cap V} + f(u) = t_{|U \cap V}$, so there exists $t_1 \in \mathscr{G}(U \cup V)$ such that $t_{1|U} = t$ and $t_{1|V} = t''$. We have $g(t_1)_{|U} = g(t) = s_{|U}$ and $g(t_1)_{|V} = g(t'') = g(t') = s_{|V}$, so $g(t_1) = s_{|U \cup V}$. As $U \subsetneq U \cup V$, this contradicts the maximality of (U, t).

(h). (i) We can find an open cover $(U_i)_{i \in I}$ of X and sections $t_i \in \mathscr{G}(U_i)$ with $g(t_i) = s_{|U_i}$ for every $i \in I$. As X is paracompact, after replacing the cover $(U_i)_{i \in I}$ by a refinement, we may assume that it is locally finite. As X is a separable metric space, its topology has a countable basis. So we may assume that the cover $(U_i)_{i \in I}$ is countable.

(ii) We take $a_0 = t_{0|F_0}$. Suppose that $n \geq 0$ and that we have found a_n. Let $\Omega \supset F_0 \cup \ldots \cup F_n$ be an open subset of X and $a'_n \in \mathscr{G}(\Omega)$ be a representative of a_n. We have $g(a'_n)_{|U_{n+1} \cap (F_0 \cup \ldots \cup F_n)} = s_{|U_{n+1} \cap (F_0 \cup \ldots \cup F_n)} = g(t_{n+1})_{|U_{n+1} \cap (F_0 \cup \ldots \cup F_n)}$, so, after shrinking Ω, we may assume that $g(a'_n) = g(t_{n+1|\Omega \cap U_{n+1}})$. Then there exists $b \in \mathscr{F}(\Omega \cap U_{n+1})$ such that $f(b) = a'_n - t_{n+1|\Omega \cap U_{n+1}}$. As $\mathscr{F}$ is soft, there exists $b' \in \mathscr{F}(X)$ such that $b'_{|F_0 \cup \ldots \cup F_n} = b_{|F_0 \cup \ldots \cup F_n}$. After shrinking Ω again, we may assume that $b'_{|\Omega} = b$. Let $t'_{n+1} = t_{n+1} + f(b'_{|U_{n+1}})$. Then $g(t'_{n+1}) = g(t_{n+1}) = s_{|U_{n+1}}$ and $t'_{n+1|\Omega \cap U_{n+1}} = t_{n+1|\Omega \cap U_{n+1}} + f(b_{\Omega \cap U_{n+1}}) = a'_{n|\Omega \cap U_{n+1}}$. So there exists $a'_{n+1} \in \mathscr{G}(U_{n+1} \cup \Omega)$ such that $a'_{n+1|\Omega} = a'_n$ and $a'_{n+1|U_{n+1}} = t_{n+1}$, and we have $g(a'_{n+1}) = s_{|\Omega \cup U_{n+1}}$. We take for $a_{n+1} \in \mathscr{G}(F_0 \cup \ldots \cup F_{n+1})$ the element represented by $a'_{n+1} \in \mathscr{G}(\Omega \cup U_{n+1})$.

[31] Actually, with a little more work we could show that every injective sheaf is flabby, so any functorial resolution of $\mathscr{F}$ by injective sheaves (which exists because $\mathrm{Sh}(X, R)$ is a Grothendieck abelian category) is a resolution by injective and flabby sheaves. But it is simpler to use the functor G.

(iii) As $X = \bigcup_{n \geq 0} V_n$, the family $(a_{n|V_0 \cup \ldots \cup V_n})_{n \geq 0}$ glues to a section $a \in \mathscr{G}(X)$ such that $g(a_{|V_n}) = s_{|V_n}$ for every n, hence $g(a) = s$.

(i). Let $\mathscr{C}$ be the full subcategory of $\mathrm{Sh}(X, R)$ whose objects are flabby sheaves of R-modules on X. Suppose that $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence in $\mathrm{Sh}(X, R)$ with $\mathscr{F}$ and $\mathscr{G}$ flabby. We claim that $\mathscr{H}$ is also flabby. Indeed, let U be an open subset of X. By question (g) (whose proof adapts immediately to show that $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective), we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}(X) \arrow[r] \arrow[d, "u"'] & \mathscr{G}(X) \arrow[r] \arrow[d, "v"'] & \mathscr{H}(X) \arrow[r] \arrow[d, "w"'] & 0 \\ 0 \arrow[r] & \mathscr{F}(U) \arrow[r] & \mathscr{G}(U) \arrow[r] & \mathscr{H}(U) \arrow[r] & 0 \end{tikzcd}$$

where the morphisms u and v are surjective. So w is also surjective.

We show by induction on n that, for every $n \geq 1$ and every $\mathscr{F} \in \mathrm{Ob}(\mathscr{C})$, we have $\mathrm{H}^n(X, \mathscr{F}) = 0$. Suppose that $n = 1$. Let $\mathscr{F}$ be an object of $\mathscr{C}$. Choose a monomorphism $\mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an injective sheaf. Let $\mathscr{H} = \mathscr{G}/\mathscr{F}$. The long exact sequence of cohomology gives an exact sequence

$$\mathscr{G}(X) \to \mathscr{H}(X) \to \mathrm{H}^1(X, \mathscr{F}) \to \mathrm{H}^1(X, \mathscr{G}).$$

But $\mathrm{H}^1(X, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective and $\mathscr{G}(X) \to \mathscr{H}(X)$ is surjective because $\mathscr{F}$ is flabby (by question (g)), so $\mathrm{H}^1(X, \mathscr{F}) = 0$.

Now suppose the result known for $n \geq 1$, and let us prove it for $n + 1$. Let $\mathscr{F}$ be a flabby sheaf on X. By question (f), there exists a monomorphism $\mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an injective flabby sheaf. Let $\mathscr{H} = \mathscr{G}/\mathscr{F}$. We have shown that $\mathscr{H}$ is flabby. The long exact sequence of cohomology gives an exact sequence

$$\mathrm{H}^n(X, \mathscr{H}) \to \mathrm{H}^{n+1}(X, \mathscr{F}) \to \mathrm{H}^{n+1}(X, \mathscr{G}).$$

But $\mathrm{H}^{n+1}(X, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective, and $\mathrm{H}^n(X, \mathscr{H}) = 0$ by the induction hypothesis, so $\mathrm{H}^{n+1}(X, \mathscr{F}) = 0$.

The proof for soft sheaves on a separable metric space is exactly the same, once we have proved that a quotient of soft sheaves is soft; this is the same proof as for a quotient of flabby sheaves, using question (h) instead of question (g).

$\square$

A.8 Problem set 8

A.8.1 Right multiplicative systems

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$. Let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$ and $W_{\mathscr{I}}$ be the set of morphisms of $\mathscr{I}$ that are in W. Suppose that W is a right multiplicative system and that, for every $s : X \to Y$ in W such that $X \in \mathrm{Ob}(\mathscr{I})$, there exists a morphism $f : Y \to Z$ with $Z \in \mathrm{Ob}(\mathscr{I})$ and $f \circ s \in W$.

Show that $W_{\mathscr{I}}$ is a right multiplicative system.

Solution. Conditions (S1) and (S2) of Definition V.2.2.1 are clear. We check condition (S3). Let $f : X \to Y$ and $s : X \to X'$ be morphisms of $\mathscr{I}$ such that $s \in W$. Then there exist a morphism $g : X' \to Y'$ in $\mathscr{C}$ and a morphism $t : Y \to Y'$ in W such that $t \circ f = g \circ s$. Moreover, by the hypotheses of the proposition, there exists $h : Y' \to Y''$, with $Y'' \in \mathrm{Ob}(\mathscr{I})$, such that $h \circ t \in W$. As $\mathscr{I}$ is a full subcategory of $\mathscr{C}$, we get a commutative diagram in $\mathscr{I}$:

$$\begin{tikzcd}
X' \arrow[r, "h \circ g"] & Y'' \\
X \arrow[u, "s"] \arrow[r, "f"'] & Y \arrow[u, "h \circ t"']
\end{tikzcd}$$

We now check condition (S4). Let $f, g : X \to Y$ be two morphisms of $\mathscr{I}$, and let $s : X' \to X$ be a morphism of $W_{\mathscr{I}}$ such that $f \circ s = g \circ s$. As W is a right multuplicative system, there exists $t : Y \to Y'$ in W such that $t \circ f = t \circ g$. Take $h : Y' \to Y''$ such that $Y'' \in \mathrm{Ob}(\mathscr{I})$ and $h \circ t \in W$. Then $h \circ t \in W_{\mathscr{I}}$, and we have $(h \circ t) \circ f = (h \circ t) \circ g$.

□

A.8.2 Isomorphisms in triangulated categories

Let $(\mathscr{D}, T)$ be a triangulated category, and let $f : X \to Y$ be a morphism of $\mathscr{D}$. Show that f is an isomorphism if and only if there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$.

Solution. Suppose that f is an isomorphism. By (TR2), there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$. By (TR4), the commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "f^{-1}"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}$$

can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d, "f^{-1}"] & Z \arrow[r] \arrow[d, "g"] & T(X) \\ X \arrow[r, "\mathrm{id}_X"'] & X \arrow[r] & 0 \arrow[r] & T(X) \end{tikzcd}$$

By Corollary V.1.1.12 , the morphism g is an isomorphism, so $Z = 0$.

Conversely, suppose that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$. Then, for every object W of $\mathscr{D}$, applying $\mathrm{Hom}_{\mathscr{D}}(W, \cdot)$ to the triangle $X \to Y \to Z \to T(X)$ and using Proposition V.1.1.11(ii) shows that $f_* : \mathrm{Hom}_{\mathscr{D}}(W, X) \to \mathrm{Hom}_{\mathscr{D}}(W, Y)$ is an isomorphism. By the Yoneda lemma (Corollary I.3.2.9), the morphism f is an isomorphism.

□

A.8.3 Null systems

Let $(\mathscr{D}, T)$ be a triangulated. Remember that a null system in $\mathscr{D}$ is a set $\mathscr{N}$ of objects of $\mathscr{D}$ such that:

(N1) $0 \in \mathscr{N}$;

(N2) for every $X \in \mathrm{Ob}(\mathscr{C})$, we have $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$;

(N3) if $X \to Y \to Z \to T(X)$ is a distinguished triangle and if $X, Y \in \mathscr{N}$, then $Z \in \mathscr{N}$.

We fix a null system $\mathscr{N}$, and we denote by $W_{\mathscr{N}}$ the set of morphisms $f : X \to Y$ in $\mathscr{D}$ such that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z \in \mathscr{N}$.

(a). If $X \in \mathscr{N}$ and Y is isomorphic to X, show that $Y \in \mathscr{N}$.

(b). Show that $W_{\mathscr{N}}$ contains all the isomorphisms of $\mathscr{D}$.

(c). Show that $W_{\mathscr{N}}$ is stable by composition.

(d). Show that $W_{\mathscr{N}}$ satisfies conditions (S3) and (S4) of Definition V.2.2.1 .

(e). Show that $W_{\mathscr{N}}$ is also a left multiplicative system.

Solution.

(a). Let $f : X \to Y$ be an isomorphism. By problem A.8.2, the triangle $X \xrightarrow{f} Y \to 0 \to T(X)$ is distinguished. By axiom (TR3), the triangle $0 \to X \xrightarrow{f} Y \to T(0) = 0$ is also distinguished and so, by (N1) and (N3), we have $Y \in \mathscr{N}$.

(b). This follows immediately from problem A.8.2 and from (N0).

(c). Let $f : X \to Y$ and $g : Y \to Z$ be in $W_{\mathscr{N}}$. Choose distinguished triangles $X \xrightarrow{f} Y \to Z' \to T(X)$ and $Y \xrightarrow{g} Z \to Z' \to T(Y)$ with $Z', Y' \in \mathscr{N}$. Let $X \xrightarrow{g \circ f} Y \to Z' \to T(X)$ be a distinguished triangle. By the octahedral axiom (axiom (TR5)), there exists a distinguished triangle $Z' \to Y \to X' \to T(X')$. By (N3), we have $X' \in \mathscr{N}$, and so $g \circ f \in W_{\mathscr{N}}$.

(d). We show condition (S3). Let $f : X \to Y$ and $s : X \to X'$ be morphisms in $\mathscr{D}$ with $s \in W_{\mathscr{N}}$. By the definition of $W_{\mathscr{N}}$ and axioms (TR3) and (N2), we can find a distinguished triangle $Z \xrightarrow{h} X \to X' \to T(Z)$ with $Z \in \mathscr{N}$. By (TR2), we can find a diatinguished triangle $Z \xrightarrow{f \circ h} Y \xrightarrow{t} Y' \to T(Z)$, and $t \in W_{\mathscr{N}}$ by (TR3) and (N2). Finally, by (TR4), we can complete the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "\mathrm{id}_Z"'] & X \arrow[r, "s"] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, dashed, "g"'] & T(Z) \arrow[d, "\mathrm{id}_{T(Z)}"'] \\ Z \arrow[r, "f \circ h"'] & Y \arrow[r, "t"'] & Y' \arrow[r] & T(Z) \end{tikzcd}$$

In other words, we can find a morphism $g : X' \to Y'$ such that $g \circ s = t \circ f$. This finishes the proof of (S3).

We show condition (S4). Let $f, g : X \to Y$ be two morphisms of $\mathscr{D}$, and suppose that there exists $s : X' \to X$ such that $f \circ s = g \circ s$ and $s \in W_{\mathscr{N}}$. If $h = f - g$, then we have $h \circ s = 0$. Choose a distinguished triangle $X' \xrightarrow{s} X \xrightarrow{u} Z \to T(X')$ with $Z \in \mathscr{N}$. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, Y)$ to this distinguished triangle, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(Z, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, X').$$

As the image $h \circ s$ of $h \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$ by the second morphism of this sequence is 0, there exists $k \in \mathrm{Hom}_{\mathscr{D}}(Z, Y)$ such that $h = k \circ u$. Consider a distinguished triangle $Z \xrightarrow{k} Y \xrightarrow{t} Y' \to T(Z)$. As $Z \in \mathscr{N}$, we have $t \in W_{\mathscr{N}}$. Also, as $t \circ k = 0$ (by Proposition V.1.1.11(i)), we have $t \circ h = 0$, so $t \circ f = t \circ g$.

(e). We know that $\mathscr{D}^{\mathrm{op}}$ is also a triangulated category, and $\mathscr{N}^{\mathrm{op}} = \{X \in \mathrm{Ob}(\mathscr{D}^{\mathrm{op}}) \mid X \in \mathscr{N}\}$ is a null system in $\mathscr{D}^{\mathrm{op}}$; indeed, axioms (N1) and (N2) obviously hold, and axiom (N3) for $\mathscr{N}^{\mathrm{op}}$ follows from (N3) for $\mathscr{N}$ thanks to (TR3) and (N2). Also, again thanks to (TR3) and (N2), the set of morphisms $W_{\mathscr{N}^{\mathrm{op}}}$ determined by $\mathscr{N}^{\mathrm{op}}$ is equal to $(W_{\mathscr{N}})^{\mathrm{op}}$. So, by question (d), the set $(W_{\mathscr{N}})^{\mathrm{op}}$ is a right mutliplicative system. But this is equivalent to the fact that $W_{\mathscr{N}}$ is a left multiplicative system.

$\square$

A.8.4 Localization of functors

Let $\mathscr{C}$ be a category, let W be a set of morphisms of $\mathscr{C}$, and let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$; denote by $W_{\mathscr{I}}$ the set of morphisms of $\mathscr{I}$ that are in W. We fix a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ of $\mathscr{C}$ by W, and we denote by $\iota : \mathscr{I} \to \mathscr{C}$ the inclusion functor. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose that:

(a) W is a right multiplicative system;

(b) for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $s : X \to Y$ in W such that $Y \in \mathrm{Ob}(\mathscr{I})$;

(c) for every $s \in W_{\mathscr{I}}$, the morphism $F(s)$ is an isomorphism.

Show that, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, the map

$$\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F \circ \iota, G \circ Q \circ \iota)$$

induced by composition on the right by ι is bijective.

Solution. Let $u_1, u_2 : F \to G \circ Q$ be morphism of functors such that $\alpha(u_1) = \alpha(u_2)$. Let $X \in \mathrm{Ob}(\mathscr{C})$, and choose a morphism $s : X \to X'$ such that $X' \in \mathrm{Ob}(\mathscr{I})$. Then we have commutative diagrams

$$\begin{tikzcd} F(X) \arrow[r, "u_1(X)"] \arrow[d, "F(s)"'] & G \circ Q(X) \arrow[d, "G\circ Q(s)", "\wr"'] \\ F(X') \arrow[r, "u_1(X')"'] & G \circ Q(X') \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(X) \arrow[r, "u_2(X)"] \arrow[d, "F(s)"'] & G \circ Q(X) \arrow[d, "G\circ Q(s)", "\wr"'] \\ F(X') \arrow[r, "u_2(X')"'] & G \circ Q(X') \end{tikzcd}$$

and $u_1(X') = u_2(X')$ because $X' \in \mathrm{Ob}(\mathscr{I})$, so $u_1(X) = u_2(X)$. This shows that $u_1 = u_2$, and hence that α is injective.

We show that α is surjective. Let $v : F \circ \iota \to G \circ Q \circ \iota$ be a morphism of fucntors. Let $X \in \mathrm{Ob}(\mathscr{C})$, and let $s : X \to X'$ be a morphism of W such that $X' \in \mathrm{Ob}(\mathscr{I})$. Then $G \circ Q(s)$ is an isomorphism, and we set $u(X) = (G \circ Q(s))^{-1} \circ v(X') \circ F(s)$. We must check that this does not depend on the choice of s. Let $s' : X \to X''$ be another morphism of W such that $X'' \in \mathrm{Ob}(\mathscr{I})$. By condition (S3), we can find a commutative square

$$\begin{tikzcd} X \arrow[r, "s'"] \arrow[d, "s"'] & X'' \arrow[d, "t"] \\ X' \arrow[r, "t'"'] & Y \end{tikzcd}$$

with $t \in W$. After composing with a morphism $Y \to Y'$ in W such that $Y' \in \mathrm{Ob}(\mathscr{I})$, we may assume that $Y \in \mathrm{Ob}(\mathscr{I})$. The images of s, t and s' by $G \circ Q$ are isomorphisms, so $G \circ Q(t')$ is also an isomorphism. As v is a morphism of functors, we have

$$\begin{aligned}(G\circ Q(s'))^{-1}\circ v(X'')\circ F(s') &= (G\circ Q(s'))^{-1}\circ(G\circ Q(t))^{-1}\circ v(Y)\circ F(t)\circ F(s')\\ &= (G\circ Q(s))^{-1}\circ(G\circ Q(t'))^{-1}\circ v(Y)\circ F(t')\circ F(s)\\ &= (G\circ Q(s))^{-1}\circ v(X')\circ F(s).\end{aligned}$$

So $u(X)$ is well-defined. It remains to show that teh family $(u(X))_{X\in\mathrm{Ob}(\mathscr{C})}$ is a morphism of functors from F to $G\circ Q$. Let $f:X\to Y$ be a morphism of $\mathscr{C}$. We choose morphisms $s:X\to X'$ and $t:Y\to Y'$ un W such that $X',Y'\in\mathrm{Ob}(\mathscr{I})$. By condition (S3), we can find morphisms $f':X'\to Z$ and $s':Y'\to Z$ such that $s'\in W$ and that $s'\circ t\circ f=f'\circ s$. After composing s' and g by a morphism $Z\to Z'$ in W such that $Z'\in\mathrm{Ob}(\mathscr{I})$, we may assume that $Z\in\mathrm{Ob}(\mathscr{I})$. Then, using the fact that v is a morphism of functors and the definition of u, we get

$$\begin{aligned}(G\circ Q(f))\circ u(X) &= (G\circ Q(f))(G\circ Q(s))^{-1}\circ v(X')\circ F(s)\\ &= (G\circ Q(t))^{-1}\circ(G\circ Q(s'))^{-1}\circ(G\circ Q(g))\circ v(X')\circ F(s)\\ &= (G\circ Q(t))^{-1}\circ(G\circ Q(s'))^{-1}\circ v(Z)\circ F(g)\circ F(s)\\ &= (G\circ Q(t))^{-1}\circ(G\circ Q(s'))^{-1}\circ v(Z)\circ F(s')\circ F(t)\circ F(f)\\ &= (G\circ Q(t))^{-1}\circ v(Y')\circ F(t)\circ F(f)\\ &= u(Y)\circ F(f).\end{aligned}$$

This shows that u is a morphism of functors.

□

A.8.5 Localization of a triangulated category

Let $(\mathscr{D},T)$ be a triangulated category, let $\mathscr{N}$ be a null system in $\mathscr{D}$, and let $W=W_{\mathscr{N}}$ be the corresponding multiplicative system. (See problem A.8.3.) We write $Q:\mathscr{D}\to\mathscr{D}/\mathscr{N}$ for $Q:\mathscr{D}\to\mathscr{D}[W^{-1}]$.

(a). Show that there exists an auto-equivalence $T_{\mathscr{N}}:\mathscr{D}/\mathscr{N}\to\mathscr{D}/\mathscr{N}$ such that $T_{\mathscr{N}}\circ Q\simeq Q\circ T$.

We say that a triangle in $\mathscr{D}/\mathscr{N}$ is distinguished if it is isomorphic to the image by Q of a distinguished triangle of $\mathscr{D}$. Axiom (TR0) of Definition V.1.1.4 is obvious.

(b). Show that axioms (TR1)-(TR5) also hold.

Solution.

(a). The functor T preserves $W_{\mathscr{N}}$ (by (TR3) and (N2)), so the functor $\mathscr{D}\xrightarrow{T}\mathscr{D}\xrightarrow{Q}\mathscr{D}/\mathscr{N}$ sends elements to $W_{\mathscr{N}}$ to isomorphisms, so it factors through a factor $T_{\mathscr{N}}:\mathscr{D}/\mathscr{N}\to\mathscr{D}/\mathscr{N}$.

If we want to justify the construction of $T_{\mathscr{N}}$ in Theorem V.3.1.4, we can say this: If $X \in \mathrm{Ob}(\mathscr{D}/\mathscr{N}) = \mathrm{Ob}(\mathscr{D})$, we set $T_{\mathscr{N}}(X) = T(X)$. Let $u : X \to Y$ be a morphism of $\mathscr{D}$, and chose a diagram
$$\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "t"'] \end{tikzcd}$$
in $\mathscr{D}$ representing u, with $t \in W_{\mathscr{N}}$. We take $T_{\mathscr{N}}(u)$ to be the morphism from $T_{\mathscr{N}}(X)$ to $T_{\mathscr{N}}(Y)$ represented by the diagram
$$\begin{tikzcd} & T(Y') & \\ T(X) \arrow[ur, "T(f)"] & & T(Y) \arrow[ul, "T(t)"'] \end{tikzcd}$$
. This makes sense because $T(t) \in Z_{\mathscr{N}}$, by (TR3) and (N2). If we choose two representatives
$$\begin{tikzcd} & Y_1' & \\ X \arrow[ur, "f_1"] & & Y \arrow[ul, "t_1"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y_2' & \\ X \arrow[ur, "f_2"] & & Y \arrow[ul, "t_2"'] \end{tikzcd}$$
of u, then we have a commutative diagram
$$\begin{tikzcd} & Y_1' \arrow[d] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3' & Y \arrow[ul, "t_1"'] \arrow[l, "t_3"'] \arrow[dl, "t_2"] \\ & Y_2' \arrow[u] & \end{tikzcd}$$
with $t_3 \in W_{\mathscr{N}}$. Then applying T gives a commutative diagram
$$\begin{tikzcd} & T(Y_1') \arrow[d] & \\ T(X) \arrow[ur, "T(f_1)"] \arrow[r, "T(f_3)"] \arrow[dr, "T(f_2)"'] & T(Y_3') & T(Y) \arrow[ul, "T(t_1)"'] \arrow[l, "T(t_3)"'] \arrow[dl, "T(t_2)"] \\ & T(Y_2') \arrow[u] & \end{tikzcd}$$
so
$$\begin{tikzcd} & T(Y_1') & \\ T(X) \arrow[ur, "T(f_1)"] & & T(Y) \arrow[ul, "T(t_1)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & T(Y_2') & \\ T(X) \arrow[ur, "T(f_2)"] & & T(Y) \arrow[ul, "T(t_2)"'] \end{tikzcd}$$
represent the same morphism from $T(X)$ to $T(Y)$ in $\mathscr{D}/\mathscr{N}$. So $T_{\mathscr{N}}$ is well-define, and it is easy to see that it is a functor.

(b)(TR1) Let $X \in \mathrm{Ob}(\mathscr{D}/\mathscr{N})$. Then the triangle $X \overset{\mathrm{id}_X}{\to} X \to 0 \to T_{\mathscr{N}}(X)$ in $\mathscr{D}/\mathscr{N}$ is isomorphic to the image by Q of the distinguished triangle $X \overset{\mathrm{id}_X}{\to} X \to 0 \to T(X)$ in $\mathscr{D}$, so it is distinguished.

(TR2) Let $u : X \to Y$ be a morphism in $\mathscr{D}/\mathscr{N}$, and choose morphisms $f : X \to Y'$ and $s : Y \to Y'$ in $\mathscr{D}$ such that $s \in W_{\mathscr{N}}$ and $u = Q(s)^{-1} \circ Q(f)$. Choose a distinguished triangle $X \xrightarrow{f} Y' \xrightarrow{g} Z \to T(X)$ in $\mathscr{D}$. Then the triangle $X \xrightarrow{u} Y \xrightarrow{Q(g\circ s)} Z \to T_{\mathscr{N}}(X)$ in $\mathscr{D}/\mathscr{N}$ is isomorphic to the image by Q of $X \xrightarrow{f} Y' \xrightarrow{g} Z \to T(X)$, so it is distinguished.

(TR3) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T_{\mathscr{N}}(X)$ be a triangle in $\mathscr{D}/\mathscr{N}$. If it isn distinguished, then it is isomorphic to the image by Q of a distinguished triangle $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} T(X')$ in $\mathscr{D}$, and then the triangle $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T_{\mathscr{N}}(f)} T_{\mathscr{N}}(Y)$ is isomorphic to the image by Q of $Y' \xrightarrow{g'} Z \xrightarrow{h'} T(X') \xrightarrow{-T(f')} T(Y')$, hence it is also distinguished. The proof of the converse is similar.

(TR4) Consider a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] & T_{\mathscr{N}}(X) \arrow[d, "T(u)"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T_{\mathscr{N}}(X')
\end{tikzcd}$$

in $\mathscr{D}/\mathscr{N}$, where the rows are distinguished triangles. By the definition of distinguished triangles in $\mathscr{D}/\mathscr{N}$, we may assume that f, g, h, f', g', h' are morphisms of $\mathscr{D}$. We write $u = Q(s)^{-1} \circ Q(a)$, where $a : X \to X''$ and $s : X' \to X''$ are morphisms of $\mathscr{D}$ such that $s \in W_{\mathscr{N}}$. As $W_{\mathscr{N}}$ is a multiplicative system, we can find a commutative square

$$\begin{tikzcd}
X'' \arrow[r, "k"] & T \\
X' \arrow[u, "s"] \arrow[r, "f'"'] & Y' \arrow[u, "s'"']
\end{tikzcd}$$

with $s' \in W_{\mathscr{N}}$. Write $v = Q(t')^{-1} \circ Q(b')$, with $b' : Y \to Y'''$ and $t' : Y' \to Y''$ are morphisms of $\mathscr{D}$ such that $t' \in W_{\mathscr{N}}$. Then

$$Q(s')^{-1}\circ Q(k\circ a) = Q(f')\circ Q(s)^{-1}\circ Q(s) = Q(f')\circ u = v\circ Q(f) = Q(t')^{-1}\circ Q(b'\circ f),$$

so, by the description of the Hom in the localization after Definition V.2.2.3, there exists a commutative diagram

$$\begin{tikzcd}
 & Y''' \arrow[d, "b''"] & \\
X \arrow[ur, "b'\circ f"] \arrow[r, "k'"] \arrow[dr, "k\circ a"'] & Y'' & Y' \arrow[ul, "t'"'] \arrow[l, "t"'] \arrow[dl, "s'"] \\
 & T \arrow[u, "c"] &
\end{tikzcd}$$

with $t \in W_{\mathscr{N}}$. Let $b = b'' \circ b' : Y \to Y''$. Then

$$Q(t)^{-1} \circ Q(b) = Q(t')^{-1} \circ Q(b') = v.$$

Let $f'' = c \circ k : X'' \to Y''$. Then

$$f'' \circ a = c \circ k \circ a = k' = b'' \circ b' \circ f = b \circ f$$

and

$$t \circ f' = c \circ s' \circ f' = c \circ k \circ s'' = f'' \circ s'',$$

so we have constructed a commutative diagram in $\mathscr{D}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[r, "h"] & T_{\mathscr{N}}(X) \arrow[d, "T_{\mathscr{N}}(a)"] \\
X'' \arrow[r, "f''"] & Y'' \arrow[r, dashed, "g''"] & Z'' \arrow[r, dashed, "h''"] & T_{\mathscr{N}}(X'') \\
X' \arrow[r, "f'"'] \arrow[u, "s"'] & Y' \arrow[r, "g'"'] \arrow[u, "t"'] & Z' \arrow[r, "h'"'] & T_{\mathscr{N}}(X') \arrow[u, "T_{\mathscr{N}}(s)"']
\end{tikzcd}$$

and, by axiom (TR2), we can extend $f'' : X'' \to Y''$ to a distinguished triangle $X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} T(X'')$. Completing s and t to distinguished triangles, we get a commutative diagram where the first two rows and columns are distinguished triangles and $N_1, N_2 \in \mathscr{N}$:

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "s"'] \arrow[dr, dashed] & Y' \arrow[r, "g'"] \arrow[d, "t"'] & Z' \arrow[r, "h'"] & T(X') \arrow[d, "T(s)"] \\
X'' \arrow[r, "f''"] \arrow[d] & Y'' \arrow[r, "g''"] \arrow[d] \arrow[dr, dashed] & Z'' \arrow[r, "h''"] & T(X'') \arrow[d] \\
N_1 \arrow[d] & N_2 \arrow[d] & A & T(N_1) \arrow[d] \\
T(X') \arrow[r, "T(f')"'] & T(Y') \arrow[r, "g'"'] & T(Z') \arrow[r, "T(h')"'] & T^2(X')
\end{tikzcd}$$

We also complete $t \circ f' = f'' \circ s$ to a distinguished triangle $X' \to Y'' \to A \to T(X')$. By the octahedral axiom (TR5) applied to the triangles based on $(f', t, t \circ f')$ and on $(s, f'', f''circs)$, we have distinguished triangles

$$Z' \to A \to N_2 \to T(Z')$$

and

$$N_1 \to A \to Z'' \to T(N_1).$$

Applying the octahedral axiom again for the morphisms $Z' \to A$, $A \to Z''$ and their composition, we get a commutative diagram where the rows and the third columns are distinguished triangles:

$$
\begin{tikzcd}
Z' \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & T(Z') \arrow[d, equal] \\
Z' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d, equal] & N_3 \arrow[r] \arrow[d] & T(Z') \arrow[d] \\
A \arrow[r] & Z'' \arrow[r] & T(N_1) \arrow[r] \arrow[d] & T(A) \\
& & T(N_2) &
\end{tikzcd}
$$

In particular, we have $N_3 \in \mathscr{N}$. So we have completed the commutative square

$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "s"'] & Y' \arrow[d, "t"] \\
X'' \arrow[r, "f''"'] & Y''
\end{tikzcd}
$$

to a morphism of triangles

$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "s"'] & Y' \arrow[r, "g'"] \arrow[d, "t"] & Z' \arrow[r, "h'"] \arrow[d] & T(X') \arrow[d, "T(s)"] \\
X'' \arrow[r, "f''"'] & Y'' \arrow[r, "g''"'] & Z'' \arrow[r, "h''"'] & T(X'')
\end{tikzcd}
$$

such that the morphism $Z' \to Z''$ is in $W_{\mathscr{N}}$. Moreover, by (TR4), we can complete the commutative square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\
X'' \arrow[r, "f''"'] & Y''
\end{tikzcd}
$$

to a morphism of triangles

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d] & T(X) \arrow[d, "T(a)"] \\
X'' \arrow[r, "f''"'] & Y'' \arrow[r, "g''"'] & Z'' \arrow[r, "h''"'] & T(X'')
\end{tikzcd}
$$

So we have constructed a commutative diagram in $\mathscr{D}$ whose rows are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d] & T(X) \arrow[d, "T(a)"] \\
X'' \arrow[r, "f''"'] & Y'' \arrow[r, "g''"'] & Z'' \arrow[r, "h''"'] & T(X'') \\
X' \arrow[r, "f'"] \arrow[u, "s"'] & Y' \arrow[r, "g'"] \arrow[u, "t"] & Z' \arrow[r, "h'"] \arrow[u] & T(X') \arrow[u, "T(s)"]
\end{tikzcd}$$

and such that s, t and the morphism $Z' \to Z''$ are in $W_{\mathscr{N}}$. Taking the image of this by Q, we get a morphism of distinguished triangles in $\mathscr{D}/\mathscr{N}$ extending the pair (u, v).

(TR5) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathscr{D}/\mathscr{N}$. After replacing Y and Z by isomorphic objects, we may assume that f and g are morphisms of $\mathscr{D}$. Applying (TR5) in $\mathscr{D}$ to distinguished triangles based on the morphisms $(f, g, g \circ f)$ and taking the image of the resulting diagram by Q, we get (TR5) in $\mathscr{D}/\mathscr{N}$.

$\square$

A.8.6 More group cohomology

The description of group cohomology in Subsection IV.3.5 can be useful in this problem.

We define elements u, v, r and s of the symmetric group $\mathfrak{S}_4$ by $u = (12)(34)$, $v = (14)(23)$, $r = (123)$ and $s = (13)$. The Klein four group is the normal subgroup K of $\mathfrak{S}_4$ generated by u and v.

Let k be a field of characteristic 2.

(a). Show that $\mathfrak{S}_4 / K \simeq \mathfrak{S}_3$.

(b). Show that there is a unique representation $\tau : \mathfrak{S}_4 \to \mathrm{GL}_2(k)$ such that $\tau(u) = \tau(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tau(r) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $M = M_2(k)$, with the action of $\mathfrak{S}_4$ given by $g \cdot A = \tau(g) A \tau(g)^{-1}$, for $g \in \mathfrak{S}_4$ and $A \in M_2(k)$. We identify $\mathfrak{S}_3$ with the subgroup of $\mathfrak{S}_4$ generated by r and s. We have a short exact sequence of groups

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \mathfrak{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

where the generator $1 \in \mathbb{Z}/3\mathbb{Z}$ is sent to $r \in \mathfrak{S}_3$.

(c). If N is any representation of $\mathbb{Z}/3\mathbb{Z}$ on a k-vector space, show that $\mathrm{H}^p(\mathbb{Z}/3\mathbb{Z}, N) = 0$ for every $p \geq 1$. (You might find Remark IV.3.5.1 useful.)

(d). If N is any representation of $\mathfrak{S}_3$ on a k-vector space, show that we have canonical isomorphisms $\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, N^{\mathbb{Z}/3\mathbb{Z}}) \overset{\sim}{\to} \mathrm{H}^p(\mathfrak{S}_3, N)$ for every $p \geq 0$.

(e). Show that $\mathrm{H}^p(\mathfrak{S}_3, M) = 0$ for every $p \geq 1$.

(f). Show that we have canonical isomorphisms

$$\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, \mathrm{H}^1(K, M)^{\mathbb{Z}/3\mathbb{Z}}) \overset{\sim}{\to} \mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)),$$

for every $p \geq 0$.

(g). Show that $\mathrm{H}^1(K, M) = \mathrm{Hom}_{\mathbf{Grp}}(K, M)$, and that the action of $\mathfrak{S}_3$ on $\mathrm{H}^1(K, M)$ is given by $(g \cdot \varphi)(x) = g \cdot \varphi(g^{-1}xg)$, if $g \in \mathfrak{S}_3$, $x \in K$ and $\varphi \in \mathrm{H}^1(K, M)$.

(h). Show that $\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K, M))$ is a 1-dimensional k-vector space, and that $\mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)) = 0$ if $p \geq 1$.

(i). Show that we have canonical isomorphisms $\mathrm{H}^1(\mathfrak{S}_4, M) \overset{\sim}{\to} \mathrm{H}^1(K, M)^{\mathfrak{S}_3}$ and $\mathrm{H}^2(\mathfrak{S}_4, M) \overset{\sim}{\to} \mathrm{H}^2(K, M)^{\mathfrak{S}_3}$.

(j). Let N be a k-vector space with trivial action of K. Show that the map $Z^2(K, N) \to N^3$ sending a 2-cocycle $\eta : K^2 \to N$ to $(\eta(u,u)-\eta(1,1), \eta(v,v)-\eta(1,1), \eta(uv,uv)-\eta(1,1))$ induces an isomorphism $\mathrm{H}^2(K, N) \overset{\sim}{\to} N^3$.

(k). Show that $\mathrm{H}^2(\mathfrak{S}_4, M)$ is a 2-dimensional k-vector space.

Solution.

(a). We have $K = \langle u, v\rangle = \{1, u, v, uv\}$, with $uv = (13)(24)$, so the elements of K are 1 and the permutation in $\mathfrak{S}_4$ that are the product of two transpositions with disjoint supports. This implies that K is a normal subgroup of $\mathfrak{S}_4$. Also, it is easy to see that the subgroup H of $\mathfrak{S}_4$ generated by r and s is equal to the group $\{\sigma \in \mathfrak{S}_4 \mid \sigma(4) = 4\}$, which is isomorphic to $\mathfrak{S}_3$. We have $H \cap K = \{1\}$, so the composition $H \subset \mathfrak{S}_4 \to \mathfrak{S}_4/K$ is injective; as $|H| = 6 = 24/4 = |\mathfrak{S}_4/K|$, this composition is an isomorphism, so $\mathfrak{S}_3 \simeq H \overset{\sim}{\to} \mathfrak{S}_4/K$.

(b). The uniqueness of τ follows from the fact that the set $\{u, v, r, s\}$ generates $\mathfrak{S}_4$.

Let us show the existence of τ. Consider the bijection $\mathbb{F}_2^2 - \{0\} \simeq \{1, 2, 3\}$ sending $\begin{pmatrix}1\\0\end{pmatrix}$ to 3, $\begin{pmatrix}0\\1\end{pmatrix}$ to 1 and $\begin{pmatrix}1\\1\end{pmatrix}$ to 2. This induces an injective morphism of groups $\psi : \mathrm{GL}_2(\mathbb{F}_2) \to \mathfrak{S}_3$ sending $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ to s and $\begin{pmatrix}0&1\\1&1\end{pmatrix}$ to r. As $|\mathrm{GL}_2(\mathbb{F}_2)| = 6 = |\mathfrak{S}_3|$, the morphism ψ is an automorphism, and we get the representation $\tau : \mathfrak{S}_4 \to \mathrm{GL}_2(k)$ as the composition

$$\mathfrak{S}_4 \to \mathfrak{S}_4/K \simeq \mathfrak{S}_3 \overset{\psi^{-1}}{\to} \mathrm{GL}_2(\mathbb{F}_2) \subset \mathrm{GL}_2(k).$$

(c). Let Γ be any finite group of odd order. We will show that, for every $k[\Gamma]$-module N and any $p \geq 1$, we have $\mathrm{H}^p(\Gamma, N) = 0$. By Remark IV.3.5.1, we can calculate $\mathrm{H}^p(\Gamma, N)$ as a derived functor on the category $\mathscr{A} = {}_{k[\Gamma]}\mathbf{Mod}$. We claim that the abelian category $\mathscr{A}$ is semisimple (that is, every short exact sequence splits), which implies that every additive functor on $\mathscr{A}$ is exact, hence has trivial higher derived functors.

The semisimplicity of $\mathscr{A}$ follows from Maschke's theorem, whose proof in this case goes like so: Let $0 \to N_1 \xrightarrow{u} N_2 \xrightarrow{v} N_3 \to 0$ be an exact sequence of left $k[\Gamma]$-modules. As k is a field, there exists a k-linear map $w_0 : N_3 \to N_2$ such that $v \circ w_0 = \mathrm{id}_{N_3}$. Define $z : N_3 \to N_2$ by

$$w(x) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma \cdot w_0(\gamma^{-1}\cdot x),$$

where we use the fact that $|\Gamma|$ is odd to see that it is invertible in k. Then an easy calculation shows that w is $k[\Gamma]$-linear and $w \circ v = \mathrm{id}_{N_3}$.

(d). Consider the Hochschild-Serre spectral sequence for the extension $1 \to \mathbb{Z}/3 \to \mathfrak{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 1$ and the $k[\mathfrak{S}_3]$-module N:

$$E_2^{pq} = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, \mathrm{H}^q(\mathbb{Z}/3\mathbb{Z}, N)) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{S}_3, N).$$

By question (c), we have $E_2^{pq} = 0$ if $q \neq 0$, so the spectral sequence degenerates at the second page, and $E_\infty^{pq} = E_2^{pq}$. So, for every $p \geq 0$, we get an isomorphism

$$\mathrm{H}^p(\mathfrak{S}_3, N) \simeq E_\infty^{p,0} = E_2^{p,0} = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, N^{\mathbb{Z}/3\mathbb{Z}}).$$

(e). We use the formula of question (d). By definition of the action of $\mathfrak{S}_4$ on M, the k-vector space $M^{\mathbb{Z}/3\mathbb{Z}}$ is the centralizer of $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ in $M_2(k)$, that is, the space $\left\{\begin{pmatrix} a & b \\ b & a+b \end{pmatrix},\ a, b \in k\right\}$, with the action of the nontrivial element s of $\mathbb{Z}/2\mathbb{Z}$ given by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $a, b \in k$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & b \\ b & a \end{pmatrix}.$$

So we get

$$M^{\mathfrak{S}_3} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in k\right\},$$

$$(1+s)\cdot M^{\mathbb{Z}/3\mathbb{Z}} = (s-1)\cdot M^{\mathbb{Z}/3\mathbb{Z}} = \left\{\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ b \in k\right\}$$

(remember that $2 = 0$ in k), and

$$\{x \in M^{\mathbb{Z}/3\mathbb{Z}} \mid (1+s)\cdot x = 0\} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in k\right\}.$$

By problem A.7.2(a)(ii), we get $\mathrm{H}^p(\mathfrak{S}_3, M) = \mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, M^{\mathbb{Z}/3\mathbb{Z}}) = 0$ if $p \geq 1$, and $\mathrm{H}^0(\mathfrak{S}_3, M) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in k\right\}$.

(f). Apply (d) to the $k[\mathfrak{S}_3]$-module $\mathrm{H}^1(K,M)$, where the action of $\mathfrak{S}_3$ comes from the isomorphism $\mathfrak{S}_3 \simeq \mathfrak{S}_4/K$ of (a).

(g). We use the description of $\mathrm{H}^1(K,M)$ given in Subsection IV.3.5. As K acts trivially on M, a remark in this subsection gives $\mathrm{H}^1(K,M) = Z^1(K,M) = \mathrm{Hom}_{\mathbf{Grp}}(K,M)$. Moreover, if we make $G = \mathfrak{S}_4$ act on $\mathbb{Z}^{K^{n+1}}$ via its action by diagonal conjugation on K^{n+1}, then the unnormalized bar resolution $X^\bullet \to \mathbb{Z}$ of $\mathbb{Z}$ as a left $\mathbb{Z}[K]$-module is G-equivariant. So we get actions of G on the groups $C^n(K,M)$ that preserve the subgroups $Z^n(K,M)$ and $B^n(K,M)$, and induce the action of G on $\mathrm{H}^n(K,M)$. By definition of the action of G on $X^\bullet$, the action of G on $C^n(K,M) \xrightarrow{\sim} \mathscr{F}(K^n,M)$ (the set of functions from K^n to M) is given by $(g\cdot\eta)(k_1,\dots,k_n) = g\cdot\eta(g^{-1}k_1g,\dots,g^{-1}k_ng)$, for $g \in G$, $\eta : K^n \to M$ and $k_1,\dots,k_n \in K$. This implies in particular the second statement of (g).

(h). We have $r^{-1}ur = uv$ and $r^{-1}vr = u$, so, by (g), we have an isomorphism $\mathrm{H}^1(K,M) = \mathrm{Hom}_{\mathbf{Grp}}(K,M) \xrightarrow{\sim} M^2$ sending $c : K \to M$ to $(c(u),c(v))$, and the action of $r \in G$ on $\mathrm{H}^1(K,M)$ corresponding to the following action on M^2: if $x,y \in M$, then $r\cdot(x,y) = (\tau(r)(x+y)\tau(r)^{-1}, \tau(r)x\tau(x)^{-1})$. If $x = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then we have

$$\tau(r)x\tau(r)^{-1} = \begin{pmatrix} c+d & c \\ a+b+c+d & a+c\end{pmatrix}.$$

So a straightforward calculation shows that

$$\mathrm{H}^1(K,M)^{\mathbb{Z}/3\mathbb{Z}} \xrightarrow{\sim} \left\{(x,y) \in M^2 \mid \exists a,b \in k \text{ with } x = \begin{pmatrix} a & b \\ a+b & a\end{pmatrix} \text{ and } y = \begin{pmatrix} b & a+b \\ a & b\end{pmatrix}\right\}.$$

Moreover, we have $sus = v$ and $svs = u$, so the action of $s \in G$ on $\mathrm{H}^1(K,M)$ corresponds to the following action on M^2: if $x,y \in M$, then $s\cdot(x,y) = (\tau(s)y\tau(s), \tau(s)x\tau(s))$. If $x = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then we have

$$\tau(s)x\tau(s) = \begin{pmatrix} d & c \\ b & a\end{pmatrix}.$$

So, if $N = \mathrm{H}^1(K,M)^{\mathbb{Z}/3\mathbb{Z}}$, we have

$$\begin{aligned} N^{\mathbb{Z}/2\mathbb{Z}} = \{n \in N \mid (1-s)\cdot n = 0\} &= \{n \in N \mid (1+s)\cdot n = 0\} \\ &= (1-s)\cdot N = (1+s)\cdot N \\ &= \left\{\left(\begin{pmatrix} a & a \\ 0 & a\end{pmatrix}, \begin{pmatrix} a & 0 \\ a & a\end{pmatrix}\right),\ a \in k\right\}. \end{aligned}$$

By question (f) and problem A.7.2(a)(ii), we get

$$\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K,M)) = \left\{\left(\begin{pmatrix} a & a \\ 0 & a\end{pmatrix}, \begin{pmatrix} a & 0 \\ a & a\end{pmatrix}\right),\ a \in k\right\}$$

and, if $p \geq 1$, then

$$\mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K,M)) = 0.$$

(i). Consider the Hochschild-Serre spectral sequence for the extension $1 \to K \to \mathfrak{S}_4 \to \mathfrak{S}_3 \to 1$ and the $k[\mathfrak{S}_4]$-module M:

$$E_2^{pq} = \mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^q(K, M)) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{S}_4, M).$$

By questions (e) and (h), we have $E_2^{pq} = 0$ if $q \in \{0, 1\}$ and $p \neq 0$. So the second page of the spectral sequence looks like this:

$\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^2(K, M))$	$\mathrm{H}^1(\mathfrak{S}_3, \mathrm{H}^2(K, M))$	$\mathrm{H}^2(\mathfrak{S}_3, \mathrm{H}^2(K, M))$	$\mathrm{H}^3(\mathfrak{S}_3, \mathrm{H}^2(K, M))$
$\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K, M))$	0	0	0
$\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^0(K, M))$	0	0	0

In particular, if $r \geq 2$ and $q \in \{0, 1, 2\}$, then $d_r^{0,q} : E_r^{0,q} \to E_r^{r,q-r+1}$ is zero, because $E_r^{r,q-r+1} = 0$, hence $E_{r+1}^{0,q} = E_r^{0,q}$. So we get $E_\infty^{0,q} = E_2^{0,q}$ if $q \in \{0, 1, 2\}$, and $E_\infty^{1,0} = E_\infty^{1,1} = E_\infty^{2,0} = 0$ (because the corresponding E_2 terms are 0). This gives isomorphisms

$$\mathrm{H}^0(\mathfrak{S}_4, M) \xrightarrow{\sim} E_\infty^{0,0} = \mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^0(K, M)),$$
$$\mathrm{H}^1(\mathfrak{S}_4, M) \xrightarrow{\sim} E_\infty^{0,1} = \mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K, M)),$$

and

$$\mathrm{H}^2(\mathfrak{S}_4, M) \xrightarrow{\sim} E_\infty^{0,2} = \mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^2(K, M)).$$

(j). Let $\eta \in C^2(K, N)$. As K acts trivially on N, the function η is a 2-cocycle if and only if, for all $g_1, g_2, g_3 \in K$, we have

$$0 = \eta(g_2, g_3) - \eta(g_1g_2, g_3) + \eta(g_1, g_2g_3) - \eta(g_1, g_2).$$

As N is a k-vector space and k has characteristic 2, this relation can also be written as

(*) $$0 = \eta(g_2, g_3) + \eta(g_1g_2, g_3) + \eta(g_1, g_2g_3) + \eta(g_1, g_2).$$

Also, the function η is a 2-coboundary if and only if there exists a function $c : K \to M$ such that $\eta = d^1(c)$, that is, for all $g_1, g_2 \in K$,

(**) $$\eta(g_1, g_2) = c(g_1) + c(g_2) + c(g_1g_2).$$

Let $\eta \in Z^2(K, M)$. Taking $g_1 = g_2 = 1$ in equation (*), we get, for every $g \in K$, $\eta(1, 1) = \eta(1, g)$. Similarly, taking $g_2 = g_3 = 1$ in (*), we get, for every $g \in K$, $\eta(1, 1) = \eta(g, 1)$. Taking (g_1, g_2, g_3) equal to (u, v, uv), (v, u, uv), (u, uv, v), (v, uv, u), (uv, u, v) and (uv, v, u), we get the following six relations:

(1) $$\eta(u, v) + \eta(v, uv) = \eta(u, u) + \eta(uv, uv)$$

(2) $$\eta(v,u)+\eta(u,uv)=\eta(v,v)+\eta(uv,uv)$$

(3) $$\eta(u,uv)+\eta(uv,v)=\eta(u,u)+\eta(v,v)$$

(4) $$\eta(v,uv)+\eta(uv,u)=\eta(u,u)+\eta(v,v)$$

(5) $$\eta(uv,u)+\eta(u,v)=\eta(v,v)+\eta(uv,uv)$$

(6) $$\eta(uv,v)+\eta(v,u)=\eta(u,u)+\eta(uv,uv)$$

Taking (g_1,g_2,g_3) equal to (u,u,v), (v,v,u) and (uv,uv,u), (and using the fact that $\eta(1,g)=\eta(g,1)=\eta(1,1)$ for every $g\in K$), we get the following three relations:

(7) $$\eta(u,v)+\eta(u,uv)=\eta(u,u)+\eta(1,1)$$

(8) $$\eta(v,u)+\eta(v,uv)=\eta(v,v)+\eta(1,1)$$

(9) $$\eta(uv,v)+\eta(uv,u)=\eta(uv,uv)+\eta(1,1)$$

Let $\alpha : C^2(K,N^3)\to N^3$ be the morphism sending $\eta : K^2\to N$ to $(\eta(u,u)-\eta(1,1),\eta(v,v)-\eta(1,1),\eta(uv,uv)-\eta(1,1))$. We claim that $(\operatorname{Ker}\alpha)\cap Z^2(K,N)=B^2(K,N)$.

Suppose first that $\eta\in B^2(K,N)$, and write $\eta=d^1(c)$, with $c:K\to N$. Taking $g_1=g_2$ in (**) and using the fact that every element of K is of order 1 or 2, we get, for every $g\in K$, $\eta(g,g)=c(1)$. Hence $\eta(g,g)=\eta(1,1)$ for every $g\in K$, so $\alpha(\eta)=0$.

Conversely, let $\eta\in Z^2(K,N)$ such that $\alpha(\eta)=0$. Then $\eta(u,u)=\eta(v,v)=\eta(uv,uv)=\eta(1,1)$, so equations (1)-(6) imply that $\eta(u,v)=\eta(v,uv)=\eta(uv,u)$ and $\eta(v,u)=\eta(uv,v)=\eta(u,uv)$, and then equation (7) implies that $\eta(u,v)=\eta(u,uv)$, so we finally get

$$\eta(u,v)=\eta(v,uv)=\eta(uv,u)=\eta(v,u)=\eta(uv,v)=\eta(u,uv).$$

Define $c:K\to N$ by $c(u)=c(v)=0$, $c(1)=\eta(1,1)$ and $c(uv)=\eta(u,v)$. Then it is easy to check that $\eta=d^1(c)$, so $\eta\in B^2(K,M)$.

To finish the proof, we need to show that α induces a surjection $Z^2(K,N)\to N^3$. Let $(x,y,z)\in N^3$. We want to find $\eta\in Z^2(K,N)$ such that $\alpha(\eta)=(x,y,z)$. As we can always translate η by an element of $B^2(K,N)$ without changing $\alpha(\eta)$, we may take

$\eta(1,1) = \eta(u,v) = 0$. Then we must have $\eta(u,u) = x$, $\eta(v,v) = y$ and $\eta(uv,uv) = z$, and equations (1)-(9) imply that

$$\eta(v,uv) = x + z$$

$$\eta(uv,u) = y + z$$

$$\eta(u,uv) = x$$

$$\eta(uv,v) = y$$

$$\eta(v,u) = x + y + z$$

Also, if η is a 2-cocyle, we must have $\eta(1,g) = \eta(g,1) = \eta(1,1) = 0$ for every $g \in K$. This determines the values of η on all of K^2, and it is easy to check that the function η that we defined is indeed a 2-cocycle.

(k). We know that $\mathrm{H}^2(\mathfrak{S}_4, M) \simeq \mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^2(K,M))$ by question (i), so we need to calculate the action of $\mathfrak{S}_3$ on $\mathrm{H}^2(K,M)$; we will use the isomorphism $\alpha : \mathrm{H}^2(K,M) \xrightarrow{\sim} M^3$ of question (j). By the proof of question (g), an element $g \in \mathfrak{S}_4$ acts on a 2-cocycle $\eta \in Z^2(K,M)$ by $(g\cdot\eta)(k_1,k_2) = g\cdot\eta(g^{-1}k_1g, g^{-1}k_2g)$. Let $\eta \in Z^2(K,M)$, and let $(x,y,z) = \alpha(\eta)$. We have $sus = v$, $svs = u$, $s(uv)s = uv$, $r^{-1}ur = uv$, $r^{-1}vr = u$ and $r^{-1}(uv)r = v$, so

$$\alpha(s\cdot\eta) = (s\cdot y, s\cdot x, s\cdot z)$$

and

$$\alpha(r\cdot\eta) = (r\cdot z, r\cdot x, r\cdot y).$$

So η represents an element of $\mathrm{H}^2(K,M)^{\mathfrak{S}_3}$ if and only if $s\cdot y = x$, $s\cdot x = y$, $s\cdot z = z$, $r\cdot z = x$, $r\cdot x = y$ and $r\cdot y = z$. We already calculate the action of r and s on M in the solution of question (h). The relation $s\cdot z = z$ is equivalent to the fact that $z = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, for $a,b \in k$. Then we get

$$x = r\cdot z = \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}$$

and

$$y = r\cdot x = \begin{pmatrix} a+b & 0 \\ b & a+b \end{pmatrix}.$$

We have $z = r\cdot y$ because $r^3 = 1$, and it is clear that $x = s\cdot y$ and $y = s\cdot x$. So the k-vector space

$$\mathrm{H}^2(K,M)^{\mathfrak{S}_3} \simeq \left\{ \left(\begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}, \begin{pmatrix} a+b & 0 \\ b & a+b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right),\ a,b \in k \right\}$$

is 2-dimensional.

□

A.9 Problem set 9

A.9.1 Abelian subcatgeories of triangulated categories

Let $\mathscr{D}$ be a triangulated category. We denote the shift functors by $X \longmapsto X[1]$, and we write triangles as $X \to Y \to Z \to X[1]$ or $X \to Y \to Z \xrightarrow{+1}$. For every $X, Y \in \mathrm{Ob}(\mathscr{D})$ and every $n \in \mathbb{Z}$, we write $\mathrm{Hom}^n_{\mathscr{D}}(X,Y) = \mathrm{Hom}(X, Y[n])$.

(a). Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$ be two distinguished triangles of $\mathscr{D}$, and let $g : Y \to Y'$ be a morphism.

(i) Show that the following conditions are equivalent:

(1) $v' \circ g \circ u = 0$;

(2) there exists $f : X \to X'$ such that $u' \circ f = g \circ u$;

(3) there exists $h : Z \to Z'$ such that $h \circ v = v' \circ g$;

(4) there exist $f : X \to X'$ and $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

(ii) Suppose that the conditions (i) hold and that $\mathrm{Hom}^{-1}_{\mathscr{D}}(X, Z') = 0$. Show that the morphisms f and h of (i)(2) and (i)(3) are unique.

(b). Let $\mathscr{C}$ be a full subcategory of $\mathscr{D}$, and suppose that $\mathrm{Hom}^n(X,Y) = 0$ if $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $n < 0$.

(i) Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Take a distinguished triangle $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ in $\mathscr{D}$, and suppose that we have a distinguished triangle $N[1] \to S \to C \xrightarrow{+1}$ with $N, C \in \mathrm{Ob}(\mathscr{C})$. In particular, we get morphisms $\alpha : N[1] \to S \to X[1]$ and $\beta : Y \to S \to X$.

Show that $\alpha[-1] : N \to X$ is a kernel of f and that $\beta : Y \to C$ is a cokernel of f.

We say that a morphism f of $\mathscr{C}$ is *admissible* if there exist distinguished triangles satisfying the conditions of (i). We say that a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of morphisms of $\mathscr{C}$ is an *admissible short exact sequence* if there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ in $\mathscr{D}$.

(ii) Suppose that $\mathscr{C}$ as a zero object. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$ with $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, show that f and g are admissible, that f is a kernel of g and that g is a cokernel of f.

(iii) If $f : X \to Y$ is an admissible monomorphism (resp. epimorphism) in $\mathscr{C}$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$, show that f has a cokernel (resp. a kernel) in $\mathscr{C}$ and that $Z \simeq \mathrm{Coker}(f)$ (resp. $Z[-1] \simeq \mathrm{Ker}(f)$).

(iv) Suppose that every morphism of $\mathscr{C}$ is admissible and $\mathscr{C}$ is an additive subcategory of $\mathscr{D}$. Show that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible.

(v) Suppose that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible. Show that every morphism of $\mathscr{C}$ is admissible.

Solution.

(a). (i) Obviously, point (4) implies (2) and (3). Also, as $v' \circ u' = 0$ and $v \circ u = 0$ by Proposition V.1.1.11(i) , points (2) and (3) each imply (1). Also, by axiom (TR4), we have that (2) implies (4). So it remains to show that (1) implies (2). Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(X, \cdot)$ to the distinguished triangle $X' \to Y' \to Z' \stackrel{+1}{\to}$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X, X') \stackrel{u'\circ(\cdot)}{\to} \mathrm{Hom}_{\mathscr{D}}(X, Y') \stackrel{v'\circ(\cdot)}{\to} \mathrm{Hom}_{\mathscr{D}}(X, Z').$$

So, if $v' \circ (g \circ u) = 0$ (that is, if (1) holds), then there exists $f \in \mathrm{Hom}_{\mathscr{D}}(X, X')$ such that $u' \circ f = g \circ u$ (that is, (2) holds).

(ii) In the exact sequence of (i), the kernel of $u' \circ (\cdot) : \mathrm{Hom}_{\mathscr{D}}(X, X') \to \mathrm{Hom}_{\mathscr{D}}(X, Y')$ is the image of the morphism $w'[-1] \circ (\cdot) : \mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) \to \mathrm{Hom}_{\mathscr{D}}(X, X')$. This gives the uniqueness of f in (2) (if it exists). To show the uniqueness of h, suppose that we have two morphisms $h, h' : Z \to Z'$ such that $h \circ v = v' \circ g = h' \circ v$, so that $(h - h') \circ v = 0$. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, Z')$ to the distinguished triangle $X \to Y \to Z \stackrel{+1}{\to}$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X[1], Z') = \mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) = 0 \to \mathrm{Hom}_{\mathscr{D}}(Z, Z') \to \mathrm{Hom}_{\mathscr{D}}(Z, Y').$$

So the morphism $(\cdot) \circ v : \mathrm{Hom}_{\mathscr{D}}(Z, Z') \to \mathrm{Hom}_{\mathscr{D}}(Y, Z')$ is injective, which shows that $h = h'$.

(b). (i) We show that β is a cokernel of f. Let $g : Y \to Z$ be a morphism of $\mathscr{C}$ such that $g \circ f = 0$. We want to show that there exists a unique morphism $g' : C \to Z$ such that $g' \circ \beta = g$. By (TR1) and (TR3), we have a distinguished triangle $0 \to Z \stackrel{\mathrm{id}_Z}{\to} Z \to 0[1] = 0$. Applying question (a) to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[d, "g"'] & S \arrow[r] \arrow[d, dashed, "h"] & X[1] \\
0 \arrow[r] & Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0
\end{tikzcd}$$

and using the fact that $\mathrm{Hom}^{-1}_{\mathscr{D}}(X, Z) = 0$ (because $X, Z \in \mathrm{Ob}(\mathscr{C})$), we see that there exists a unique morphism $h : S \to Z$ making the diagram commute. This already implies the uniqueness of g' (if it exists). To show the existence of g', we apply question (a) again to the diagram

$$\begin{tikzcd}
N[1] \arrow[r] & S \arrow[r] \arrow[d, "h"] & C \arrow[r] \arrow[d, dashed, "g'"] & N[2] \\
0 \arrow[r] & Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0
\end{tikzcd}$$

The hypothesis of (a) is satisfied, because the composition of h and of $N[1] \to S$ is an element of $\mathrm{Hom}_{\mathscr{D}}(N[1], Z) = \mathrm{Hom}_{\mathscr{D}}^{-1}(N, Z) = 0$.

We show that $\alpha[-1]$ is a kernel of f. The proof is similar. Let $g : Z \to X$ be a morphism of $\mathscr{C}$ such that $f \circ g = 0$. We want to show that there exists a unique morphism $g' : Z \to N$ such that $\alpha[-1] \circ g' = g$. First, we apply question (a) to the diagram

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, dashed, "h"'] & Z \arrow[r] \arrow[d, "g"'] & 0 \arrow[r] & 0 \\
S[-1] \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r] & S
\end{tikzcd}$$

Using the fact that $\mathrm{Hom}_{\mathscr{D}}^{-1}(Z, Y) = 0$ (because $Y, Z \in \mathrm{Ob}(\mathscr{C})$), we see that there is a unique morphism $h : Z \to S[-1]$ making the diagram commute. This impluies the uniqueness of g'. To show the existence of g', we apply question (a) to the diagram

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, dashed, "g'"'] & Z \arrow[r] \arrow[d, "h"'] & 0 \arrow[r] & 0 \\
N \arrow[r] & S[-1] \arrow[r] & C[-1] \arrow[r] & N[2]
\end{tikzcd}$$

The hypothesis of (a) is satisfied, because the composition of h and of $S[-1] \to C[-1]$ is an element of $\mathrm{Hom}_{\mathscr{D}}(Z, C[-1]) = 0$.

(ii) The morphism f is admissible, because we take $S = Z$ in question (i), and then we have a distinguished triangle $0 \to S \to Z \to 0$. Similarly, the morphism g is admissible, because we can take $S = X[1]$ in (i), and then we have a distinguished triangle $X[1] \to S \to 0 \to X[2]$. Also, question (i) immediately implies that g is a cokernel of f and that f is a kernel of g.

(iii) Let $f : X \to Y$ be an admissible morphism in $\mathscr{C}$, and let $X \xrightarrow{f} Y \to S = Z \xrightarrow{+1}$ and $N[1] \to S \to C \xrightarrow{+1}$ be distinguished triangles as in question (i); by that question, we have $\operatorname{Ker} f = N$ and $\operatorname{Coker} f = C$. If f is a monomorphism, this implies that $N = 0$, so the morphism $S \to C$ is an isomorphism, which shows that S is isomorphic to the cokernel of f. If f is an epimorphism, then we have $C = 0$, so the morphism $N[1] \to S$ is an isomorphism, which shows that $S[-1]$ is isomorphic to the kernel of f.

(iv) By question (i), every morphism of $\mathscr{C}$ has a kernel and a cokernel. Let $f : X \to Y$ be a morphism of $\mathscr{C}$; we need to check that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism, or in other words that the canonical morphism $X \to \mathrm{Im}(f)$ is a cokernel of $\ker(f) \to X$. Let $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ and $N[1] \to S \to C \xrightarrow{+1}$ be distinguished triangles as in question (i), and let $\alpha[-1] : N \to X$ and $\beta : Y \to C$ be the morphisms defined in that question. Applying the octahedral axiom to the morphisms $Y \to S$ and $S \to C$ and to their composition β, we get a commutative diagram where the rows and the third column are distinguished triangles:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
Y \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d, equal] & I[1] \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
S \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N[2] \arrow[r] \arrow[d] & S[1] \arrow[d] \\
Y[1] \arrow[r] & S[1] \arrow[r] & X[2] \arrow[r] & Y[2]
\end{tikzcd}$$

As β is the cokernel of f, it is an epimorphism, so, by question (iii), the morphism $I \to Y$ is isomorphic to $\mathrm{Ker}(\beta) \to Y$, that is, to $\mathrm{Im}(f) \to Y$. As we have a distinguished triangle $N \to X \to I \to N[1]$, question (iii) shows that $X \to I$ is isomorphic to $X \to \mathrm{Coker}(\alpha[-1])$, that is, to $X \to \mathrm{Coim}(f)$, so we are done.

Finally, we show that every short exact sequence of $\mathscr{C}$ is admissible. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathscr{C}$, and let $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ be a distinguished triangle in $\mathscr{D}$. As f is an admissible monomorphism and $g : Y \to Z$ is a cokernel of f, question (iii) implies that there exists a commutative triangle

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[dr] & Z \arrow[d, "\wr"] \\
& S
\end{tikzcd}$$

where $Z \to S$ is an isomorphism. This implies that $X \xrightarrow{f} Y \xrightarrow{g} Z$ extends to a distinguished triangle.

(v) Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Let $N = \mathrm{Ker}(f)$, $C = \mathrm{Coker}(f)$ and $I = \mathrm{Im}(f)$. We have exact sequences $0 \to N \to X \to I \to 0$ and $0 \to I \to Y \to C \to 0$, that extend to distinguished triangles in $\mathscr{D}$ by the hypothesis. Applying the octohedral axiom to the morphism $X \to I$ and $I \to Y$ and to their composition $X \xrightarrow{f} Y$, we get a commutative diagram where the rows and the third column are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & N[1] \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & X[1] \arrow[d] \\
I \arrow[r] & Y \arrow[r] & C \arrow[r] \arrow[d] & I[1] \\
 & & N[2] &
\end{tikzcd}$$

This gives the two triangles of (i) and shows that f is admissible.

$\square$

A.9.2 t-structures

We use the convention of problem A.9.1. A *t-structure* on $\mathscr{D}$ is the date of two full subcategories $\mathscr{D}^{\leq 0}$ and $\mathscr{D}^{\geq 0}$ such that (with the convention that $\mathscr{D}^{\leq n} = \mathscr{D}^{\leq 0}[-n]$ and $\mathscr{D}^{\geq n} = \mathscr{D}^{\geq 0}[-n]$);

(0) If $X \in \mathrm{Ob}(\mathscr{D})$ is isomorphic to an object of $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$), then X is in $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$).

(1) For every $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and every $Y \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$, we have $\mathrm{Hom}(X, Y) = 0$.

(2) We have $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 0} \supset \mathscr{D}^{\geq 1}$.

(3) For every $X \in \mathrm{Ob}(\mathscr{D})$, there exists a distinguished triangle $A \to X \to B \xrightarrow{+1}$ with $A \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$.

We fix a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ on $\mathscr{D}$.

(a). Show that the distinguished triangle of condition (3) is unique up to unique isomorphism.

(b). For every $n \in \mathbb{Z}$, show that the inclusion functor $\mathscr{D}^{\leq n} \subset \mathscr{D}$ has a right adjoint $\tau^{\leq n}$ and the inclusion functor $\mathscr{D}^{\geq n} \subset \mathscr{D}$ has a left adjoint $\tau^{\geq n}$. (Hint: It suffice to treat the case $n = 0$.)

(c). For every $n \in \mathbb{Z}$, show that there is a unique morphism $\delta : \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ such that the triangle $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{\delta} (\tau^{\leq n}X)[1]$ is distinguished, where the other two morphisms are given by the counit and unit of the adjunctions of (b).

(d). Let $a, b \in \mathbb{Z}$ such that $a \leq b$, and let $X \in \mathrm{Ob}(\mathscr{D})$. Show that there exists a unique morphism $\alpha : \tau^{\geq a}\tau^{\leq b}X \to \tau^{\leq b}\tau^{\geq a}X$ such that the following diagram commutes:

$$\begin{tikzcd}
\tau^{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\geq a}X \\
\tau^{\geq a}\tau^{\leq b}X \arrow[rr, "\alpha"'] & & \tau^{\leq b}\tau^{\geq a}X \arrow[u]
\end{tikzcd}$$

(where all the other morphisms are counit or unit morphisms of the adjunctions of (b)), and that α is an isomorphism. (Hint: Apply the octahedral axiom to $\tau^{\leq a-1}X \xrightarrow{f} \tau^{\leq b}X \xrightarrow{g} X$.)

(e). If $a, b \in \mathbb{Z}$ are such that $a \leq b$, show that, for every $X \in \mathrm{Ob}(\mathscr{D})$, we have $\tau^{\geq a}\tau^{\leq b}X \in \mathrm{Ob}(\mathscr{D}^{\geq a}) \cap \mathrm{Ob}(\mathscr{D}^{\leq b})$.

Let $\mathscr{C} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$; that is, $\mathscr{C}$ is the full subcategory of $\mathscr{D}$ such that $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{D}^{\leq 0}) \cap \mathrm{Ob}(\mathscr{D}^{\geq 0})$. We denote the functor $\tau^{\leq 0}\tau^{\geq 0} : \mathscr{D} \to \mathscr{C}$ by H^0. The category $\mathscr{C}$ is called the *heart* or *core* of the t-structure.

(f). Show that $\mathscr{C}$ is an abelian category.

(g). Show that, if $X \to Y \to Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$ such that $X, Z \in \mathrm{Ob}(\mathscr{C})$, then Y is also in $\mathscr{C}$.

(h). The goal of this question is to show that the functor $\mathrm{H}^0 : \mathscr{D} \to \mathscr{C}$ is a cohomological functor. Let $X \to Y \to Z \xrightarrow{+1}$ be a distinguished triangle in $\mathscr{D}$.

(i) If $X, Y, Z \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: A sequence of morphisms $A \to B \to C \to 0$ in an abelian category $\mathscr{A}$ is exact if and only if, for every object D of $\mathscr{A}$, the sequence of abelian groups $0 \to \mathrm{Hom}_{\mathscr{A}}(C, D) \to \mathrm{Hom}_{\mathscr{A}}(B, D) \to \mathrm{Hom}_{\mathscr{A}}(A, D)$ is exact.)

(ii) If $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: Construct a distinguished triangle $X \to \tau^{\leq 0}Y \to \tau^{\leq 0}Z \xrightarrow{+1}$.)

(iii) If $Z \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$, show that the sequence $0 \to \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

(iv) In general, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

Solution.

(a). Suppose that we have two distinguished triangles $A \to X \to B \xrightarrow{+1}$ and $A' \to X \to B' \xrightarrow{+1}$ with $A, A' \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B, B' \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. We have $B'[-1] \in \mathrm{Ob}(\mathscr{D}^{\geq 2}) \subset \mathrm{Ob}(\mathscr{D}^{\geq 1})$, so, by condition (1), $\mathrm{Hom}_{\mathscr{D}}(A, B') = 0$ and $\mathrm{Hom}^{-1}_{\mathscr{D}}(A, B') = \mathrm{Hom}_{\mathscr{D}}(A, B'[-1]) = 0$. So question (i) of problem A.9.1 implies that id_X extends to a unique morphism of distinguished triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & B \arrow[r] \arrow[d, "v"'] & A[1] \arrow[d, "u[1]"] \\ A' \arrow[r] & X \arrow[r] & B' \arrow[r] & A'[1] \end{tikzcd}$$

Exchanging the roles of (A, B) and (A', B'), we get that id_X also extends to a unique morphism of distinguished triangles

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & B' \arrow[r] \arrow[d, "v'"'] & A'[1] \arrow[d, "u'[1]"] \\ A \arrow[r] & X \arrow[r] & B \arrow[r] & A[1] \end{tikzcd}$$

So we have two endomorphisms of the distinguished triangle $A \to X \to B \xrightarrow{+1}$ extending id_X, the endomorphisms given by $(u' \circ u, \mathrm{id}_X, v' \circ v)$ and $(\mathrm{id}_A, \mathrm{id}_X, \mathrm{id}_B)$. For the same reason as before, these morphisms must be equal, so $u' \circ u = \mathrm{id}_A$ and $v' \circ v = \mathrm{id}_B$. We see similarly that $u \circ u' = \mathrm{id}_{A'}$ and $v \circ v' = \mathrm{id}_{B'}$.

(b). As the shift is an auto-equivalence of $\mathscr{D}$, we may assume that $n = 0$.

To show that the inclusion functor $\mathscr{D}^{\leq 0} \to \mathscr{D}$ has a right adjoint, it suffices by Proposition I.4.7 to show that the functor $F_Y : \mathrm{Hom}_{\mathscr{D}}(\cdot, Y) : (\mathscr{D}^{\leq 0})\mathrm{op} \to \mathbf{Set}$ is representable for every $Y \in \mathrm{Ob}(\mathscr{D})$. Let $Y \in \mathrm{Ob}(\mathscr{D})$, and let $A \to Y \to B \xrightarrow{+1}$ be a distinguished triangle with $A \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. Let $X \in \mathrm{Ob}(\mathscr{D})$. Then we have an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X, B[-1]) \to \mathrm{Hom}_{\mathscr{D}}(X, A) \to \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, B).$$

If $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, then $\mathrm{Hom}_{\mathscr{D}}(X, B[-1]) = \mathrm{Hom}_{\mathscr{D}}(X, B) = 0$ by condition (1) (because $B[-1] \in \mathrm{Ob}(\mathscr{D}^{\geq 2}) \subset \mathrm{Ob}(\mathscr{D}^{\geq 1})$), so the morphism $\mathrm{Hom}_{\mathscr{D}}(X, A) \to \mathrm{Hom}_{\mathscr{D}}(X, Y)$ is an isomorphism. This shows that F_Y is representable by the couple $(A, A \to Y)$ (note that the morphism $A \to Y$ is an element of $F_Y(A)$).

Similary, To show that the inclusion functor $\mathscr{D}^{\geq 0} \to \mathscr{D}$ has a left adjoint, it suffices by Proposition I.4.7 to show that the functor $G_X : \mathrm{Hom}_{\mathscr{D}}(X, \cdot) : \mathscr{D}^{\geq 0} \to \mathbf{Set}$ is representable for every $X \in \mathrm{Ob}(\mathscr{D})$. As in the previous paragraph, we see that, if $A \to X \to B \xrightarrow{+1}$ is a distinguished triangle with $A \in \mathrm{Ob}(\mathscr{D}^{\leq -1})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$ (to get such a triangle, use condition (3) for $X[-1]$ and then apply the functor $[1]$), then G_X is representable by the pair $(B, X \to B)$.

(c). As in question (b), it suffices to treat the case $n = 0$. Let $X \in \mathrm{Ob}(\mathscr{D})$, and let $A \to X \to B \to A[1]$ be a distinguished triangle such that $A \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. We have seen in the solution of question (b) that the morphism $\tau^{\leq 0} X \to X$ is isomorphic to $A \to X$, and the morphism $X \to \tau^{\geq 1} X$ is isomorphic to $X \to B$, so the morphism $B \to A[1]$ induces a morphism $\delta : \tau^{\geq 1}X \to (\tau^{\leq 0}X)[1]$ that makes the triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \xrightarrow{\delta} (\tau^{\leq 0}X)[1]$ distinguished.

(d). Let $X \in \mathrm{Ob}(\mathscr{D})$. As $\mathscr{D}^{\leq a} \subset \mathscr{D}^{\leq b}$, the canonical morphism $\tau^{\leq b}X \to X$ induces an isomorphism $\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq a}X, \tau^{\leq b}X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\tau^{\leq a}X, \tau^{\leq b}X)$, so the canonical morphism $\tau^{\leq a}X \to X$ factors through a morphism $\tau^{\leq a}X \to \tau^{\leq b}X$; applying the functor $\tau^{\leq a}$, we get a sequence of morphisms

$$\tau^{\leq a}X \to \tau^{\leq a}\tau^{\leq b}X \to \tau^{\leq b}X \to X.$$

Hence, if Y is an object of $\mathscr{D}^{\leq a}$, then the bijection $\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(Y, X)$ is equal to the composition

$$\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}\tau^{\leq b}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq b}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, X),$$

where the second and third maps are bijection. This shows that $\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}\tau^{\leq b}X)$ is bijective for every $Y \in \mathrm{Ob}(\mathscr{D}^{\leq a})$, i.e. that the morphism $\tau^{\leq a}X \to \tau^{\leq a}\tau^{\leq b}X$ is an isomorphism. Similarly, we have a canonical isomorphism $\tau^{\geq b}\tau^{\geq a}X \xrightarrow{\sim} \tau^{\geq b}X$ for every $X \in \mathrm{Ob}(\mathscr{D})$.

Note also that, by question (c), if $c \in \mathbb{Z}$, then an object X of $\mathscr{D}$ is in $\mathscr{D}^{\leq c}$ (resp. $\mathscr{D}^{\geq c}$) if and only if $\tau^{\geq c+1}X = 0$ (resp. $\tau^{\leq c-1}X = 0$). In particular, if $X \in \mathrm{Ob}(\mathscr{D})$, then we have $\tau^{\geq b+1}\tau^{\geq a}\tau^{\leq b}X = \tau^{\geq b+1}\tau^{\leq b}X = 0$ and $\tau^{\leq a-1}\tau^{\leq b}\tau^{\geq a}X = \tau^{\leq a-1}\tau^{\geq a}X = 0$ (where the first isomorphisms are proved in the previous paragraph), so $\tau^{\geq a}\tau^{\leq b}X \in \mathscr{D}^{\leq b}$ and $\tau^{\leq b}\tau^{\geq a}X \in \mathrm{Ob}(\mathscr{D}^{\geq a})$.

Now let $X \in \mathrm{Ob}(\mathscr{D})$. By definition of $\tau^{\geq a}$, the morphism $\tau^{\leq b}X \to X \to \tau^{\geq a}X$ factors uniquely as

$$\tau^{\leq b}X \to \tau^{\geq a}\tau^{\leq b}X \xrightarrow{(1)} \tau^{\geq a}X.$$

As $\tau^{\geq a}\tau^{\leq b}X \in \mathrm{Ob}(\mathscr{D}^{\leq b})$, the morphism (1) factors uniquely as

$$\tau^{\geq a}\tau^{\leq b}X \xrightarrow{(2)} \tau^{\leq b}\tau^{\geq a}X \to \tau^{\geq a}X.$$

It remains to show that (2) is an isomorphism. Applying the octahedral axiom to the canonical morphism $\tau^{\leq a-1}X \to \tau^{\leq b}X \to X$ (and their composition), we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
\tau^{\leq a-1}X \arrow[r] \arrow[d, equal] & \tau^{\leq b}X \arrow[r] \arrow[d] & \tau^{\geq a}\tau^{\leq b}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq a-1}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \tau^{\geq a}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq b}X \arrow[r] & X \arrow[r] & \tau^{\geq b+1}X \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

So, by question (c), we have a morphism of distinguished triangles

$$\begin{tikzcd}
\tau^{\geq a}\tau^{\leq b}X \arrow[r] \arrow[d, "(2)"'] & \tau^{\geq a}X \arrow[r] \arrow[d, equal] & \tau^{\geq b+1}X \arrow[r, "+1"] \arrow[d, equal] & {} \\
\tau^{\leq b}\tau^{\geq a}X \arrow[r] & \tau^{\geq a}X \arrow[r] & \tau^{\geq b+1}X \arrow[r, "+1"] & {}
\end{tikzcd}$$

This shows that (2) is an isomorphism.

(e). We already showed this in the solution of question (d).

(f). We already know that $\mathscr{C}$ is a full addditive subcategory of $\mathscr{D}$, because it is the intersection of two full additive subcategories. We also have $\operatorname{Hom}^n(X,Y)=0$ if $X,Y\in\operatorname{Ob}(\mathscr{C})$ and $n<0$ by properties (1) and (2) of a t-structure. So, by question (b)(iv) of problem A.9.1, it suffices to show that every morphism of $\mathscr{C}$ is admissible. Let $f:X\to Y$ be a morphism of $\mathscr{C}$, and complete it to a distinguished triangle $X\to Y\to S\overset{+1}{\to}$. Let $N=\tau^{\leq -1}S[-1]$ and $C=\tau^{\geq 0}S$. By question (c), we have a distinguished triangle $N[1]\to S\to C\overset{+1}{\to}$, so it suffices to show that N and C are in $\mathscr{C}$. By question (e), it suffices to show that $S\in\operatorname{Ob}(\mathscr{D}^{\leq 0}\cap\mathscr{D}^{\geq -1})$.

Note that we have a distinguished triangle $Y\to S\to X[1]\overset{+1}{\to}$. Let $S'=\tau^{\geq 1}S$. As $Y\in\operatorname{Ob}(\mathscr{D}^{\leq 0})$ and $X[1]\in\operatorname{Ob}(\mathscr{D}^{\leq -1})\subset\operatorname{Ob}(\mathscr{D}^{\leq 0})$, condition (1) in the definition of a t-structure implies that $\operatorname{Hom}_{\mathscr{D}}(Y,S')=\operatorname{Hom}_{\mathscr{D}}(X[1],S')=0$, and, applying the cohomological functor $\operatorname{Hom}_{\mathscr{D}}(\cdot,S')$ to the distinguished triangle $Y\to S\to X[1]\overset{+1}{\to}$, we deduce that $\operatorname{Hom}_{\mathscr{D}}(S,S')=0$. As $\operatorname{Hom}_{\mathscr{D}}(S,S')=\operatorname{Hom}_{\mathscr{D}^{\geq 1}}(S',S')$, this implies that $S'=0$, hence that $S\in\operatorname{Ob}(\mathscr{D}^{\leq 0})$. The proof that $S\in\operatorname{Ob}(\mathscr{D}^{\geq -1})$ is similar.

(g). We showed in the solution of (f) that, if X and Z are in $\mathscr{D}^{\leq 0}$ (resp. in $\mathscr{D}^{\geq 0}$), then so is Y. This immediately implies the result.

(h). (i) We first prove the hint. If $A\to B\to C\to 0$ is exact, the exactness of $0\to\operatorname{Hom}_{\mathscr{A}}(C,D)\to\operatorname{Hom}_{\mathscr{A}}(B,D)\to\operatorname{Hom}_{\mathscr{A}}(A,D)$ for every D simply follows from the left exactness of the functor $\operatorname{Hom}_{\mathscr{A}}(\cdot,D)$. Suppose that we have morphisms $A\overset{f}{\to}B\overset{g}{\to}C$ such that $0\to\operatorname{Hom}_{\mathscr{A}}(C,D)\to\operatorname{Hom}_{\mathscr{A}}(B,D)\to\operatorname{Hom}_{\mathscr{A}}(A,D)$ is exact for every D. Taking $D=\operatorname{Coker} g$, we see that the canonical morphism $u:C\to\operatorname{Coker} g$ is sent to $0=u\circ g\in\operatorname{Hom}_{\mathscr{A}}(B,\operatorname{Coker} g)$, so $u=0$, so $\operatorname{Coker} g=0$ and g is surjective. Also, taking $D=C$, we see that id_C goes to $g\in\operatorname{Hom}_{\mathscr{D}}(B,C)$, then to $g\circ f\in\operatorname{Hom}_{\mathscr{A}}(A,C)$, so we have $g\circ f=0$. It remains to show that the inclusion $\operatorname{Im} f\subset\operatorname{Ker} g$ is an isomorphism. Take $D=B/\operatorname{Im} f$ and let $v:B\to D$ be the canonical projection. Then $v\circ f=0$, so, by hypothesis, there exists a morphism $w:C\to D$ such that $v=w\circ g$. In particular, we have $\operatorname{Ker} g\subset\operatorname{Ker} v=\operatorname{Im} f$.

Now we prove the statement of (i). As $X\in\mathscr{D}^{\leq 0}$, then we have $\mathrm{H}^0(X)=\tau^{\geq 0}X$, hence $\operatorname{Hom}_{\mathscr{C}}(\mathrm{H}^0(X),D)\simeq\operatorname{Hom}_{\mathscr{D}}(X,D)$ for every $D\in\operatorname{Ob}(\mathscr{C})$, and similarly for Y and Z. Also, if $D\in\operatorname{Ob}(\mathscr{C})$, then axiom (1) of t-structures implies that $\operatorname{Hom}_{\mathscr{D}}(X[1],D)=0$. So, if $D\in\operatorname{Ob}(\mathscr{C})$, applying the cohomological functor

$\mathrm{Hom}_{\mathscr{D}}(\cdot, D)$ to the distinguished triangle $X \to Y \to Z \overset{+1}{\to}$ gives an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X[1], D) = 0 \to \mathrm{Hom}_{\mathscr{C}}(\mathrm{H}^0(Z), D) \to \mathrm{Hom}_{\mathscr{C}}(\mathrm{H}^0(Y), D) \to \mathrm{Hom}_{\mathscr{D}}(\mathrm{H}^0(X), D).$$

This shows that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$.

(ii) For every $T \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, T)$ to $X \to Y \to Z \overset{+1}{\to}$ and using the fact that $\mathrm{Hom}_{\mathscr{D}}(X, T) = \mathrm{Hom}_{\mathscr{D}}(X[1], T) = 0$ (because $X, X[1] \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$) gives an isomorphism $\mathrm{Hom}_{\mathscr{D}}(Z, T) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(Y, T)$, hence an isomorphism $\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}Z, T) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}Y, T)$. This implies that the functor $\tau^{\geq 1}$ sends the morphism $Y \to Z$ to an isomorphism. Applying the octahedral axiom to the morphisms $Y \to Z \to \tau^{\geq 1}Z$, we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X[1] \arrow[r, "+1"] \arrow[d] & {} \\
Y \arrow[r] \arrow[d] & \tau^{\geq 1}Z \arrow[r] \arrow[d, equal] & \tau^{\geq 0}Y[1] \arrow[r, "+1"] \arrow[d] & {} \\
Z \arrow[r] & \tau^{\geq 1}Z \arrow[r] & \tau^{\leq 0}Z \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

So we have a distinguished triangle $X \to \tau^{\leq 0}Y \to \tau^{\leq 0}Z \overset{+1}{\to}$. Applying question (i) gives an exact sequence

$$\mathrm{H}^0(X) \to \mathrm{H}^0(\tau^{\leq 0}Y) \to \mathrm{H}^0(\tau^{\leq 0}Z) \to 0$$

in $\mathscr{C}$. As the morphism $\mathrm{H}^0(\tau^{\leq 0}Y) \to \mathrm{H}^0(Y)$ induced by $\tau^{\leq 0}Y \to Y$ is an isomorphism (by definition of H^0) and similarly for Z, we are done.

(iii) This is just the result of question (ii) in the opposite category. (Note that $(\mathscr{D}^{\geq 0}, \mathscr{D}^{\leq 0})$ is a t-structure on $\mathscr{D}^{\mathrm{op}}$.)

(iv) Applying the octahedral axiom to the morphisms $\tau^{\leq 0}X \to X \to Y$, we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
\tau^{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau^{\geq 1}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq 0}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & T \arrow[r, "+1"] \arrow[d] & {} \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

Question (ii) for the second row gives an exact sequence

$$\mathrm{H}^0(\tau^{\leq 0}X) = \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(T) \to 0,$$

and question (iii) for the distinguished triangle $T \to Z \to \tau^{\geq 1}X[1] \overset{+1}{\to}$ gives an exact sequence

$$0 \to \mathrm{H}^0(T) \to \mathrm{H}^0(Z).$$

Putting these two sequences together, we see that the sequence

$$\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$$

is exact.

□

A.9.3 The canonical t-structure

Let $\mathscr{A}$ be an abelian category.

(a). Let $n \in \mathbb{Z}$. If $X \in \mathrm{Ob}(\mathrm{D}^{\leq n}(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^{\geq n+1}(\mathscr{A}))$, show that $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) = 0$.

(b). Show that $(\mathrm{D}^{\leq 0}(\mathscr{A}), \mathrm{D}^{\geq 0}(\mathscr{A}))$ is a t-structure on $\mathrm{D}(\mathscr{A})$, that its heart is equivalent to $\mathscr{A}$, and that the associated functor $\mathrm{H}^0 : \mathrm{D}(\mathscr{A}) \to \mathscr{A}$ is the 0th cohomology functor.

Solution.

(a). After replacing X and Y by isomorphic objects in $\mathrm{D}(\mathscr{A})$, we may assume that $X^m = 0$ for $m > n$ and $X^m = 0$ for $m \leq n$. Let $u : X \to Y$ be a morphism in $\mathrm{D}(\mathscr{A})$. Then we have morphisms $f : X \to Z$ and $s : Y \to Z$ in $K(\mathscr{A})$ such that s is a quasi-isomorphism and $u = s^{-1} \circ f$ in $\mathrm{D}(\mathscr{A})$. As $Y^m = 0$ for $m \leq n$, the morphism $s' = \tau^{\geq n+1}s : Y = \tau^{\geq n+1}Y \to Z' = \tau^{\geq n+1}Z$ is also a quasi-isomorphism, and we have a commutative diagram:

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f"] \arrow[dr, "\tau^{\geq n+1}f"'] & & Y \arrow[ul, "s"'] \arrow[dl, "s'"] \\
& Z' &
\end{tikzcd}$$

By Theorem V.2.2.4 , this implies that $s^{-1} \circ f = s'^{-1} \circ \tau^{\geq n+1}f$ as morphisms in $\mathrm{D}(\mathscr{A})$. But $X^m = 0$ for $m \geq n+1$, so $\tau^{\geq n+1}f = 0$, and finally $u = 0$.

(b). Let $\mathscr{D}^{\leq 0} = \mathrm{D}^{\leq 0}(\mathscr{A})$ and $\mathscr{D}^{\geq 0} = \mathrm{D}^{\geq 0}(\mathscr{A})$. Note that, for every $n \in \mathbb{Z}$, we have $\mathscr{D}^{\leq n} = \mathrm{D}^{\leq n}(\mathscr{A})$ and $\mathscr{D}^{\geq n} = \mathrm{D}^{\geq n}(\mathscr{A})$. We check conditions (0)-(3) in the definition of a t-structure. Condition (0) is clear, condition (1) follows from question (a), condition (2) follows from the description of $\mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 1}$ that we just gave, and condition (3) follows from Proposition V.4.2.7(i).

The fact that the heart of the t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is canonically equivalent to $\mathscr{A}$ is proved in Remark V.4.2.5. Finally, the isomorphisms of functors $\mathrm{H}^0 \simeq \tau^{\leq 0}\tau^{\geq 0}$ is Proposition V.4.2.7(ii).

$\square$

A.9.4 Torsion

Let $\mathscr{D} = \mathscr{D}(\mathbf{Ab})$, and let

$$^*\mathrm{D}^{\leq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i > 1, \text{ and } \mathrm{H}^1(X) \text{ is torsion}\}$$

and

$$^*\mathrm{D}^{\geq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i < 0, \text{ and } \mathrm{H}^0(X) \text{ is torsionfree}\}.$$

Let $\mathscr{C} = {}^*\mathrm{D}^{\leq 0} \cap {}^*\mathrm{D}^{\geq 0}$.

(a). Show that $({}^*\mathrm{D}^{\leq 0}, {}^*\mathrm{D}^{\geq 0})$ is a t-structure on $\mathscr{D}$.

(b). Let $f : A \to B$ be a morphism of torsionfree abelian groups. We can see A and B as objects of $\mathscr{C}$ (concentrated in degree 0), and then f is also a morphism of $\mathscr{C}$.

  (i) Show that f is a monomorphism in $\mathscr{C}$ if and only if f is injective (and $\mathbf{Ab}$) and $B/f(A)$ is torsionfree.

  (ii) Show that f is an epimorphism in $\mathscr{C}$ if and only if $f \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective.

  (iii) Calculate the kernel, the cokernel and the image of f in $\mathscr{C}$.

(c). For every $n \geq 1$, show that $\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.

(d). If A and B are finitely generated abelian groups, show that $\mathrm{Ext}^n_{\mathbf{Ab}}(A, B) = 0$ for every $n \geq 2$. [32]

(e). Let $X \in \mathrm{Ob}(\mathscr{C})$. Suppose that $\mathrm{H}^i(X)$ is a finitely generated abelian group for every $i \in \mathbb{Z}$. If $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, show that $X = 0$.

(f). Give an example of a nonzero $X \in \mathrm{Ob}(\mathscr{C})$ such that $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$.

(g). Let $X \in \mathrm{Ob}(\mathscr{D})$. If $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0} \cap \mathrm{D}^b(\mathbf{Ab}))$ and $\mathrm{H}^i(X)$ is finitely generated for every $i \in \mathbb{Z}$), show that $R\,\mathrm{Hom}_{\mathbf{Ab}}(X, \mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$).

[32] This actually holds for any abelian groups.

(h). Let $X \in \mathrm{Ob}(\mathscr{D})$ be a bounded complex of finitely generated abelian groups. If $R\operatorname{Hom}_{\mathbf{Ab}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$), show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$).

Solution.

(a). Note that ${}^*\mathrm{D}^{\leq 0} \subset \mathrm{D}^{\leq 1}(\mathscr{A})$ and ${}^*\mathrm{D}^{\geq 0} \subset \mathrm{D}^{\geq 0}(\mathscr{A})$.

Condition (0) is obvious. Let $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ and $Y \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 1})$. Then $X \in \mathrm{D}^{\leq 1}(\mathscr{A})$ and $Y \in \mathrm{D}^{\geq 1}(\mathscr{A})$, so we have isomorphisms

$$\operatorname{Hom}_{\mathscr{D}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X,\tau^{\leq 1}Y) = \operatorname{Hom}_{\mathbf{Ab}}(\mathrm{H}^1(X),\mathrm{H}^1(Y)).$$

As $\mathrm{H}^1(X)$ is torsion and $\mathrm{H}^1(Y)$ is torsionfree, this last group is equal to 0. This proves condition (1). Condition (2) is clear.

If $X \in \mathrm{Ob}(\mathcal{C}(\mathbf{Ab}))$, we set

$$B^1(X)' = \{z \in Z^1(X) \mid \exists n \in \mathbb{Z}-\{0\},\ nz \in B^1(X)\},$$

and we define ${}^*\tau^{\leq 0}X$ and ${}^*\tau^{\geq 1}X$ by

$${}^*\tau^{\leq 0}(X) = (\ldots \to X^{-2} \to X^{-1} \to X^0 \to B^1(X)' \to 0 \to \ldots)$$

and

$${}^*\tau^{\geq 1}(X) = (\ldots \to 0 \to 0 \to X^1/B^1(X)' \to X^2 \to X^3 \to \ldots).$$

These constructions are clearly functorial in X, and we have obvious morphisms ${}^*\tau^{\leq 0}X \to X$ and $X \to {}^*\tau^{\geq 1}X$. If we apply the functor H^n to the first morphism, then we get the identity of $\mathrm{H}^n(X)$ if $n \leq 0$, the inclusion $0 \to \mathrm{H}^n(X)$ if $n \geq 2$, and the inclusion $\mathrm{H}^1(X)_{\mathrm{tors}} \to \mathrm{H}^1(X)$ if $n = 1$. If we apply the functor H^n to the second morphism, then we get the identity of $\mathrm{H}^n(X)$ if $n \geq 2$, the unique map $\mathrm{H}^n(X) \to 0$ if $n \leq 0$, and the projection $\mathrm{H}^1(X) \to \mathrm{H}^1(X)/\mathrm{H}^1(X)_{\mathrm{tors}}$ if $n = 1$. In particular, if $X \to Y$ is a quasi-isomorphism, then so are the morphisms ${}^*\tau^{\leq 0}X \to {}^*\tau^{\leq 0}Y$ and ${}^*\tau^{\geq 1}X \to {}^*\tau^{\geq 1}Y$, so the functors ${}^*\tau^{\leq 0}$ and ${}^*\tau^{\geq 1}$ induce endofunctors of $\mathrm{D}(|Ab)$, that we will still write ${}^*\tau^{\leq 0}$ and ${}^*\tau^{\geq 1}$. Finally, for every $X \in \mathrm{Ob}(\mathcal{C}(\mathbf{Ab}))$, the sequence $0 \to {}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \to 0$ is exact in $\mathcal{C}(\mathbf{Ab})$, so it induces a distinguished triangle ${}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \xrightarrow{+1}$ in $\mathrm{D}(\mathbf{Ab})$. As ${}^*\tau^{\leq 0}X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ and ${}^*\tau^{\geq 1}X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 1})$ by construction, this proves condition (3).

(b). Let $f : A \to B$ be a morphism of torsionfree abelian groups. We denote by $\operatorname{Ker}_{\mathscr{C}} f$, $\operatorname{Coker}_{\mathscr{C}} f$ etc the kernel, cokernel etc of f in the category $\mathscr{C}$, and by $\operatorname{Ker} f$, $\operatorname{Coker} f$ etc the kernel, cokernel etc of f in the category $\mathbf{Ab}$.

We solve question (iii) first, by using the formulas for $\operatorname{Ker}_{\mathscr{C}} f$ and $\operatorname{Coker}_{\mathscr{C}} f$ from question (b)(i) of problem A.9.1. First we complete $f : A \to B$ to the distinguished triangle $A \xrightarrow{f} B \to \mathrm{Mc}(f) \xrightarrow{+1}$. By definition of the mapping cone, the complex $\mathrm{Mc}(f)$ is equal to

$$\ldots \to 0 \to A \xrightarrow{f} B \to 0 \to \ldots,$$

with B in degree 0. Then we have $\operatorname{Ker}_{\mathscr{C}} f[1] = {}^*\tau^{\leq -1}\mathrm{Mc}(f)$ and $\operatorname{Coker}_{\mathscr{C}} f = {}^*\tau^{\geq 0}\mathrm{Mc}(f)$. By the formulas that we proved in the solution of question (a), this shows that $\operatorname{Ker}_{\mathscr{C}} f$ is the complex

$$\ldots \to 0 \to A \to I \to 0 \to \ldots$$

with A in degree 0, and $\operatorname{Coker}_{\mathscr{C}} f$ is the complex

$$\ldots \to 0 \to B/I \to 0 \to \ldots$$

with B/I in degree 0, where

$$I = \{x \in B \mid \exists n \in \mathbb{Z} - \{0\},\ nx \in \operatorname{Im} f\}.$$

Note that the abelian group B/I is torsionfree, so we can apply what we just did to calculate the kernel (in $\mathscr{C}$) of the canonical projection $B \to B/I$, which is $\operatorname{Im}_{\mathscr{C}} f$. We get that

$$\operatorname{Im}_{\mathscr{C}} f = (\ldots \to 0 \to B \to B/I \to 0 \to \ldots),$$

where B is in degree 0; this is quasi-isomorphic to the object I of $\mathbf{Ab}$, seen as complex concentrated in degree 0 (the quasi-isomorphism is given by the inclusion $I \subset B$); note that this is an object of $\mathscr{C}$ because I is torsionfree.

Now we can solve (i) and (ii) easily. For example, the morphism f is an epismorphism in $\mathscr{C}$ if and only if $I = \operatorname{Im} f = B$, that is, if and only if $B/\operatorname{Im} f$ is torsion, which is equivalent to the fact that $f \otimes_{\mathbb{Z}} \mathbb{Z}$ is surjective (in the category of $\mathbb{Q}$-vector spaces). On the other hand, the morphism f is a monomorphism in $\mathscr{C}$ if and only if $\operatorname{Ker}_{\mathscr{C}} f = 0$, which means that the complex $\ldots \to 0 \to A \to I \to 0 \to \ldots$ is quasi-isomorphic to 0, i.e. has zero cohomology. This happens if and only if the morphism $A \to I$ is injective (i.e., as $I \subset B$, the morphism f itself is injective in $\mathbf{Ab}$) and $I = \operatorname{Im} f$ (i.e. $B/\operatorname{Im} f$ is torsionfree).

(c). The exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ in $\mathbf{Ab}$, so we can calculate the $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ by applying the functor $\operatorname{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ to the complex $\ldots \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to 0 \to \ldots$ (with the second $\mathbb{Z}$ in degree 0). We get $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \operatorname{Ker}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = 0$, $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \operatorname{Coker}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ if $i \notin \{0, 1\}$.

(d). Using the resolution of $\mathbb{Z}/n\mathbb{Z}$ from the solution of question (c), we get that, if B is any abelian group, then $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = \{x \in B \mid nx = 0\}$, $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = B/nB$, and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = 0$ if $i \notin \{0, 1\}$. Also, as $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, we have $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, B) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) = B$ and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, B) = 0$ for $i \neq 0$.

Let A be a finitely generated abelian group and B be an abelian group. Then $A = A_0 \oplus A_1$ where A_0 is finitely generated free abelian group and A_1 is a finitely generated torsion abelian group, i.e. a direct sum of groups $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$. So, if $i \geq 1$, we have $\operatorname{Ext}^i_{\mathbb{Z}}(A, B) = \operatorname{Ext}^i_{\mathbb{Z}}(A_1, B)$, and $\operatorname{Ext}^i_{\mathbb{Z}}(A_1, B) = 0$ if $i \geq 2$.

(e). As $\mathrm{H}^i(X) = 0$ for $i \notin \{0,1\}$, we have $\tau^{\leq -1}X \simeq 0$ and $\tau^{\geq 2}X \simeq 0$, so, by Proposition V.4.2.7, the canonical morphisms $\tau^{\leq 0}X \to \mathrm{H}^0(X)$ and $\mathrm{H}^1(X)[-1] \to \tau^{\geq 1}X$ are isomorphisms. In particular, using the remark after Definition V.4.5.1, we get, for every $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X[-i], \mathbb{Z}) = \mathrm{Hom}_{\mathscr{D}}(\mathrm{H}^1(X), \mathbb{Z}[1+i]) = \mathrm{Ext}^{1+i}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = 0,$$

which is 0 by question (d) if $i \geq 1$. On the other hand, we have

$$\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X[i], \mathbb{Z}) = \mathrm{Ext}^{-i}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}),$$

which is equal to 0 if $i \geq 1$. So applying $\mathrm{Hom}_{\mathscr{D}}(\cdot, \mathbb{Z})$ to the distinguished triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \overset{+1}{\to}$ gives an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) \to \mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}) \to 0.$$

Hence, of $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, then $\mathrm{Hom}_{\mathbb{Z}}(H^0(X), \mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = 0$.

As X is an object of $\mathscr{C}$, we know that $\mathrm{H}^0(X)$ is torsionfree and $\mathrm{H}^1(X)$ is torsion. Moreover, by assumption, both $\mathrm{H}^0(X)$ and $\mathrm{H}^1(X)$ are finitely generated. So we have $\mathrm{H}^0(X) \simeq \mathbb{Z}^n$ for some $n \in \mathbb{N}$, and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}) \simeq \mathbb{Z}^n \simeq \mathrm{H}^0(X)$ (non canonically). On the other hand, we have $\mathrm{H}^1(X) \simeq \bigoplus_{s=1}^r \mathbb{Z}/n_s\mathbb{Z}$ for some integers $n_1, \ldots, n_r \geq 2$. By question (c), we get that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) \simeq \mathrm{H}^1(X)$ (also non canonically). So, if $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, then $\mathrm{H}^0(X) = 0$ and $\mathrm{H}^1(X) = 0$, which shows that $X \simeq 0$ in $\mathscr{D}$, hence in $\mathscr{C}$.

(f). Let $X = \mathbb{Q}$ (concentrated in degree 0). Then $X \not\simeq 0$, but $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

(g). Suppose that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$. Then $X \in \mathrm{Ob}(\mathrm{D}^{\leq 1}(\mathbf{Ab}))$, so we have an exact triangle

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \simeq \mathrm{H}^1(X)[-1] \overset{+1}{\to}.$$

Applying the triangulated functor $R\,\mathrm{Hom}_{\mathbf{Ab}}(\cdot, \mathbb{Z})$, we get an exact triangle in $\mathrm{D}(\mathbf{Ab})$:

$$R\,\mathrm{Hom}_{\mathscr{D}}(\mathrm{H}^1(X)[-1], \mathbb{Z}) = R\,\mathrm{Hom}_{\mathscr{D}}(\mathrm{H}^1(X), \mathbb{Z})[1] \to R\,\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}) \overset{+1}{\to}.$$

If $i \leq -1$, then $\mathbb{Z}[-i] \in \mathrm{D}^{\geq 1}(\mathbf{Ab})$, so $\mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z})) = \mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}[-i]) = 0$. This shows that $R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}) \in \mathrm{Ob}(\mathrm{D}^{\geq 0}(\mathbf{Ab}))$. For every $i \in \mathbb{Z}$, we have

$$\mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})) = \mathrm{Ext}^{i+1}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}).$$

This is equal to 0 if $i \leq -2$; if $i = -1$, then $\mathrm{Ext}^{i+1}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z})$ is also equal to 0, since $\mathrm{H}^1(X)$ is torsion. So $R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})$ is also in $\mathrm{Ob}(\mathrm{D}^{\geq 0}(\mathbf{Ab}))$. As we have an exact sequence

$$\mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})) \to \mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z})) \to \mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}))$$

for every $i \in \mathbb{Z}$, we conclude that $\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})) = 0$ for $i \leq -1$, i.e. that $R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$.

Suppose that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0} \cap \mathrm{D}^b(\mathbf{Ab}))$ and that the $\mathrm{H}^i(X)$ are finitely generated. We have $\mathrm{H}^i(X) = 0$ if $i \leq -1$ or if i is big enough, and $\mathrm{H}^0(X)$ is torsionfree. In particular, the canonical morphism $X \to \tau^{\geq 0}X$ is an isomorphism and $\tau^{\geq i}X \simeq 0$ for i big enough. So it suffices to prove that, if $i \geq 0$ is an integer such that $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i+1}X,\mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$, then $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i}X,\mathbb{Z})$ is also in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. We have an exact triangle

$$\mathrm{H}^i(X)[-i] \to \tau^{\geq i}X \to \tau^{\geq i+1}X \overset{+1}{\to},$$

so we get an exact triangle

$$R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i+1}X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i}X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i],\mathbb{Z}) \overset{+1}{\to},$$

and it suffices to prove that $R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i],\mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. Let $j \geq 1$. Then

$$\mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i],\mathbb{Z})) = \mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X),\mathbb{Z}[i])) = \operatorname{Ext}^{i+j}_{\mathbb{Z}}(\mathrm{H}^i(X),\mathbb{Z}).$$

If $i \geq 1$, then $i+j \geq 2$, so this group is zero by question (d). If $i = 0$, then $\mathrm{H}^i(X)$ is a free $\mathbb{Z}$-module, so $\operatorname{Ext}^j_{\mathbb{Z}}(\mathrm{H}^i(X),\mathbb{Z}) = 0$ for every $j \geq 1$. In both cases, we get that $\mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i],\mathbb{Z})) = 0$.

(h). If $n \in \mathbb{Z}$, let ${}^*\tau^{\leq n}$ and ${}^*\tau^{\geq n}$ be the truncation functors for the t-structure $({}^*\mathrm{D}^{\leq n},{}^*\mathrm{D}^{\geq n})$, and define ${}^*\mathrm{H}^n : \mathscr{D} \to \mathscr{C}$ by ${}^*\mathrm{H}^n(X) = ({}^*\tau^{\leq n}{}^*\tau^{\geq n}X)[n] = {}^*\mathrm{H}^0(X[n])$.

Let $X \in \mathrm{Ob}(\mathscr{D})$ satisfying the conditions of the question. Then $\mathrm{H}^n(X) = 0$ for all but finitely many $n \in \mathbb{Z}$, so there exists $N \in \mathbb{N}$ such that $\mathrm{H}^n(X) = 0$ for $|n| \geq N$. Then $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq N})$ and $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq -N})$, so ${}^*\tau^{\leq n}X \overset{\sim}{\to} X$ for $n \geq N+1$ and $X \overset{\sim}{\to} {}^*\tau^{\geq n}X$ for $n \geq -N-1$.

First we show the following claim: If $R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}) = 0$, then $X = 0$. Indeed, suppose that $X \neq 0$, and let n be the biggest integer such that ${}^*\tau^{\leq n}X \to X$ is not an isomorphism (such a n exists because ${}^*\tau^{\leq n}X = 0$ for n small enough). We have an exact triangle

$${}^*\tau^{\leq n}X \to X \to {}^*\mathrm{H}^{n+1}(X)[-n-1] \overset{+1}{\to}$$

with ${}^*\mathrm{H}^{n+1}(X) \neq 0$, hence an exact triangle

$$R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1],\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq n}X,\mathbb{Z}) \overset{+1}{\to}.$$

By question (g), we have $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq n}X,\mathbb{Z}) \in \mathrm{Ob}(\mathrm{D}^{\geq -n}(\mathbf{Ab}))$, so the morphism

$$\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1],\mathbb{Z})) \to \mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}))$$

is an isomorphism. As $\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1],\mathbb{Z})) = \operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X),\mathbb{Z}) \neq 0$ by question (e), we conclude that $\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})) \neq 0$, hence $R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}) \neq 0$.

Suppose that $R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$. We want to show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$. We have a distinguished triangle

$$ {}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \xrightarrow{+1}, $$

hence a distinguished triangle

$$ R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq 0}X,\mathbb{Z}) \xrightarrow{+1}. $$

Also, by question (g), we have $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z}) \in \mathrm{D}^{\leq -1}(\mathbf{Ab})$ and $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq 0}X,\mathbb{Z}) \in \mathrm{D}^{\geq 0}(\mathbf{Ab})$. In particular, if $i \leq -1$, then $\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z})) \xrightarrow{\sim} \mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})) = 0$. This implies that $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z}) = 0$, hence that ${}^*\tau^{\geq 1}X = 0$ by the claim we proved in the previous paragraph. So ${}^*\tau^{\leq 0}X \to X$ is an isomorphism.

Now suppose that $R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. We want to show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$. We have a distinguished triangle

$$ {}^*\tau^{\leq -1}X \to X \to {}^*\tau^{\geq 0}X \xrightarrow{+1}, $$

hence a distinguished triangle

$$ R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 0}X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) \xrightarrow{+1}. $$

Also, by question (g), we have $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) \in \mathrm{D}^{\geq 1}(\mathbf{Ab})$ and $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 0}X,\mathbb{Z}) \in \mathrm{D}^{\leq 0}(\mathbf{Ab})$. In particular, if $i \geq 1$, then $0 = \mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(X,\mathbb{Z})) \xrightarrow{\sim} \mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}))$. This implies that $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) = 0$, hence that ${}^*\tau^{\geq 1}X = 0$. So ${}^*\tau^{\leq 0}X \to X$ is an isomorphism.

□

A.9.5 Weights

Let $\mathscr{A}$ be an abelian category. Suppose that we have a family $(\mathscr{A}_n)_{n\in\mathbb{Z}}$ of full abelian subcategories of $\mathscr{A}$ such that:

(1) If $n \neq m$, then $\operatorname{Hom}_{\mathscr{A}}(A,B) = 0$ for any $A \in \mathrm{Ob}(\mathscr{A}_n)$ and $B \in \mathrm{Ob}(\mathscr{A}_m)$.

(2) Any object A of $\mathscr{A}$ has a *weight filtration*, that is, an increasing filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A = 0$ for $n << 0$, $\mathrm{Fil}_n A = A$ for $n >> 0$ and $\mathrm{Fil}_n A/\mathrm{Fil}_{n+1}A \in \mathrm{Ob}(\mathscr{A}_n)$ for every $n \in \mathbb{Z}$.

For every $n \in \mathbb{Z}$, we denote by $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$) the full subcategory of $\mathscr{A}$ whose objects are the $A \in \mathrm{Ob}(\mathscr{A})$ having a weight filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A = A$ (resp. $\mathrm{Fil}_n A = 0$).

(a). If $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$.

(b). Show that the inclusion functor $\mathscr{A}_{\leq n} \subset \mathscr{A}$ has a right adjoint ${}^w\tau^{\leq n}$, and that the inclusion functor $\mathscr{A}_{\geq n} \subset \mathscr{A}$ has a left adjoint ${}^w\tau^{\geq n}$.

(c). If $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Ext}^i_{\mathscr{A}}(A, B) = 0$ for every $i \in \mathbb{Z}$.

(d). Define two full subcategories ${}^w\,\mathrm{D}^{\leq n}$ and ${}^w\,\mathrm{D}^{\geq n}$ of $\mathrm{D}^b(\mathscr{A})$ by:

$$\mathrm{Ob}({}^w\,\mathrm{D}^{\leq n}) = \{X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A})) \mid \forall i \in \mathbb{Z},\ \mathrm{H}^i(X) \in \mathscr{A}_{\leq n}\}$$

and

$$\mathrm{Ob}({}^w\,\mathrm{D}^{\geq n+1}) = \{X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A})) \mid \forall i \in \mathbb{Z},\ \mathrm{H}^i(X) \in \mathscr{A}_{\geq n+1}\}.$$

Show that $({}^w\,\mathrm{D}^{\leq n}, {}^w\,\mathrm{D}^{\geq n+1})$ is a t-structure on $\mathrm{D}^b(\mathscr{A})$, and that the heart of this t-structure is $\{0\}$.

Solution.

(a). If $\mathrm{Fil}_\bullet A$ is a filtration on an object A of $\mathscr{A}$ such that $\mathrm{Fil}_n A = A$ for $n >> 0$ and $\mathrm{Fil}_n A = 0$ for $n << 0$, the length of $\mathrm{Fil}_\bullet$ is by definition the integer $n_1 - n_2$, where n_1 is the smallest integer such that $\mathrm{Fil}_{n_1} A = A$ and n_2 is the biggest integer such that $\mathrm{Fil}_{n_2} A = 0$. For example, if the length of $\mathrm{Fil}_\bullet A$ is 0, then there exists $n \in \mathbb{Z}$ such that $\mathrm{Fil}_n A = A$ and $\mathrm{Fil}_n A = 0$, so $A = 0$.

If A has a weight filtration $\mathrm{Fil}_\bullet A$ of length 1, then there exists $n \in \mathbb{Z}$ such that $\mathrm{Fil}_n A = A$ and $\mathrm{Fil}_{n-1} A = 0$, so $A = \mathrm{Fil}_n A/\mathrm{Fil}_{n-1} A \in \mathrm{Ob}(\mathscr{A}_n)$. Conversely, if $A \in \mathrm{Ob}(\mathscr{A}_n)$ for some $n \in \mathbb{Z}$, then it has a weight filtration $\mathrm{Fil}_\bullet A$ of length 1, given by $\mathrm{Fil}_k A = A$ for $k \geq n$ and $\mathrm{Fil}_k A = 0$ for $k \leq n - 1$.

For every subset I of $\mathbb{N}$, we denote by $\mathscr{A}_I$ the full subcategory of $\mathscr{A}$ whose objects are the $A \in \mathrm{Ob}(\mathscr{A})$ having a weight filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A/\mathrm{Fil}_{n-1} A = 0$ if $n \not\in I$.

We prove a more general statement than that of the question: if I and J are disjoint subsets of $\mathbb{N}$, if $A \in \mathrm{Ob}(\mathscr{A}_I)$ and $B \in \mathrm{Ob}(\mathscr{A}_J)$, then $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$. Choose weight filtrations $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet B$ on A and B such that $\mathrm{Fil}_n A = \mathrm{Fil}_{n-1} A$ if $n \not\in I$ and $\mathrm{Fil}_n B = \mathrm{Fil}_{n-1} B$ if $n \not\in J$. We prove that $\mathrm{Hom}_{\mathscr{A}}(A, b) = 0$ by induction on the sum of the lengths ℓ_A and ℓ_B of $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet B$. If $\ell_A + \ell_B \leq 1$, then one of the filtrations has length 0, so one of A of B is 0, so the result if clear. If $\ell_A + \ell_B \geq 3$, then one of the filtrations has length ≥ 2. If for example $\ell_A \geq 2$, then $\mathrm{Fil}_\bullet A$ induces a weight filtration of length $\ell_A - 1$ on $A' = \mathrm{Fil}_{n-1} A$, and $A'' = \mathrm{Fil}_n A/\mathrm{Fil}_{n-1} A \in \mathrm{Ob}(\mathscr{A}_n)$ has a weight filtration of length 1. As A' and A'' are both in $\mathscr{A}_I$, the induction hypothesis implies that $\mathrm{Hom}_{\mathscr{A}}(A', B) = \mathrm{Hom}_{\mathscr{A}}(A'', B) = 0$. Moreover, the exact sequence

$$0 \to A' \to A \to A'' \to 0$$

induces an exact sequence

$$\mathrm{Hom}_{\mathscr{A}}(A'', B) \to \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{A}}(A', B),$$

so $\mathrm{Hom}_{\mathscr{A}}(A,B)=0$. The case where $\ell_B\geq 2$ is similar. It remains to treat the case where $\ell_A+\ell_B=2$. If $\ell_A=0$ (resp. $\ell_B=0$), then $A=0$ (resp. $B=0$), so the result is obvious. Finally, suppose that $\ell_A=1$ and $\ell_B=1$. Then there exist $n_A\in I$ and $n_B\in J$ such that $A\in\mathrm{Ob}(\mathscr{A}_{n_A})$ and $B\in\mathrm{Ob}(\mathscr{A}_{n_B})$; as $I\cap J=\varnothing$, we have $n_A\neq n_B$, so $\mathrm{Hom}_{\mathscr{A}}(A,B)=0$ by assumption (1).

(b). We show the existence of ${}^w\tau^{\leq n}$. It suffices to show that, for every $B\in\mathrm{Ob}(\mathscr{A})$, the functor $\mathscr{A}_{\leq n}\to\mathbf{Set}$, $A\mapsto\mathrm{Hom}_{\mathscr{A}}(A,B)$ is representable. Fix $B\in\mathrm{Ob}(\mathscr{A})$, and let $\mathrm{Fil}_\bullet B$ be a weight filtration on B. Then $\mathrm{Fil}_\bullet B$ induces a weight filtration on $B/\mathrm{Fil}_n B$, which shows that $B/\mathrm{Fil}_n B\in\mathrm{Ob}(\mathscr{A}_{\geq n+1})$. Let $A\in\mathrm{Ob}(\mathscr{A}_{\leq n})$. Applying $\mathrm{Hom}_{\mathscr{A}}(A,\cdot)$ to the exact sequence

$$0\to\mathrm{Fil}_n B\to B\to B/\mathrm{Fil}_n B\to 0$$

and using question (a), we see that the canonical morphism $\mathrm{Hom}_{\mathscr{A}}(A,\mathrm{Fil}_n B)\to\mathrm{Hom}_{\mathscr{A}}(A,B)$ is an isomorphism. This shows that the couple $(\mathrm{Fil}_n B,\mathrm{Fil}_n B\subset B)$ represents the functor $\mathscr{A}_{\leq n}\to\mathbf{Set}$, $A\mapsto\mathrm{Hom}_{\mathscr{A}}(A,B)$. In particular, we get ${}^w\tau^{\leq n}B=\mathrm{Fil}_n B$. By uniqueness of the right adjoint, this implies that the weight filtration on B is unique.

If $A\in\mathrm{Ob}(\mathscr{A})$ and $\mathrm{Fil}_\bullet A$ is its weight filtration, a similar proof shows that the pair $(A/\mathrm{Fil}_{n-1}A, A\to A/\mathrm{Fil}_{n-1}A)$ represents the functor $\mathscr{A}_{\geq n}\to\mathbf{Set}$, $B\mapsto\mathrm{Hom}_{\mathscr{A}}(A,B)$. This shows the existence of ${}^w\tau^{\geq n}$ and the fact that ${}^w\tau^{\geq n}A=A/\mathrm{Fil}_{n-1}A$.

Note also that the formulas for ${}^w\tau^{\leq n}$ and ${}^w\tau^{\geq n+1}$ imply that, for every $A\in\mathrm{Ob}(\mathscr{A})$, the following sequence is exact:

$$0\to{}^w\tau^{\leq n}A\to A\to{}^w\tau^{\geq n+1}A\to 0.$$

(c). For later use, we prove that the functors ${}^w\tau^{\leq n}$ are exact. Let $f:A\to B$ be a morphism of $\mathscr{A}$; we want to prove that, for every $n\in\mathbb{Z}$, the canonical morphisms ${}^w\tau^{\leq n}(\mathrm{Ker}\,f)\to\mathrm{Ker}({}^w\tau^{\leq n}f)$ and $\mathrm{Coker}({}^w\tau^{\leq n}f)\to{}^w\tau^{\leq n}(\mathrm{Coker}\,f)$ are isomorphisms; this implies in particular that $\mathrm{Ker}({}^w\tau^{\leq n}),\mathrm{Coker}({}^w\tau^{\leq n})\in\mathrm{Ob}(\mathscr{A}_{\leq n})$ and that $\mathrm{Ker}({}^w\tau^{\geq n+1}),\mathrm{Coker}({}^w\tau^{\geq n+1})\in\mathrm{Ob}(\mathscr{A}_{\geq n+1})$, so that $\mathscr{A}_{\leq n}$ and $\mathscr{A}_{\geq n+1}$ are abelian subcategories of $\mathscr{A}$. We prove the claim by induction on $\ell_A+\ell_B$, where ℓ_A (resp. ℓ_B) is the length of the weight filtration of A (resp. B). If $n\in\mathbb{Z}$, applying the snake lemma to the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & {}^w\tau^{\leq n}A \arrow[r]\arrow[d] & A \arrow[r]\arrow[d, "f"] & {}^w\tau^{\geq n+1}A \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & {}^w\tau^{\leq n}B \arrow[r] & B \arrow[r] & {}^w\tau^{\geq n+1}B \arrow[r] & 0
\end{tikzcd}$$

we get an exact sequence

$$0\to\mathrm{Ker}({}^w\tau^{\leq n}f)\to\mathrm{Ker}(f)\to\mathrm{Ker}({}^w\tau^{\geq n+1}f)\xrightarrow{\delta}\mathrm{Coker}({}^w\tau^{\leq n})\to\mathrm{Coker}(f)\to\mathrm{Coker}({}^w\tau^{\geq n+1})\to 0$$

The claim that we want to prove is equivalent to the fact that $\delta = 0$. Indeed, if $\delta = 0$ then we immediately get the result, and if $\mathrm{Coker}({}^w\tau^{\leq n}f) \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $\mathrm{Ker}({}^w\tau^{\geq n+1}f) \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$ then the solution of (a) implies that $\delta = 0$. We first show that the result holds if at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0. If ${}^w\tau^{\geq n+1}A$ or ${}^w\tau^{\leq n}B$ is 0, then $\delta = 0$. Suppose that ${}^w\tau^{\leq n}A = 0$ and ${}^w\tau^{\geq n+1}B = 0$; then $A \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\leq n})$, so $f = 0$ by the solution of (a), and the result is clear. If $\ell_A, \ell_B \leq 1$, then there exist $n_A, n_B \in \mathbb{Z}$ such that $A \in \mathrm{Ob}(\mathscr{A}_{n_A})$ and $B \in \mathrm{Ob}(\mathscr{A}_{n_B})$, and then, for every $n \in \mathbb{Z}$, at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0, so we are done. Suppose that $\ell_A \geq 2$, and let $n \in \mathbb{Z}$. If both ${}^w\tau^{\leq n}A$ and ${}^w\tau^{\geq n+1}A$ are nonzero, then they both have weight filtrations of lengths $< \ell_A$; by the induction hypothesis, applied to ${}^w\tau^{\leq n}f$ and ${}^w\tau^{\geq n+1}f$, we have $\mathrm{Ker}({}^w\tau^{\geq n+1}f) \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$ and $\mathrm{Coker}({}^w\tau^{\leq n}f) \in \mathrm{Ob}(\mathscr{A}_{\leq n})$, so $\delta = 0$ and we are done. If ${}^w\tau^{\geq n+1}A = 0$, then $\delta = 0$, and again we are done. Suppose that ${}^w\tau^{\leq n}A = 0$. If ${}^w\tau^{\leq n}B$ and ${}^w\tau^{\geq n+1}B$ are both nonzero, then again we can use the induction hypothesis to finish the proof; if at least one of them is), then at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0, so we are done. The case where $\ell_B \geq 2$ is similar.

Now fix $n \in \mathbb{Z}$ and let $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$. If $i \leq -1$, then $\mathrm{Ext}^i_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(A, B[i]) = 0$ by Corollary V.4.2.8. If $i = 0$, then $\mathrm{Ext}^i_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(A,B) = 0$ by question (a). Suppose that $i \geq 1$. We use the description of $\mathrm{Ext}^i_{\mathscr{A}}(A,B)$ given by Proposition V.4.5.3. So let $x \in \mathrm{Ext}^i_{\mathscr{A}}(A,B)$, and let $c = (0 \to B \overset{f}{\to} E_{i-1} \overset{f_{i-1}}{\to} \dots \overset{f_1}{\to} E_0 \overset{f_0}{\to} A \to 0)$ be a Yoneda extension of A by B representing x. Applying ${}^w\tau^{\leq n}$ to this exact sequence, we get an exact sequence $0 \to 0 \to F_{i-1} \to \dots \to F_0 \to A \to 0$, where $F_j = {}^w\tau^{\leq n}E_j$ for every $j \in \{0, \dots, i-1\}$. We denote the obvious inclusion $F_j \to E_j$ by u_j. So we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f"] & E_{i-1} \arrow[r] & E_{i-2} \arrow[r] & \dots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r, "\mathrm{id}_B + 0"'] \arrow[u, equal] & B \oplus F_{i-1} \arrow[r] \arrow[u, "f + u_{i-1}"] & F_{i-2} \arrow[r] \arrow[u, "u_{i-2}"] & \dots \arrow[r] & F_0 \arrow[r] \arrow[u, "u_0"] & A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the morphism $B \oplus F_{i-1} \to F_{i-2}$ is equal to 0 on B and to ${}^w\tau^{\leq n}f_{i-1}$ on F_{i-1}. So the second row also represents $x \in \mathrm{Ext}^i_{\mathscr{A}}(A,B)$. To show that $x = 0$, it suffices to show that the morphism

$$g : (\dots \to 0 \to B \to B \oplus F_{i-1} \to F_{i-2} \dots F_0 \to 0 \to \dots) \to B[i]$$

(with F_0 in degree 0 on the left hand side) is equal to 0. But the complex $(\dots \to 0 \to B \to B \oplus F_{i-1} \to F_{i-2} \dots F_0 \to 0 \to \dots)$ is the direct sum of $(\dots \to 0 \to B \to B \to 0 \dots 0 \to 0 \to \dots)$ (with the first B in degree $-i$) and of $(\dots \to 0 \to 0 \to F_{i-1} \to F_{i-2} \dots F_0 \to 0 \to \dots)$ (with F_0 in degree 0), the morphism g is 0 on the second of these summands, and the first of these summands is quasi-isomorphic to 0, so $g = 0$ in $\mathrm{D}(\mathscr{A})$.

(d). Fix $n \in \mathbb{Z}$. Note that we have proved in the solution of (c) that $\mathscr{A}_{\leq n}$ and $\mathscr{A}_{\geq n+1}$ are abelian subcategories of $\mathscr{A}$.

By condition (1), if $m \in \mathbb{Z}$ and $A \in \mathrm{Ob}(\mathscr{A})$ is isomorphic to an object of $\mathscr{A}_m$, then $A \in \mathrm{Ob}(\mathscr{A}_m)$. By the existence of weight filtrations (condition (2)), if $A \in \mathrm{Ob}(\mathscr{A})$ is isomorphic to an object of $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$), then A is in $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$). This implies that $({}^w\mathrm{D}^{\leq n}, {}^w\mathrm{D}^{\geq n+1})$ satisfies condition (0) of the definition of a t-structure.

We clearly have ${}^w\mathrm{D}^{\leq n}[k] = {}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n+1}[k] = {}^w\mathrm{D}^{\geq n+1}[k]$ for every $k \in \mathbb{Z}$, so condition (2) of the definition of a t-structure is clear.

For every $A \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))$, we define the cohomological amplitude of A to be $n_1 - n_2$, where n_1 (resp. n_2) is the biggest (resp. smallest) integer $n \in \mathbb{Z}$ such that $\mathrm{H}^n(A) \neq 0$. If the cohomological amplitude of A is 0 then $A = 0$, and if it is 1, then there exists $n \in \mathbb{Z}$ such that $\mathrm{H}^i(A) = 0$ for $i \neq 0$, so that $A \simeq \mathrm{H}^n(A)[-n]$.

Let $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. We claim that $\mathrm{Ext}^i_{\mathscr{A}}(A, B) = 0$ for every $i \in \mathbb{Z}$. (In particular, we get condition (1) of the definition of a t-structure.) We prove this by induction on $c_A + c_B$, where c_A (resp. c_B) is the cohomological amplitude of A (resp. B). If $c_A, c_B \leq 1$, then the claim follows from question (c). Suppose that $c_A \geq 2$. Then there exists $n \in \mathbb{Z}$ such that $\tau^{\leq n}A \to A$ and $A \to \tau^{\geq n+1}A$ are not isomorphisms, hence $\tau^{\leq n}A$, $\tau^{\geq n+1}A$ have cohomological amplitude $< c_A$. Let $i \in \mathbb{Z}$. Applying the cohomological functor $\mathrm{Ext}^i_{\mathscr{A}}(\cdot, B) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(\cdot, B[i])$ to the exact triangle

$$\tau^{\leq A} \to A \to \tau^{\geq n+1}A \xrightarrow{+1},$$

we get an exact sequence

$$\mathrm{Ext}^i_{\mathscr{A}}(\tau^{\geq n+1}A, B) \to \mathrm{Ext}^i_{\mathscr{A}}(A, B) \to \mathrm{Ext}^i_{\mathscr{A}}(\tau^{\leq n}A, B).$$

As $\mathrm{Ext}^i_{\mathscr{A}}(\tau^{\leq 1}A, B) = \mathrm{Ext}^i_{\mathscr{A}}(\tau^{\geq n+1}A, B) = 0$ by the induction hypothesis, this implies that $\mathrm{Ext}^i_{\mathscr{A}}(A, B) = 0$. The case wgere $c_B \geq 2$ is similar.

We check condition (3) of the definition of a t-structure. Let $X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))$. We start with a remark: Suppose that there exists an exact triangle $(*)\quad A \to X \to B \xrightarrow{+1}$ with $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. Let $i \in \mathbb{Z}$. Then we have an exact sequence

$$\mathrm{H}^{i-1}(B) \to \mathrm{H}^i(A) \to \mathrm{H}^i(X) \to \mathrm{H}^i(B) \to \mathrm{H}^{i+1}(A),$$

in which $\mathrm{H}^i(A)$, $\mathrm{H}^{i+1}(A)$ are in $\mathscr{A}_{\leq n}$ and $\mathrm{H}^{i-1}(B)$, $\mathrm{H}^i(B)$ are in $\mathscr{A}_{\geq n+1}$. By the solution of question (a), the morphisms $\mathrm{H}^{i-1}(B) \to \mathrm{H}^i(A)$ and $\mathrm{H}^i(B) \to \mathrm{H}^{i+1}(A)$ are zero, so we get an exact sequence

$$0 \to \mathrm{H}^i(A) \to \mathrm{H}^i(X) \to \mathrm{H}^i(b) \to 0,$$

which proves that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}(\mathrm{H}^i(X))$ and $\mathrm{H}^i(B) = {}^w\tau^{\geq n+1}(\mathrm{H}^i(X))$.

Now we prove by induction on the cohomological amplitude c of X that there exists an exact triangle $(*)$. If $c = 0$, then $X = 0$ and we can take $A = B = 0$. Suppose that $c = 1$. Then there exists $m \in \mathbb{Z}$ such that $X \simeq \mathrm{H}^m(X)[-m]$. As ${}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n+1}$ are stable by all functor $[k]$, it suffices to prove the existence of the exact triangle $(*)$ for $\mathrm{H}^m(X)$, so we may assume that $X \in \mathrm{Ob}(\mathscr{A})$. Then we can take for $(*)$ the exact triangle associated to the exact sequence $0 \to {}^w\tau^{\leq n}X \to X \to {}^w\tau^{\geq n+1}X \to 0$. Suppose that $c \geq 2$. Then there exists $m \in \mathbb{Z}$ such that $\tau^{\leq m}X \to X$ and $X \to \tau^{\geq m+1}X$ are not isomorphisms, hence $X' = \tau^{\leq m}X$, $X'' = \tau^{\geq m+1}X$ have cohomological amplitude $< c$. By the induction hypothesis, we have exact triangles $A' \to X' \to B' \xrightarrow{+1}$ and $A'' \to X'' \to B'' \xrightarrow{+1}$, with $A', A'' \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B', B'' \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. By question (a) of problem A.9.1, there exists a unique morphism of exact triangles

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & B'' \arrow[r, "+1"] \arrow[d] & {} \\
A'[1] \arrow[r] & X'[1] \arrow[r] & B'[1] \arrow[r, "+1"] & {}
\end{tikzcd}$$

extending the morphism $X'' \to X'[1]$. We complete the morphism $A'' \to A'[1]$ to an exact triangle $A' \to A \to A'' \to A'[1]$. By axiom (**??**) of triangulated categories, we can find a morphism $A \to X$ such that the diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r, "+1"] \arrow[d] & {} \\
X' \arrow[r] & X \arrow[r] & X'' \arrow[r, "+1"] & {}
\end{tikzcd}$$

is a morphism of exact triangles. Finally, we complete the morphism $A \to X$ to an exact triangle $A \to X \to B \xrightarrow{+1}$. We claim that this is the desired exact triangle $(*)$. To prove this claim, it suffices to show that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}(\mathrm{H}^i(X))$ for every $i \in \mathbb{Z}$; indeed, by the long exact sequence of cohomology, this implies that, for every $i \in \mathbb{Z}$, the morphism $\mathrm{H}^i(X) \to \mathrm{H}^i(B)$ is surjective and identifies $\mathrm{H}^i(B)$ to ${}^w\tau^{\geq n+1}(\mathrm{H}^i(X))$, and so we will have $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. To prove the claim, let $i \in \mathbb{Z}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{H}^{i-1}(A'') \arrow[r] \arrow[d] & \mathrm{H}^i(A') \arrow[r] \arrow[d] & \mathrm{H}^i(A) \arrow[r] \arrow[d] & \mathrm{H}^i(A'') \arrow[r] \arrow[d] & \mathrm{H}^{i+1}(A') \arrow[d] \\
\mathrm{H}^{i-1}(X'') \arrow[r] & \mathrm{H}^i(X') \arrow[r] & \mathrm{H}^i(X) \arrow[r] & \mathrm{H}^i(X'') \arrow[r] & \mathrm{H}^{i+1}(X')
\end{tikzcd}$$

If $i \leq m$, then $\mathrm{H}^i(X'') = \mathrm{H}^{i-1}(X'') = 0$, so $\mathrm{H}^j(A'') = {}^w\tau^{\leq n}\mathrm{H}^j(X'') = 0$ for $j \in \{i, i-1\}$, so the diagram becomes a commutative square whose horizontal arrows are isomorphisms:

$$\begin{tikzcd}
{}^w\tau^{\leq n}\mathrm{H}^i(X') \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^i(A) \arrow[d] \\
\mathrm{H}^i(X') \arrow[r, "\sim"] & \mathrm{H}^i(X)
\end{tikzcd}$$

which shows that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}\mathrm{H}^i(X)$. If $i \geq m+1$, then $\mathrm{H}^i(X') = \mathrm{H}^{i+1}(X') = 0$, so $\mathrm{H}^j(A') = {}^w\tau^{\leq n}\mathrm{H}^j(X') = 0$ for $j \in \{i, i+1\}$, so the diagram becomes a commutative square whose horizontal arrows are isomorphisms:

$$\begin{tikzcd} \mathrm{H}^i(A) \arrow[r, "\sim"] \arrow[d] & {}^w\tau^{\leq n}\mathrm{H}^i(X'') \arrow[d] \\ \mathrm{H}^i(X) \arrow[r, "\sim"] & \mathrm{H}^i(X'') \end{tikzcd}$$

which shows again that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}\mathrm{H}^i(X)$.

Finally, we calculate the heart of the t-structure $({}^w\mathrm{D}^{\leq n}, {}^w\mathrm{D}^{\geq n+1})$. Let $X \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n}) \cap \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. For every $i \in \mathbb{Z}$, the object $\mathrm{H}^i(X)$ of $\mathscr{A}$ is in $\mathrm{Ob}(\mathscr{A}_{\leq n}) \cap \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, so $\mathrm{id}_{\mathrm{H}^i(X)} = 0$ by question (a), so $\mathrm{H}^i(X) = 0$. This shows that $X = 0$.

$\square$

A.10 Problem set 10

A.10.1 The Dold-Kan correspondence

You need to look at the results of problems PS3.1 and PS3.2 to do this problem.

Remember the simplicial category Δ and the category of simplicial sets $\mathbf{sSet}$ from problem PS1.9 and problem PS2.2. Let $\mathscr{C} = \mathrm{kar}((\mathbb{Z}[\Delta])^{\oplus})$ (see problems PS3.1 and PS3.2), so that $\mathscr{C}$ is an additive pseudo-abelian category.

The category $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$ is called the category of *simplicial abelian groups* and denoted by $\mathbf{sAb}$; it is an abelian category, where kernel, cokernels and images are calculated in the obvious way (that is, $\mathrm{Ker}(X \to Y) = (\mathrm{Ker}(X_n \to Y_n))_{n\in\mathbb{N}}$ etc).

By the Yoneda lemma, the functor $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ is fully faithful; by problems PS3.1 and PS3.2, we have an equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \simeq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab}) = \mathbf{sAb}$. So we get a fully faithful functor $\mathscr{C} \to \mathbf{sAb}$, and we identify $\mathscr{C}$ with the essential image of this functor.

If X is a simplical set, we denote by $\mathbb{Z}^{(X)}$ the "free simplicial abelian group on X" : it is the simplicial abelian group sending $[n]$ to the free abelian group $\mathbb{Z}^{(X_n)}$ and $\alpha : [n] \to [m]$ to the unique group morphism from $\mathbb{Z}^{(X_m)}$ to $\mathbb{Z}^{(X_n)}$ extending $\alpha^* : X_m \to X_n$. If $u : X \to Y$ is a morphism of simplicial sets, we simply write $u : \mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)}$ for the morphism of simplicial abelian groups induced by u. If $\alpha : [n] \to [m]$, we also use α to denote the morphism $\Delta_n \to \Delta_m$ that is the image of α by the Yoneda embedding $h_\Delta : \Delta \to \mathbf{sSet}$.

(a). For every $n \in \mathbb{N}$, show that the simplicial abelian group $\mathbb{Z}^{(\Delta_n)}$ is in $\mathscr{C}$. (Hint : It's the image of the object $[n]$ of Δ. Follow the identifications !)

Let $n \geq 1$. Remember from problem PS1.9 that we have defined morphisms $\delta_0, \delta_1, \ldots, \delta_n : [n-1] \to [n]$ in Δ by the condition that δ_i is the unique increasing map $[n-1] \to [n]$ such that $i \notin \mathrm{Im}(\delta_i)$. According to our previous conventions, we get morphisms $\delta_i : \Delta_{n-1} \to \Delta_n$ in $\mathbf{sSet}$ and $\delta_i : \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$ in $\mathbf{sAb}$. Remember also that, for $k \in [n]$, the *horn* Λ_k^n is the union of the images of the δ_i, for $i \in [n] - \{k\}$.

(b). Show that $\mathbb{Z}^{(\Lambda_k^n)} = \sum_{i \in [n]-\{k\}} \mathrm{Im}(\delta_i)$, where the sum is by definition the image of the canonical morphism $\bigoplus_{i \in [n]-\{k\}} \mathrm{Im}(\delta_i) \to \mathbb{Z}^{(\Delta_n)}$ and we have identified $\mathbb{Z}^{(\Lambda_k^n)}$ to its image in $\mathbb{Z}^{(\Delta_n)}$.

If $f : [n] \to X$ is a map from $[n]$ to a set X, we also use the notation $(f(0) \to f(1) \to \ldots \to f(n))$ to represent f. Let $n \in \mathbb{N}$, and let S_n be the set of sequences $(a_1, \ldots, a_n) \in [n]$ such that $a_i \in \{i-1, i\}$ for every $i \in \{1, \ldots, n\}$; if $\underline{a} = (a_1, \ldots, a_n)$, we write $f_{\underline{a}} = (0 \to a_1 \to \ldots \to a_n) \in \mathrm{Hom}_{\mathbf{Set}}([n],[n])$ and $\varepsilon(\underline{a}) = (-1)^{\mathrm{card}(\{i | a_i \neq i\})}$.

(c). For every $\underline{a} \in S_n$, show that $f_{\underline{a}} \in \mathrm{Hom}_\Delta([n],[n])$.

(d). Let $p_n = \sum_{\underline{a} \in S_n} \varepsilon(\underline{a}) f_{\underline{a}} \in \mathrm{End}_{\mathscr{C}}(\mathbb{Z}^{(\Delta_n)})$. Show that p_n is a projector.

(e). Show that $\mathbb{Z}^{(\Lambda_0^n)} = \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) = \mathrm{Ker}(p_n)$. In particular, $\mathbb{Z}^{(\Lambda_0^n)}$ is an object of $\mathscr{C}$.

(f). Let $I_n = \mathrm{Im}(p_n)$. This is also an object of $\mathscr{C}$. Show that we have an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \mathbb{Z}^{(\Lambda_0^n)} \oplus I_n$ in $\mathscr{C}$.

(g). If X is an object of $\mathbf{sAb}$ and $f : X \to I_n$ is a surjective morphism (that is, such that f_r is surjective for every $r \geq 0$), show that there exists a morphism $g : I_n \to X$ such that $f \circ g = \mathrm{id}_{I_n}$.

For every $k \in [n]$, define a simplicial subset $\Delta_n^{\leq k}$ of Δ_n by taking $\Delta_n^{\leq k}([m])$ equal to the set of nondecreasing $\alpha : [m] \to [n]$ such that either $\mathrm{card}(\mathrm{Im}(\alpha)) \leq k$, or $\mathrm{card}(\mathrm{Im}(\alpha)) = k+1$ and $0 \in \mathrm{Im}(\alpha)$. In particular, question (h)(i) says that $\Delta_n^{\leq n-1} = \Lambda_0^n$. (On the geometric realizations, $|\Delta_n|$ is a simplex of dimension n with vertices numbered by $0, 1, \ldots, n$, and $|\Delta_n^{\leq k}|$ is the union of its faces of dimension $\leq k$ that contain the vertex 0.)

(h). (i) For every $k \in [n]$ and every $m \in \mathbb{N}$, show that

$$\Lambda_k^n([m]) = \{\alpha : [m] \to [n] \mid \text{either } \mathrm{card}(\mathrm{Im}(\alpha)) \leq n-1, \text{ or } \mathrm{card}(\mathrm{Im}(\alpha)) = n \text{ and } k \in \mathrm{Im}(\alpha)\}.$$

(ii) For every $m \in \mathbb{N}$, show that the set

$$\{\alpha : [m] \to [n] \mid \mathrm{Im}(\alpha) \supset [n] - \{0\}\}$$

is a basis of the $\mathbb{Z}$-module $I_n([m])$.

(iii) For every $k \in \{1, \ldots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} / \mathbb{Z}^{(\Delta_n^{\leq k-1})} \simeq I_k^{\binom{n}{k}}.$$

(iv) For every $k \in \{1, \dots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} \simeq \mathbb{Z}^{(\Delta_n^{\leq k-1})} \oplus I_k^{\binom{n}{k}}.$$

(i). Show that there is an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \bigoplus_{k=0}^n I_k^{\binom{n}{k}}$ in $\mathscr{C}$.

(j). For all $n, m \in \mathbb{N}$, show that $\mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$ is a free $\mathbb{Z}$-module of finite type. We denote its rank by $a_{n,m}$.

(k). Show that $a_{n,n} \geq 1$ and $a_{n,n+1} \geq 1$ for every $n \in \mathbb{N}$. (Hint for the second: $\delta_0 : [n] \to [n+1]$.)

(l). Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} \binom{n}{k} \binom{m}{l}.$$

(m). Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{m} \binom{n+1}{k} \binom{m}{k}.$$

(n). Show that $a_{n,n} = a_{n,n+1} = 1$ for every $n \in \mathbb{N}$ and $a_{n,m} = 0$ if $m \notin \{n, n+1\}$.

(o). Let $\mathscr{I}$ be the full subcategory of $\mathscr{C}$ whose objects are the I_n for $n \in \mathbb{N}$. If $\mathscr{A}$ is an additive category, we consider the category $\mathcal{C}^{\leq 0}(\mathscr{A})$ of complexes of objects of $\mathscr{A}$ that are concentrated in degree ≤ 0 (that is, complexes $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{A}))$ such that $X^n = 0$ for $n \geq 1$).

Give an equivalence of categories from $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ to $\mathcal{C}^{\leq 0}(\mathscr{A})$.

(p). Deduce an equivalence of categories from $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$ to $\mathcal{C}^{\leq 0}(\mathscr{A})$, if $\mathscr{A}$ is a pseudo-abelian additive category. This is called the *Dold-Kan equivalence*.

(q). Suppose that $\mathscr{A}$ is an abelian category, and let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. For $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$, we denote the morphism $X_\bullet(\delta_i^n)$ by $d_i^n : X_n \to X_{n-1}$. The *normalized chain complex* of $X_\bullet$ is the complex $N(X_\bullet)$ in $\mathcal{C}^{\leq 0}(\mathscr{A})$ given by: for every $n \geq 0$,

$$N(X_\bullet)^{-n} = \bigcap_{1 \leq i \leq n} \mathrm{Ker}(d_i^n)$$

and $d_{N(X_\bullet)}^{-n}$ is the restriction of d_0^n. This defines a functor $N : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathcal{C}^{\leq 0}(\mathscr{A})$. Show that this functor is isomorphic to the equivalence of categories of the previous question.

Solution.

(a). We denote the faithful functor $\Delta \to \mathscr{C}$ by ι. Let $n \in \mathbb{N}$. If $m \in \mathbb{N}$, we have

$$h_{\mathscr{C}}(\iota([n]))(\iota([m])) = \mathrm{Hom}_{\mathscr{C}}(\iota([m]), \iota([n])) = \mathbb{Z}^{(\mathrm{Hom}_\Delta([m],[n]))} = \mathbb{Z}^{(\Delta_n([m]))} = \mathbb{Z}^{(\Delta_n)}([m]).$$

So the image of $\iota([n])$ by the fully faithful functor $\mathscr{C} \overset{h_{\mathscr{C}}}{\to} \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \overset{\sim}{\to} \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$ is isomorphic to $\mathbb{Z}^{(\Delta_n)}$.

(b). If $X_\bullet$ is a simplicial set and if $m \in \mathbb{N}$, we denote by $(e_u)_{u \in X_m}$ the canonical basis of $\mathbb{Z}^{(X_\bullet)}([m]) = \mathbb{Z}^{(X_m)}$.

We need to show that, for every $m \in \mathbb{N}$, the subgroup $\mathbb{Z}^{(\Lambda^n_k)}([m])$ of $\mathbb{Z}^{(\Delta_n)}([m])$ is equal to $\sum_{i \in [n]-\{k\}} \mathrm{Im}(\delta_i([m]))$. Let $m \in \mathbb{N}$. For every $i \in [n]$, the morphism $\delta_i([m]) : \mathbb{Z}^{(\Delta_{n-1})}([m]) \to \mathbb{Z}^{(\Delta_n)}([m])$ is given on the canonical basis $(e_u)_{u \in \mathrm{Hom}_\Delta([m],[n])}$ of $\mathbb{Z}^{(\Delta_{n-1})}([m])$ by $\delta_i([m])(e_u) = e_{\delta_i \circ u}$. So $\sum_{i \in [n]-\{k\}} \mathrm{Im}(\delta_i([m]))$ is the $\mathbb{Z}$-submodule of $\mathbb{Z}^{(\Delta_n)}([m])$ generated by all the e_u for $u \in \mathrm{Hom}_\Delta([m],[n])$ factoring through some δ_i, $i \neq k$. This is the same as $\mathbb{Z}^{(\Lambda^n_k)}([m])$ by definition of the horn.

(c). We have to show that $f_{\underline{a}}$ is nondecreasing. Let $i \in \{0, \ldots, n-1\}$. If $i = 0$, then $f_{\underline{a}}(i) = 0 \leq i$. Then $f_{\underline{a}}(i) = a_i \in \{i-1, i\}$, so $f_{\underline{a}}(i) \leq i$. On the other hand, we have $f_{\underline{a}}(i+1) = a_{i+1} \in \{i, i+1\}$, so $f_{\underline{a}}(i+1) \geq i \geq f_{\underline{a}}(i)$.

(d). Let $\underline{a} = (a_1, \ldots, a_n) \in S_n$, and suppose that $\underline{a} \neq (1, \ldots, n)$. Then $\mathrm{Im}(f_{\underline{a}})$ is strictly contained in $[n]$, and $0 \in \mathrm{Im}(f_{\underline{a}})$. This means that there exists $i_0 \in \{1, \ldots, n\}$ such that $i_0 \notin \mathrm{Im}(f_{\underline{a}})$. Let $S'_n = \{(b_1, \ldots, b_n) \in S_n \mid b_{i_0} = i_0\}$ and $S''_n = \{(b_1, \ldots, b_n) \in S_n \mid b_{i_0} = i_0 - 1\}$. We define a map $\iota : S'_n \to S''_n$ by sending $(b_1, \ldots, b_n)$ to $(b_1, \ldots, b_{i_0-1}, i_0 - 1, b_{i_0+1}, \ldots, b_n)$. It is easy to see that ι is a bijection and that $\varepsilon(\iota(\underline{b})) = -\varepsilon(\underline{b})$ and that $f_{\underline{b}} \circ f_{\underline{a}} = f_{\varepsilon(\underline{b})} \circ f_{\underline{a}}$ for every $\underline{b} \in S'_n$. So

$$\begin{aligned} p_n \circ f_{\underline{a}} &= \sum_{\underline{b} \in S'_n} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} + \sum_{\underline{b} \in S''_n} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} \\ &= \sum_{\underline{b} \in S'_n} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} - \sum_{\underline{b} \in S'_n} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} \\ &= 0. \end{aligned}$$

On the other hand, if $\underline{a} = (1, \ldots, n)$, then $\varepsilon(\underline{a}) = 1$ and $f_{\underline{a}} = \mathrm{id}_{[n]}$, so $p_n \circ f_{\underline{a}} = p_n$.

This shows that $p_n \circ p_n = p_n$.

(e). As p_n is a projector, we know that $\mathrm{Ker}(p_n)$ exists in $\mathscr{C}$ and that $\mathrm{Ker}(p_n) = \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n)$ by problem A.3.2.

For every $\underline{a} \in S_n$ such that $\underline{a} \neq (1, \ldots, n)$, we have seen that there exists $i \in \{1, \ldots, n\} - \mathrm{Im}(f_{\underline{a}})$, and then $f_{\underline{a}}$ factors through δ_i, so the image of $f_{\underline{a}}$ in the abelian category $\mathbf{sAb}$ is contained in $\mathbb{Z}^{(\Lambda^n_0)}$. As $\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n = -\sum_{\underline{a} \in S_n - \{(1,\ldots,n)\}} \varepsilon(\underline{a}) f_{\underline{a}}$, this shows that $\mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) \subset \mathbb{Z}^{(\Lambda^n_0)}$.

If $i \in \{1, \ldots, n\}$, then the same proof as in the solution of question (d) shows that $p_n \circ \delta_i = 0$, hence $(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) \circ \delta_i = \delta_i$. As $\mathbb{Z}^{(\Lambda_0^n)} = \sum_{i=1}^n \mathrm{Im}(\delta_i)$ by question (b), this implies that $\mathbb{Z}^{(\Lambda_0^n)} \subset \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n)$.

(f). This is question (b) of problem A.3.2.

(g). Let $i = \mathrm{id}_{[n]} \in \Delta_n([n])$, let e_i be the corresponding element of $\mathbb{Z}^{(\Delta_n)}([n])$, and let $\overline{e}_i$ be its image in $I_n([n])$. As f is surjective, we can find $x \in X_n$ such that $f_n(x) = \overline{e}_i$. Let $g' : \mathbb{Z}^{(\Delta_n)} \to X$ be the morphism corresponding to x by the bijection

$$\mathrm{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, X) \simeq \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \simeq X_n,$$

and let $g = q \circ g'$, where $q : \mathbb{Z}^{(\Delta_n)} \to I_n$ is the canonical projection. We want to show that $g \circ f = \mathrm{id}_{I_n}$. By the construction of g, we have $g \circ f(\overline{e}_i) = \overline{e}_i$. Let $m \in \mathbb{N}$. Remember that we denote by $(e_u)_{u \in \mathrm{Hom}_\Delta([m],[n])}$ the canonical basis of $\mathbb{Z}^{(\Delta_n)}([m])$. The family $(q(e_u))_{u \in \mathrm{Hom}_\Delta([m],[n])}$ spans $I_n([m])$, so it suffices to show that $g_m \circ f_m(q_m(e_u)) = q(e_u)$ for every u. Let $u \in \mathrm{Hom}_\Delta([m],[n])$. Then $i \circ u = u$, so $e_u = u^*(e_i)$, and

$$f_m \circ g_m(q_m(e_u)) = u^*(f_n \circ g_n(q_n(e_i))) = u^*(q(e_i)) = q(e_u).$$

(h). (i) The set $\Lambda_k^n([m])$ is the set of nondecreasing maps $\alpha : [m] \to [n]$ that factor through some δ_i, for $i \in [n] - \{k\}$. If $\alpha : [m] \to [n]$ is a nondecreasing map, then, by definition of δ_i, the map α factors through δ_i if and only if $i \notin \mathrm{Im}(\alpha)$. This shows that $\Lambda_k^n([m])$ does not contain any surjective α, contains all the α such that $|\mathrm{Im}(\alpha)| \leq n - 1$, and contains an α such that $|\mathrm{Im}(\alpha)| = n$ if and only if $[n] - \mathrm{Im}(\alpha) \neq \{k\}$, i.e. $k \in \mathrm{Im}(\alpha)$. This is what we wanted to prove.

(ii) By (i), we have

$$\{\alpha \in \Delta_n([m]) \mid \mathrm{Im}(\alpha) \supset [n] - \{0\}\} = \Delta_n([m]) - \Lambda_0^n([m]),$$

so the family $(q_m(e_\alpha))_{\alpha \in \Delta_n([m]),\ \mathrm{Im}(\alpha) \supset [n] - \{0\}}$ is a basis of $I_n([m])$ (where q is as before the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$).

(iii) We fix $k \in \{1, \ldots, n\}$. Let Ω be the set of $A \subset [n]$ such that $0 \in A$ and $|A| = k + 1$. For every $A \in \Omega$, let $\beta_A : [k] \to [n]$ be the composition of the unique order-preserving bijection $[k] \xrightarrow{\sim} A$ and of the inclusion $A \subset [n]$; note that $\beta_A(0) = 0$. Consider the morphism $f_A : \mathbb{Z}^{(\Delta_k)} \to \mathbb{Z}^{(\Delta_n)}$ such that, for every $m \in \mathbb{N}$ and every $\alpha \in \mathrm{Hom}_\Delta([m],[k])$, we have $f_A(e_\alpha) = e_{\beta_A \circ \alpha}$. Note that $\mathrm{Im}(f_A) \subset \mathbb{Z}^{(\Delta_n^{\leq k})}$, so we can see f_A as a morphism from $\mathbb{Z}^{(\Delta_k)}$ to $\mathbb{Z}^{(\Delta_n^{\leq k})}$. Let $m \in \mathbb{N}$ and $\alpha \in \Lambda_0^k([m])$; if $|\mathrm{Im}(\alpha)| \leq k - 1$, then $|\mathrm{Im}(\beta_A \circ \alpha)| \leq k - 1$ and so $\beta_A \circ \alpha \in \Delta_n^{\leq k-1}([m])$; if $|\mathrm{Im}(\alpha)| = k$ and $0 \in \mathrm{Im}(\alpha)$, then $|\mathrm{Im}(\beta_A \circ \alpha)| = k$ and $0 \in \mathrm{Im}(\beta_A \circ \alpha)$, and so $\beta_A \circ \alpha \in \Delta_n^{\leq k-1}([m])$. This shows that $f_A(\mathbb{Z}^{(\Lambda_0^n)}) \subset \mathbb{Z}^{(\Delta_n^{\leq k-1})}$, hence that f_A induces a morphism $g_A : I_k \to \mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})}$.

Let $g = \sum_{A\in\Omega} g_A : I_k^\Omega \to \mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})}$. We claim that g is an isomorphism; this will finish the proof, because $|\Omega| = \binom{n}{k}$. Let $m \in \mathbb{N}$. For every $A \in \Omega$ and every $\alpha \in \mathrm{Hom}_\Delta([m],[k])$ such that either $|\operatorname{Im}(\alpha)| = k+1$, or $|\operatorname{Im}(\alpha)| = k$ and $0 \notin \operatorname{Im}(\alpha)$, we denote by $e_{A,\alpha} \in I_k^\Omega$ the basis element e_α of the copy of I_k corresponding to $A \in \Omega$. By (ii), this gives a basis of $(I_k^\Omega)([m])$. On the other, a basis of $(\mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})})([m])$ is given by the images of the basis elements $e_\beta \in \mathbb{Z}^{(\Delta_n^{\leq k})}([m])$ for $\beta \in \mathrm{Hom}_\Delta([m],[n])$ such that either $|\operatorname{Im}(\beta)| = k+1$ and $0 \in \operatorname{Im}(\beta)$, or $|\operatorname{Im}(\beta)| = k$ and $0 \notin \operatorname{Im}(\beta)$. To show that $g_m : (I_k^\Omega)([m]) \to (\mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})})([m])$ is an isomorphism, it suffices to notice that each $\beta \in \mathrm{Hom}_\Delta([m],[n])$ as in the previous sentence is equal to $\beta_A \circ \alpha$ for a unique $A \in \Omega$ and a unique $\alpha \in \mathrm{Hom}_\Delta([m],[k])$ (indeed, we must have $A = \operatorname{Im}(\beta)$ if $|\operatorname{Im}(\beta)| = k+1$ and $0 \in \operatorname{Im}(\beta)$, and $A = \{0\} \cup \operatorname{Im}(\beta)$ if $|\operatorname{Im}(\beta)| = k$ and $0 \notin \operatorname{Im}(\beta)$, and then A determines α because β_A is injective), and that we then have either $|\operatorname{Im}(\alpha)| = k+1$, or $|\operatorname{Im}(\alpha)| = k$ and $0 \notin \operatorname{Im}(\alpha)$.

(iv) This follows easily from (iii) and from question (g).

(i). By question (h)(iv) (and an easy induction), we have an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \bigoplus_{k=0}^n I_k^{\binom{n}{k}}$ in $\mathbf{sAb}$. As both sides are objects of $\mathscr{C}$ by question (f), and as $\mathscr{C}$ is a full subcategory of $\mathbf{sAb}$, this isomorphism is an isomorphism in $\mathscr{C}$.

(j). As I_n (resp. I_m) is a direct factor of $\mathbb{Z}^{(\Delta_n)}$ (resp. $\mathbb{Z}^{(\Delta_m)}$) by question (f), the abelian group $\mathrm{Hom}_{\mathscr{C}}(I_n, I_m) = \mathrm{Hom}_{\mathbf{sAb}}(I_n, I_m)$ admits an injective morphism into

$$\mathrm{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)}) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, \mathbb{Z}^{(\Delta_m)}) = \mathbb{Z}^{(\Delta_m)}([n]) = \mathbb{Z}^{(\mathrm{Hom}_\Delta([n],[m]))}.$$

As the latter group is free and finitely generated, so is $\mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$.

(k). We have $I_n \neq 0$ because $\Lambda_0^n \subsetneq \Delta_n$, so $0 \neq \mathrm{id}_{I_n} \in \mathrm{Hom}_{\mathscr{C}}(I_n, I_n)$, so $a_{n,n} \geq 1$.

Consider the unique nondecreasing injective map $\delta_0 : [n] \to [n+1]$ such that $0 \notin \operatorname{Im}(\delta_0)$. (In other words, we have $\delta_0(i) = i+1$ for every $i \in [n]$.) This induces a morphism $f : \mathbb{Z}^{(\Delta_n)} \to \mathbb{Z}^{(\Delta_{n+1})}$. If $m \in \mathbb{N}$ and $\alpha \in \Lambda_0^n([m])$, then $|\operatorname{Im}(\alpha)| \leq n$, so $|\operatorname{Im}(\delta_0 \circ \alpha)| \leq n$ and $\delta_0 \circ \alpha \in \Lambda_0^{n+1}([m])$. This shows that $f(\mathbb{Z}^{(\Lambda_0^n)}) \subset \mathbb{Z}^{(\Lambda_0^{n+1})}$, hence that f induces a morphism $g : I_n \to I_{n+1}$. Also, if $\alpha = \mathrm{id}_{[n]} \in \mathrm{Hom}_\Delta([n],[n])$, then $\delta_0 \circ \alpha \notin \Lambda_0^{n+1}([n])$, so the image by g of the class of e_α in $I_n([n])$ is not 0. This shows that $g \neq 0$, hence that $\mathrm{Hom}_{\mathscr{C}}(I_n, I_{n+1}) \neq 0$ and so $a_{n,n+1} \geq 1$.

(l). Let $n, m \in \mathbb{N}$. We have seen in the solution of question (j) that $\mathrm{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)})$ is a free $\mathbb{Z}$-module of rank $|\mathrm{Hom}_\Delta([n],[m])| = \binom{n+m+1}{m} = \binom{n+m+1}{n+1}$. On the other hand, by question (i), we have

$$\mathrm{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)}) \simeq \bigoplus_{k=0}^{n} \bigoplus_{l=0}^{m} (\mathrm{Hom}_{\mathbf{sAb}}(I_k, I_l))^{\binom{n}{k}\binom{m}{l}},$$

and the right hand side is a free $\mathbb{Z}$-module of rank $\sum_{k=0}^n \sum_{l=0}^m a_{k,l} \binom{n}{k}\binom{m}{l}$.

(m). Remember that Vandermonde's identity says that, for all $a, b, c \in \mathbb{N}$, we have

$$\binom{a+b}{c} = \sum_{j=0}^{c} \binom{b}{j}\binom{a}{c-j}.$$

Applying this to $a = n+1$ and $b = c = m$ and using the fact that $\binom{m}{k} = \binom{m}{m-k}$ for $0 \le k \le m$, we get

$$\binom{n+m+1}{m} = \sum_{k=0}^{m} \binom{m}{k}\binom{n+1}{k}.$$

To prove Vandermonde's identity, we consider an indeterminate t. By the binomial theorem, we have

$$(1+t)^a = \sum_{i=0}^{a} \binom{a}{i} t^i,$$

$$(1+t)^b = \sum_{j=0}^{b} \binom{b}{j} t^j$$

and

$$(1+t)^{a+b} = \sum_{c=0}^{a+b} \binom{a+b}{c} t^c.$$

As $(1+t)^{a+b} = (1+t)^a(1+t)^b$, if $c \in \mathbb{N}$, we get two formulas for the coefficient of t^c in this polynomial. The first formula is $\binom{a+b}{c}$, and the second formula is

$$\sum_{i,j\ge 0,\ i+j=c} \binom{a}{i}\binom{b}{j} = \sum_{j=0}^{c} \binom{a}{c-j}\binom{b}{j}.$$

(n). By questions (l) and (m), we have

$$\sum_{k=0}^{n}\sum_{l=0}^{m} a_{k,l}\binom{n}{k}\binom{m}{l} = \sum_{k=0}^{m}\binom{m}{k}\binom{n+1}{k} = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k} + \sum_{k=1}^{m}\binom{m}{k}\binom{n}{k-1},$$

where the second equality comes from Pascal's rule $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. By question (k) (and the obvious that all the $a_{k,l}$ are nonnegative), we have

$$\sum_{k=0}^{n}\sum_{l=0}^{m} a_{k,l}\binom{n}{k}\binom{m}{l} \ge \sum_{k=0}^{m}\binom{n}{k}\binom{m}{k} + \sum_{k=1}^{m}\binom{n}{k-1}\binom{m}{k}.$$

This implies that, for $k \in [n]$ and $l \in [m]$, we have $a_{k,l} = 0$ if $l \notin \{k, k+1\}$ and $a_{k,l} = 1$ if $l \in \{k, k+1\}$. As n and m were arbitrary, we get the conclusion.

(o). Let $F \in \mathrm{Func}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$. We define a complex $X \in \mathcal{C}^{\leq 0}(\mathscr{A})$ in the following way: For every $n \in \mathbb{N}$, we take $X^{-n} = F(I_n)$ and $d_X^{-n-1} : X^{-n-1} = F(I_{n+1}) \to X^{-n} = F(I_n)$ to the image by F of the element g_n of $\mathrm{Hom}_{\mathscr{C}}(I_n, I_{n+1})$ constructed in the solution of question (k). This defines a functor $\Phi : \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A}) \to \mathcal{C}^{\leq 0}(\mathscr{A})$.

Conversely, let X be an object of $\mathcal{C}^{\leq 0}(\mathscr{A})$. We define a functor $F : \mathscr{I}^{\mathrm{op}} \to \mathscr{A}$ in the following way: For every $n \in \mathbb{N}$, we take $F(I_n) = X^{-n}$. Let $n, m \in \mathbb{N}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$. If $m \notin \{n, n+1\}$, then $f = 0$, so we must $F(f) = 0$. If $m = n$, then, by question (n), the morphism is of the form $a \cdot \mathrm{id}_{I_n}$, where $a \in \mathbb{Z}$, and we must set $F(f) = a\mathrm{id}_{X^{-n}}$. If $m = n+1$, then, by question (n), the morphism f is of the $a \cdot g_n$ with $a \in \mathbb{Z}$, and we set $F(f) = a \cdot d_X^{-n-1} : X^{-n-1} = F(I_{n+1}) \to X^{-n} = F(I_n)$. This defines a functor $\Psi : \mathcal{C}^{\leq 0}(\mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$.

The fact that $\Phi \circ \Psi = \mathrm{id}_{\mathcal{C}^{\leq 0}(\mathscr{A})}$ follows immediately from the definitions of the functors Φ and Ψ, and the fact that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})}$ follows easily from the definition of these functors and from question (n).

(p). By problems PS3.1 and PS3.2, we have an equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \simeq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$, so we can define a functor $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathcal{C}^{\leq 0}(\mathscr{A})$ by composing a quasi-inverse of this equivalence, the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ and the equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A}) \xrightarrow{\sim} \mathcal{C}^{\leq 0}(\mathscr{A})$. Showing that this is an equivalence of categories amounts to showing that the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ is an equivalence of categories.

By the construction of the pseudo-abelian completion in problem A.3.2, every object of $\mathscr{C}$ is a direct summand of an object of $\mathbb{Z}[\Delta]^{\oplus}$, hence, by construction of the universal additive category in problem A.3.1, a direct summand of an object of the form $\bigoplus_{i \in I} \mathbb{Z}^{(\Delta_{n_i})}$, for $(n_i)_{i \in I}$ a finite family of nonnegative integers. By question (i), this implies that every object of $\mathscr{C}$ is a direct summand of an object of the form $\bigoplus_{i \in I} I_{n_i}$, for $(n_i)_{i \in I}$ a finite family of nonnegative integers.

Let $\mathscr{I}'$ be the full subcategory of $\mathscr{C}$ whose objects are finite direct sums of objects of $\mathscr{I}$; in other words, the category $\mathscr{I}'$ is the category $\mathscr{I}^{\oplus}$ defined in problem A.3.2. Then $\mathscr{I}'$ is an additive category and the preceding paragraph says that $\mathscr{C}$ is the pseudo-abelian completion of $\mathscr{I}'$. By problem A.3.2 (applied to the opposite categories), the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}'^{\mathrm{op}}, \mathscr{A})$ is an an equivalence of categories. So it remains to show that the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}'^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ is an equivalence of categories. But this is proved in problem A.3.1.

(q). Let $DK : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathcal{C}^{\leq 0}(\mathscr{A})$ be the equivalence of categories of question (p).

Let $X_\bullet \in \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. We still denote by $X_\bullet$ the corresponding functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{A}$. Let $n \in \mathbb{N}$, and let $\delta = \sum_{i=1}^n \delta_i : \bigoplus_{i=1}^n \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$, where we use the notation of question (b); by that question, we have $\mathbb{Z}^{(\Lambda_0^n)} = \mathrm{Im}(\delta)$, and by question (f), the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$ identifies I_n to $\mathrm{Coker}\,\delta$ and both $\mathrm{Im}\,\delta$ and $\mathrm{Coker}\,\delta$ are direct summands of $\mathbb{Z}^{(\Delta_n)}$. It is easy to deduce from this that, if $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$ is any additive functor, then the morphism $F(I_n) \to F(\mathbb{Z}^{(\Delta_n)})$ is a kernel of $F(\delta)$. Applying this to $F = X_\bullet : \mathscr{C}^{\mathrm{op}} \to \mathscr{A}$, we see that $F(I_n) = DK(X_\bullet)^{-n}$ is canonically isomorphic to $\mathrm{Ker}(\bigoplus_{i=1}^n : d_i^n : X_n \to X_{n-1}^n) = \bigcap_{i=1}^n \mathrm{Ker}(d_i^n) = N(X^\bullet)^{-n}$. Also, as the nonzero morphism from I_{n-1} to I_n constructed in the solution of question (k) is the restriction of $\delta_0 : \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$ (followed by the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$), its image by $X_\bullet$ is the restriction of d_0^n. So we get an isomorphism of complexes $DK(X_\bullet) \simeq N(X_\bullet)$, and this isomorphism is clearly functorial in $X_\bullet$.

□

A.10.2 The model structure on complexes

Let R be a ring, and let $\mathscr{A} = {}_R\mathbf{Mod}$. [33]

We denote by W the set of quasi-isomorphisms of $\mathcal{C}(\mathscr{A})$, by Fib the set of morphisms $f : X \to Y$ in $\mathcal{C}(\mathscr{A})$ such that $f^n : X^n \to Y^n$ is surjective for every $n \in \mathbb{Z}$ and by Cof the set of morphisms of $\mathcal{C}(\mathscr{A})$ that have the left lifting property relatively to every morphism of $W \cap \mathrm{Fib}$. We say that $X \in \mathrm{Ob}(\mathcal{C}(\mathscr{A}))$ is fibrant (resp. cofibrant) if the unique morphism $X \to 0$ (resp. $0 \to X$) is in Fib (resp. in Cof). The goal of this problem is to show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathcal{C}(\mathscr{A})$.

For every $M \in \mathrm{Ob}(\mathscr{A})$, let $K(M, n) = M[-n] \in \mathrm{Ob}(\mathcal{C}(\mathscr{A}))$, and let $D^n(M)$ be the complex X such that $X^n = X^{n+1} = M$, $d_X^n = \mathrm{id}_M$ and $X^i = 0$ if $i \notin \{n, n+1\}$. We also write $S^n = K(R, n)$ and $D^n = D^n(R)$. For every $M \in \mathrm{Ob}(\mathscr{A})$, the identity of M induces a morphism of complexes $K(M, n) \to D^{n-1}(M)$ (which is clearly functorial in M).

(a). Show that the functor $D^n : {}_R\mathbf{Mod} \to \mathcal{C}(\mathscr{A})$ is left adjoint to the functor $\mathcal{C}(\mathscr{A}) \to \mathscr{A}$, $X \longmapsto X^n$, and that the functor $K(\cdot, n) : \mathscr{A} \to \mathcal{C}(\mathscr{A})$ is left adjoint to the functor Z^n.

(b). Show that a morphism of $\mathcal{C}(\mathscr{A})$ is in Fib is and only if it has the right lifting property relatively to $0 \to D^n$ for every $n \in \mathbb{Z}$.

(c). Show that D^n is cofibrant for every $n \in \mathbb{Z}$.

(d). Show that S^n is cofibrant for every $n \in \mathbb{Z}$.

(e). Let $p : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{A})$.

  (i) If p is in $W \cap \mathrm{Fib}$, show that it has the right lifting property relatively to the canonical morphism $S^n = K(R, n) \to D^{n-1}$ for every $n \in \mathbb{Z}$.

  (ii) If p has the right lifting property relatively to the canonical morphism $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$, show that it is in $W \cap \mathrm{Fib}$.

[33] We only need $\mathscr{A}$ to have all small limits and colimits and a nice enough projective generator, but we take $\mathscr{A} = {}_R\mathbf{Mod}$ to simplify the notation.

(f). Show that the canonical morphism $S^n \to D^{n-1}$ is in Cof.

(g). Let $f : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{A})$. Let $E = X \oplus \bigoplus_{n\in\mathbb{Z},\ y\in Y^n} D^n$, let $i : X \to E$ be the obvious inclusion and let $p : E \to Y$ be the morphism that is equal to f on the summand X and that, for every $n \in \mathbb{Z}$ and $y \in Y^n$, is equal on the corresponding summand D^n to the morphism $D^n \to Y$ corresponding to $y \in Y^n = \mathrm{Hom}_R(R, Y^n)$ by the adjunction of question (a). We clearly have $p \circ i = f$.

(i) Show that i is in W.

(ii) Show that i has the left lifting property relatively to any morphism of Fib.

(iii) Show that p is in Fib.

(h). Let $f : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{A})$. Let $X_0 = X$ and $f_0 = f$. For every $i \in \mathbb{N}$, we construct morphisms of complexes $j_i : X_i \to X_{i+1}$ and $f_{i+1} : X_{i+1} \to Y$ such that j_i is a monomorphism and in Cof and $f_{i+1} \circ j_i = f_i$ in the following way: Suppose that we already have $f_i : X_i \to Y$. Consider the set $\mathscr{D}_i$ of commutative squares

$$(D) \qquad \begin{tikzcd} S^{n_D} \arrow[r, "f_D"] \arrow[d] & X_i \arrow[d, "f_i"] \\ D^{n_D-1} \arrow[r, "g_D"'] & Y \end{tikzcd}$$

(for some $n_D \in \mathbb{Z}$). Let $j_i : X_i \to X_{i+1}$ be defined by the cocartesian square

$$\begin{tikzcd} \bigoplus_{D\in\mathscr{D}_i} S^{n_D} \arrow[r, "\sum f_D"] \arrow[d] & X_i \arrow[d, "j_i"] \\ \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} \arrow[r] & X_{i+1} \end{tikzcd}$$

The morphisms $f_i : X_i \to Y$ and $\sum g_D : \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} \to Y$ induce a morphism $f_{i+1} : X_{i+1} \to Y$, and we clearly have $f_{i+1} \circ j_i = f_i$.

Finally, let $F = \varinjlim_{i\in\mathbb{N}} X_i$ (where the transition morphisms are the j_i), let $j : X \to F$ be the morphism induced by j_0 and let $q : F \to Y$ be the morphism induced by the f_i.

(i) Show that $q \circ j = f$.

(ii) Show that j is a monomorphism.

(iii) Show that j is in Cof.

(iv) Show that q is in $W \cap \mathrm{Fib}$.

(i). Show that every element of Cof is a monomorphism.

(j). Show that every element of $W \cap \mathrm{Cof}$ has the left lifting property relatively to elements of Fib. (Hint: Use question (g).)

(k). Show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathcal{C}(\mathscr{A})$.

(l). Show that the intersections of $(W, \mathrm{Fib}, \mathrm{Cof})$ with $\mathcal{C}^-(\mathscr{A})$ also give a model structure on this category.

(m). Let $f : A \to B$ be a morphism of $\mathscr{A}$. Show that f has the left lifting property relatively to epimorphisms of $\mathscr{A}$ if and only if it is injective with projective cokernel.

(n). Let $i : X \to Y$ be a morphism of $\mathcal{C}^-(\mathscr{A})$. Show that i is in Cof if and only if, for every $n \in \mathbb{Z}$, the morphism i^n is injective with projective cokernel.

Solution.

(a). Let X be an object of $\mathcal{C}(\mathscr{A})$ and M be a left R-module. Giving a morphism of complexes from $D^n(M)$ to X amounts to giving R-linear maps $f : M \to X^n$ and $g : M \to X^{n+1}$ such that $g = d_X^n \circ f$; so there is no extra condition on f, and g is determined by f. In other words, we have constructed a bijection

$$\mathrm{Hom}_{\mathcal{C}(\mathscr{A})}(D^n(M), X) \xrightarrow{\sim} \mathrm{Hom}_R(M, X^n),$$

which is clearly functorial in M and X.

On the other hand, giving a morphism of complexes from $K(M, n)$ to X amounts to giving a R-linear map $f : M \to X^n$ such that $d_X^n \circ f = 0$; this is the same as giving a R-linear map $M \to \mathrm{Ker}(d_X^n) = Z^n(X)$. In other words, we have constructed a bijection

$$\mathrm{Hom}_{\mathcal{C}(\mathscr{A})}(K(M, n), X) \xrightarrow{\sim} \mathrm{Hom}_R(M, Z^n(X)),$$

which is clearly functorial in M and X.

Moreover, these adjunctions have the following property (which is clear on their construction): Let $u : K(M, n) \to D^{n-1}(M)$ be the morphism of complexes induced by id_M. If we have a morphism $f : D^{n-1}(M) \to X$ corresponding to $x \in X^{n-1}$, then the morphism $f \circ u : K(M, n) \to X$ corresponds to $d_X^{n-1}(x) \in Z^n(X)$.

(b). Let $f : X \to Y$ be a morphism of $\mathcal{C}(\mathscr{A})$. Saying that f has the right lifting property with respect to $0 \to D^n$ means that, for every morphism $g : D^n \to Y$, there exists $h : D^n \to X$ such that $f \circ h = g$. By question (a), this is equivalent to saying that the map $\mathrm{Hom}_R(R, X^n) \to \mathrm{Hom}_R(R, Y^n)$, $h \mapsto f \circ h$ is surjective, which is equivalent to the fact that $f^n : X^n \to Y^n$ is surjective. This proves the assertion.

(c). By question (b), the morphism $0 \to D^n$ has the left lifting property with respect to every fibration, so it is a cofibration.

(d). Let $f : X \to Y$ be a morphism in $W \cap \mathrm{Fib}$, and let $n \in \mathbb{Z}$. We want to show that $0 \to S^n$ has the left lifting property relatively to f. As $\mathrm{Hom}_{\mathcal{C}(\mathscr{A})}(S^n, C) = \mathrm{Hom}_R(R, Z^n(C)) = Z^n(C)$ for every object C of $\mathcal{C}(\mathscr{A})$ (by question (a)), this is equivalent to the fact that the map $Z^n(X) \to Z^n(Y)$ induced by f^n is surjective. So let $y \in Z^n(Y)$. As f is a quasi-isomorphism, there exists $x \in Z^n(X)$ such that

$f^n(x) - y \in B^n(Y)$. Write $f^n(x) - y = d^Y_{n-1}(y')$, with $y' \in Y^{n-1}$. As f is in Fib, there exists $x' \in X^{n-1}$ such that $f^{n-1}(x') = y'$, and then we have

$$y = f^n(x) - d^Y_{n-1}(y') = f^n(x) - d^Y_{n-1}(f^{n-1}(x')) = f^n(x - d^X_{n-1}(x')).$$

Also, as $d^X_n \circ d^X_{n-1} = 0$, we still have $x - d^X_{n-1}(x') \in Z^n(X)$.

(e). By the solution of (a), saying that $p : X \to Y$ has the right lifting property relatively to $S^n \to D^{n-1}$ is equivalent to the following statement: For every $y' \in Y^{n-1}$, and for every $x \in Z^n(X)$ such that $d^{n-1}_Y(y') = p^n(x) \in Z^n(Y)$, there exists $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$ and $p^{n-1}(x') = y'$.

(i) Suppose that $p \in W \cap \mathrm{Fib}$, and let $y' \in Y^{n-1}$ and $x \in Z^n(X)$ be such that $d^{n-1}_Y(y') = p^n(x)$. In particular, we have $p^n(X) \in B^n(Y)$; as p is a quasi-isomorphism, this implies that $x \in B^n(X)$, so there exists $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$. We have

$$d^{n-1}_Y(p^{n-1}(x') - y') = p^n(d^{n-1}_X(x')) - p^n(x) = 0,$$

so $p^{n-1}(x') - y' \in Z^{n-1}(Y)$. By question (d), there exists $x'' \in Z^{n-1}(X)$ such that $p^{n-1}(x'') = p^{n-1}(x') - y'$, i.e. $y' - p^{n-1}(x' - x'')$. Moreover, as $x'' \in Z^{n-1}(X)$, we have $d^{n-1}_X(x' - x'') = d^{n-1}_X(x') = x$. So we are done.

(ii) Suppose that p has the right lifting property relatively to $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$.

We first show that p^n induces a surjective map $Z^n(X) \to Z^n(Y)$ for every $n \in \mathbb{Z}$. Indeed, let $n \in \mathbb{Z}$ and $y \in Z^n(Y)$. Then $d^n_Y(y) = 0 = p^{n+1}(0)$, so there exists $x \in X^n$ such that $d^n_X(x) = 0$, i.e. $x \in Z^n(X)$, and that $p^n(x) = y$.

Now we show that $p^n : X^n \to Y^n$ is surjective for every $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$ and $y \in Y^n$. Then $d^n_Y(y) \in Z^{n+1}(Y)$, so, by the previous paragraph, there exists $x' \in Z^{n+1}(X)$ such that $p^{n+1}(x') = d^n_Y(y)$. Then, by assumption, there exists $x \in X^n$ such that $d^n_X(x) = x'$ and $p^n(x) = y$.

We finally show that p is a quasi-isomorphism. Let $n \in \mathbb{Z}$. We already know that the map $\mathrm{H}^n(p) : \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ is surjective (because $Z^n(X) \to Z^n(Y)$ is surjective), so it remains to show that it is injective. Let $x \in Z^n(X)$, and suppose that $p^n(x) \in B^n(Y)$. Then there exists $y' \in Y^{n-1}$ such that $p^n(x) = d^{n-1}_Y(y')$, so we can also find $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$ and $p^{n-1}(x') = y'$. In particular, we have $x \in B^n(X)$.

(f). This follows from question (e) and from the definition of Cof.

(g). (i) An easy calculation shows that the complex D^n has zero cohomology for every $n \in \mathbb{Z}$. As i is the direct sum of id_X and of morphisms $0 \to D^n$, this implies that i is a quasi-isomorphism.

(ii) The morphism id_X has the left lifting property relatively to any morphism of $\mathcal{C}(\mathscr{A})$, and morphisms $0 \to D^n$ have the left lifting property relatively to morphisms of Fib

by question (b). Also, for every morphism of $\mathcal{C}(\mathscr{A})$, the set of morphisms that have the left lifting property relatively to f is stable by direct sums (this is easy, and it is also proved in Proposition VI.5.2.1).

(iii) It is clear on the definition of p that every element of Y^n is in the image of p^n, for every $n \in \mathbb{Z}$. So p is in Fib.

(h). (i) For every $i \in \mathbb{N}$, the composition $X \to X_i \xrightarrow{f_i} Y$ is equal to

$$f_i \circ (j_{i-1} \circ j_{i-2} \circ \ldots \circ f_0) = f_{i-1} \circ (\circ j_{i-2} \circ \ldots \circ f_0) = \ldots = f_1 \circ j_0 = f.$$

So $q \circ j = f$.

(ii) For every $i \in \mathbb{N}$, the morphism $X \to X_i$ (which is $j_{i-1} \circ j_{i-2} \circ \ldots \circ j_0$) is a monomorphism. As filering colimits are exact in $\mathcal{C}({}_R\mathbf{Mod})$ (because they are exact in ${}_R\mathbf{Mod}$), this implues that j is a monomorphism.

(iii) For every $i \in \mathbb{N}$, the morphism $\bigoplus_{D \in \mathscr{D}_i} S^{n_D} \to \bigoplus_{D \in \mathscr{D}_i} D^{n_D - 1}$ is in Cof by question (e). This easily implies that j_i is in Cof for every $i \in \mathbb{N}$, and then that j is is Cof (see Proposition VI.5.2.1).

(iv) By question (e), it suppose to show that q has the right lifting property with respect to $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$. So fix $n \in \mathbb{Z}$, and consider a commutative square:

$$\begin{tikzcd} S^n \arrow[r, "u"] \arrow[d] & F \arrow[d, "q"] \\ D^{n-1} \arrow[r, "v"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

We want to find $h : D^{n-1} \to Y$ making the diagram commute. Remember that $\operatorname{Hom}_{\mathcal{C}(\mathscr{A})}(S^n, F) = Z^n(F)$ by (a). As $F^n = \varinjlim_{i \in \mathbb{N}} X_i^n$, there exists $i \in \mathbb{N}$ and $x \in X_i^n$ such that the element z of $Z^n(F)$ corresponding to u is the image of x_i in F^n. As $d_F^n(z) = 0$, the image in F^{n+1} of $d_{X_i}^n(x_i)$ is 0. But the morphism $X_i \to F$ is a monomorphism (for the same reason as in (i)), so $d_{x_i}^n(x_i) = 0$, i.e. $x_i \in Z^n(X_i)$. Let $u_i : S^n \to X_i$ be the morphism corresponding to x_i. By definition of X_{i+1}, there is a morphism $h_i : D^n \to Y$ making the following diagram commute:

$$\begin{tikzcd} S^n \arrow[r, "u_i"] \arrow[d] & X_i \arrow[d, "j_i"'] \arrow[dd, bend left, "f_i"] \\ D^{n-1} \arrow[r, "h_i"'] \arrow[dr, bend right, "v"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ & Y \end{tikzcd}$$

We get the desired morphism $h : D^n \to F$ by composing h_i with the canonical morphism $X_{i+1} \to F$.

(i). Let $i : X \to Y$ be an element of Cof. By question (h), we can write $i = q \circ j$, with $j : X \to F$ a monomorphism and $q \in W \cap \mathrm{Fib}$. In particular, we have a commutative square

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] & F \arrow[d, "q"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

By definition of Cof, there exists $h : Y \to F$ such that $q \circ h = \mathrm{id}_Y$ and $h \circ i = j$. As j is a monomorphism, this implies that i is also a monomorphism.

(j). Let $j : X \to Y$ be an element of $W \cap \mathrm{Cof}$. By question (h), we can write $j = p \circ i$, where $i \in W$ has the left lifting property relatively to fibrations and $p \in \mathrm{Fib}$. As $j \in W$, we also have $p \in W$. Consider the commutative square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

As $p \in W \cap \mathrm{Fib}$ and $j \in \mathrm{Cof}$, there exists $h : Y \to A$ such that $p \circ h = \mathrm{id}_Y$ and $h \circ i = j$. So we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "j"'] \\ Y \arrow[r, "h"'] & A \arrow[r, "p"'] & Y \end{tikzcd}$$

which shows that j is a retract of i. As i has the left lifting property relatively to fibrations, so does j. (This is easy, see Proposition VI.5.2.1 for a proof.)

(k). We check the axioms. First, the sets W, Fib and Cof clearly contain the identity morphisms and are stable by composition. Also, we know that $\mathcal{C}({}_R\mathbf{Mod})$ has all small limits and colimits, which is axiom (MC1). Axiom (MC2) (the fact that W satisfies the two out of three property) and the fact that W and Fib are stable by retracts are clear. The fact that Cof is stable by retract follows from its definition as the set of morphisms having the left lifting property relatively to elements of $W \cap \mathrm{Fib}$; this finishes the proof of (MC3). The existence of the two factorizations of axiom (MC5) is proved in questions (g) and (h). Finally, consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

as in axiom (MC4). If $p \in W \cap \mathrm{Fib}$ and $i \in \mathrm{Cof}$, the existence of h follows from the definition of Cof. If $i \in W \cap \mathrm{Cof}$ and $p \in \mathrm{Fib}$, the existence of h follows from question (j).

(l). Let W^-, Fib^- and Cof^- be the intersections of W, Fib and Cof with $\mathcal{C}^-({}_R\mathbf{Mod})$. By the description of the functors $\mathrm{Hom}_{\mathcal{C}({}_R\mathbf{Mod})}(S^n, \cdot)$ and $\mathrm{Hom}_{\mathcal{C}({}_R\mathbf{Mod})}(D^n, \cdot)$ in question (a), if $f : X \to Y$ is a morphism of $\mathcal{C}^-({}_R\mathbf{Mod})$, then the algorithms of questions (g) and (h) produce factorizations of f in $\mathcal{C}^-({}_R\mathbf{Mod})$. So, to prove the statement, it suffices to check that Cof^- is the set of morphisms of $\mathcal{C}^-({}_R\mathbf{Mod})$ having the left lifting property relatively to the elements of $W^- \cap \mathrm{Fib}^-$. The fact that every morphism of Cof^- satisfies this property is clear. Conversely, let $j : A \to B$ be a morphism of $\mathcal{C}^-({}_R\mathbf{Mod})$ that has the left lifting property relatively to the elements of $W^- \cap \mathrm{Fib}^-$, and let $p : X \to Y$ be in $W \cap \mathrm{Fib}$. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

As $A, B \in \mathrm{Ob}(\mathcal{C}^-({}_R\mathbf{Mod}))$, there exists $N \in \mathbb{Z}$ such that $A = \tau^{\leq N} A$ and $B = \tau^{\leq N} B$. Also, by question (d) and the properties of the truncation functors, the morphism $\tau^{\leq N} p : \tau^{\leq N} X \to \tau^{\leq N} Y$ is still in $W \cap \mathrm{Fib}$, hence it is in $W^- \cap \mathrm{Fib}^-$. So we have a commutative square

$$\begin{tikzcd} A \arrow[r, "\tau^{\leq N} f"] \arrow[d, "j"'] & \tau^{\leq N} X \arrow[d, "\tau^{\leq N} p"] \\ B \arrow[r, "\tau^{\leq N} g"'] \arrow[ur, dashed, "h'"] & \tau^{\leq N} Y \end{tikzcd}$$

with $\tau^{\leq N} p \in W^- \cap \mathrm{Fib}^-$. By the hypothesis on j, there exists $h' B \to \tau^{\leq N} X$ making the diagram commute. Composing h' with the canonical morphism $\tau^{\leq N} X \to X$, we get a morphism $h : B \to X$ such that $p \circ h = g$ and $h \circ j = f$.

(m). Let $f : A \to B$ be a morphism of left R-modules.

Suppose that f has the left lifting property with respect to every surjective morphism of left R-modules. Denote the canonical surjection $A \to \mathrm{Im}\, f$ by q. Applying the lifting property of f to the commutative square

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "f"'] & \mathrm{Im}\, f \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd}$$

we get a morphism $h : B \to \mathrm{Im}\, f$ such that $h \circ f = q$. Applying the lifting property of f again, this time to the commutative square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[d, "q"] \\ B \arrow[r, "h"'] & \mathrm{Im}\, f \end{tikzcd}$$

we get a morphism $s : B \to A$ such that $s \circ f = \mathrm{id}_A$. So f is injective and we have $B = \operatorname{Im} f \oplus P$, with $P = \operatorname{Ker} s$. It remains to show that P is projective. Let $u : M \to N$ be a surjective morphism of left R-modules, and let $g : P \to N$ be a R-linear map. We extend it to a R-linear map $g' : B \to N$ by taking $g' = 0$ on $\operatorname{Im} f$. Then we have a commutative square

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "f"'] & M \arrow[d, "u"] \\ B \arrow[r, "g'"'] & N \end{tikzcd}$$

so there exists $h' : B \to M$ such that $u \circ h' = g'$. If $h = h'_{|P}$, we have $u \circ h = g$.

Conversely, suppose that f is injective with projective cokernel $P = \operatorname{Coker} f$. Let $p : B \to P$ be the canonical surjection. As P is projective, there exists $s : P \to B$ such that $p \circ s = \mathrm{id}_P$. Hence $B \simeq A \oplus P$, so we may assume that $B = A \oplus P$ and that $f = \begin{pmatrix} \mathrm{id}_A \\ 0 \end{pmatrix}$. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[d, "q"] \\ B \arrow[r, "v"'] & N \end{tikzcd}$$

with q a surjective map. As P is projective, there exists $h' : P \to M$ such that $q \circ h' = v_{|P}$. Let $h = \begin{pmatrix} u & h' \end{pmatrix} : B = A \oplus P \to N$. Then $h \circ f = u$ and $q \circ h = \begin{pmatrix} u & v_{|P} \end{pmatrix} = v$.

(n). We first prove that, for every $n \in \mathbb{Z}$, the functor $D^{n-1} : {}_R\mathbf{Mod} \to \mathcal{C}({}_R\mathbf{Mod})$ is right adjoint to the functor $\mathcal{C}({}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$, $X \mapsto X^n$. Let $n \in \mathbb{Z}$, let M be a left R-module and let X be an object of $\mathcal{C}({}_R\mathbf{Mod})$. Then giving a morphism of complexes $u : X \to D^{n-1}(M)$ is equivalent to giving two R-linear maps $u^{n-1} : X^{n-1} \to M$ and $u^n : X^n \to M$ such that $u^{n-1} \circ d_X^{n-2} = 0$ and $u^n \circ d_X^n = u^{n-1}$; as the second condition determines u^{n-1} and implies the first condition, this is equivalent to giving $u^n : X^n \to M$. So we have constructed a bijective map

$$\operatorname{Hom}_{\mathcal{C}({}_R\mathbf{Mod})}(X, D^{n-1}(M)) \to \operatorname{Hom}_R(X^n, M),$$

which is clearly functorial in X and M.

Let $i : A \to B$ be a morphism of $\mathcal{C}({}_R\mathbf{Mod})$. We suppose that i is in Cof, and we want to show that i^n is injective with projective kernel for every $n \in \mathbb{Z}$:

(1) Suppose first that $A = 0$, and let $n \in \mathbb{Z}$. We want to show that B^n is a projective R-module. Let $p : M \to N$ be a surjective map of left R-modules, and let $f : B^n \to N$ be a R-linear map. Then the morphism $D^{n-1}(p) : D^{n-1}(M) \to D^{n-1}(N)$ is a fibration, and it is acyclic because both $D^{n-1}(M)$ and $D^{n-1}(N)$ are acyclic complexes. Consider the morphism of complexes $u : B \to D^{n-1}(N)$ corresponding to

$f : B^n \to N$ by the adjunction of the first paragraph. As B is cofibrant, there exists a morphism $h : B \to D^{n-1}(M)$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & D^{n-1}(M) \arrow[d, "D^{n-1}(p)"] \\ B \arrow[r, "u"'] \arrow[ur, "h"] & D^{n-1}(N) \end{tikzcd}$$

and then $h^n : B^n \to M$ satisfies the identity $p \circ h^n = f$. This shows that B^n is a projective R-module.

(2) Now we treat the general case. Note that we have a cocartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & 0 \arrow[d] \\ B \arrow[r] & \mathrm{Coker}(i) \end{tikzcd}$$

By Corollary VI.1.2.4, this implies that $0 \to \mathrm{Coker}(i)$ is a cofibration, i.e. that $\mathrm{Coker}(i)$ is cofibrant. By (1), this shows that i^n has projective cokernel for every $n \in \mathbb{Z}$. To show that i^n is injective, consider the morphism $u : A \to D^{n-1}(A^n)$ corresponding to id_{A^n} by the adjunction of the first paragraph. As $D^{n-1}(A^n)$ is an acyclic complex, the morphism $D^{n-1}(A^n)$ is an acyclic fibration, so there exists $h : B \to D^{n-1}A$ such that $h \circ i = u$, and in particular we have $h^n \circ i^n = \mathrm{id}_{A^n}$, which implies that i^n is injective.

Conversely, suppose that, for every $n \in \mathbb{Z}$, the morphism i^n is injective and has projective cokernel. We want to show that i is a cofibration. Let $P = \mathrm{Coker}(i)$. As each P^n is a projective, the morphisms $i^n : A^n \to B^n$ are split injections (i.e. there exists morphisms $a^n : B^n \to A^n$ such that $a^n \circ i^n = \mathrm{id}_{A^n}$), so, without loss of generality, we may assume that $B^n = A^n \oplus P^n$ and that $i^n = \begin{pmatrix} \mathrm{id}_{A^n} \\ 0 \end{pmatrix}$. As i is a morphism of complexes, we have $d_B^n = \begin{pmatrix} d_A^n & u^n \\ 0 & d_P^n \end{pmatrix}$, with $u^n : P^n \to A^{n+1}$.

Consider a commutative square (in $\mathcal{C}({}_R\mathbf{Mod})$)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

with p an acyclic fibration. We want to show that there exists a morphism $h : B \to X$ making the diagram commute. As $g \circ i = p \circ f$, we have $g^n = \begin{pmatrix} p^n \circ f^n & v^n \end{pmatrix}$ with $v^n : P^n \to Y^n$, and the fact that g is a morphism of complexes is equivalent to the identities

(1) $$d_Y^n \circ v^n = p^{n+1} \circ f^{n+1} \circ u^n + v^{n+1} \circ d_P^n.$$

If $h : B \to X$ is a morphism such that $h \circ i = f$, then we must have $h^n = \begin{pmatrix} f^n & w^n \end{pmatrix}$, with $w^n : P^n \to X^n$. The fact that h is a morphism of complexes is equivalent to the identities

$$d_X^n \circ w^n = f^{n+1} \circ u^n + w^{n+1} \circ d_P^n, \tag{2}$$

and we have $p \circ h = g$ if and only $p^n \circ w^n = v^n$ for every $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. As $p^n : X^n \to Y^n$ is surjective and P^n is a projective R-module, there exists a R-linear map $k^n : P^n \to X^n$ such that $p^n \circ k^n = v^n$.

$$\begin{tikzcd} & X^n \arrow[d, "p^n"] \\ P^n \arrow[ur, dashed, "k^n"] \arrow[r, "v^n"'] & Y^n \end{tikzcd}$$

Let $r^n = d_X^n \circ k^n - k^{n+1} \circ d_P^n - f^{n+1} \circ u^n : P^n \to X^{n+1}$. We have

$$\begin{aligned} p^{n+1} \circ r^n &= d_Y^{n+1} \circ p^n \circ k^n - p^{n+1} \circ k^{n+1} \circ d_P^n - p^{n+1} \circ f^{n+1} \circ u^n \\ &= d_Y^{n+1} \circ v^n - v^{n+1} \circ d_P^n - p^{n+1} \circ f^{n+1} \circ u^n \\ &= 0 \quad \text{by (1).} \end{aligned}$$

Let $K = \operatorname{Ker}(p)$. We just proved that $r^n : P^n \to X^{n+1}$ factors through a R-linear map $s^n : P^n \to K^{n+1}$. Also, we have

$$r^{n+1} \circ d_P^n = d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ u^{n+1} \circ d_P^n$$

and

$$\begin{aligned} d_X^{n+1} \circ r^n &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - d_X^{n+1} \circ f^{n+1} \circ u^n \\ &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ d_A^{n+1} \circ u^n \\ &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ u^{n+1} \circ d_P^n, \end{aligned}$$

so $s^{n+1} \circ d_P^n = -d_K^{n+1} \circ s^n$. This means that the family $(s^n)_{n \in \mathbb{Z}}$ defines a morphism of complexes from P to $K[1]$. As P is a bounded above complex of projective R-modules and K is an acyclic complex, the dual of Theorem IV.3.2.1(i) says that s is homotopic to 0. This means that there exists a family of R-linear maps $(t^n : P^n \to K^n)_{n \in \mathbb{Z}}$ such that, for every $n \in \mathbb{Z}$, we have

$$s^n = t^{n+1} \circ d_P^n + d_{K[1]}^{n-1} \circ t^n = t^{n+1} \circ d_P^n - d_K^n \circ t^n.$$

For every $n \in \mathbb{Z}$, we set $w^n = k^n + t^n : P^n \to X^n$ and $h^n = \begin{pmatrix} f^n & w^n \end{pmatrix} : B^n \to X^n$. As $K^n = \operatorname{Ker}(p^n)$, we have

$$p^n \circ w^n = p^n \circ k^n = v^n,$$

so $p^n \circ h^n = g^n$. It remains to check that h is a morphism of complexes from B to X, so we check identity (2). Let $n \in \mathbb{Z}$. We have

$$\begin{aligned} d_X^n \circ w^n &= d_X^n \circ k^n + d_X^n \circ t^n \\ &= d_X^n \circ k^n + t^{n+1} \circ d_P^n - r^n \\ &= t^{n+1} \circ d_P^n + k^{n+1} \circ d_P^n + f^{n+1} \circ u^n \\ &= w^{n+1} \circ d_P^n + f^{n+1} \circ u^n, \end{aligned}$$

which is exactly (2).

□

A.11 Problem set 11

A.11.1 The model structure on complexes (continued)

(a). Let R be a ring, let $\mathscr{C} = \mathcal{C}^*({}_R\mathbf{Mod})$ with $* \in \{-, \varnothing\}$ and consider the sets of morphisms

$$I = \{S^n \to D^{n-1},\ n \in \mathbb{Z}\}$$

and

$$J = \{0 \to D^n,\ n \in \mathbb{Z}\}$$

in $\mathscr{C}$. We use the notation S^n and D^n of problem A.10.2, and we denote by W the set of quasi-isomorphisms in $\mathscr{C}$.

(i) Show that S^n is small in $\mathscr{C}$ for every $n \in \mathbb{Z}$.

(ii) Show that $I - \mathrm{inj}$ is the set of surjective quasi-isomorphisms.

(iii) Show that $J - \mathrm{inj}$ is the set of surjective morphisms.

(iv) Show that I and J are the sets of generating cofibrations and generating acyclic cofibrations of the model structure of problem A.10.2 on $\mathscr{C}$.

(v) Show that, if $f \in I - \mathrm{cell}$, then f^n is injective and $\mathrm{Coker}(f^n)$ is a free R-module for every $n \in \mathbb{Z}$.

(b). Let $\mathscr{C}' = \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$, and consider the following sets of morphisms in $\mathscr{C}'$:

$$I' = \{S^n \to D^{n-1},\ n \leq 0\} \cup \{0 \to S^0\}$$

and

$$J' = \{0 \to D^n,\ n \leq -1\}.$$

We still denote by W the set of quasi-isomorphisms in $\mathscr{C}'$.

(i) Show that $J' - \mathrm{inj}$ is the set of morphisms f such that f^n is surjective for $n \leq -1$.

(ii) Show that $I' - \mathrm{inj} = W \cap J' - \mathrm{inj}$.

(iii) Show that I' and J' satisfy the conditions of Theorem VI.5.4.5.

Solution.

(a). (i) We have to show that, for every ω-sequence $X : \mathbb{N} \to \mathscr{C}$, the canonical map

$$\varinjlim_{r \geq 0} \mathrm{Hom}_{\mathscr{C}}(S^n, X_r) \to \mathrm{Hom}_{\mathscr{C}}(S^n, \varinjlim_{r \geq 0} X_r)$$

is an isomorphism. It suppose to treat the case of $\mathscr{C} = \mathcal{C}({}_R\mathbf{Mod})$ (as $\mathcal{C}^-({}_R\mathbf{Mod})$ is a full subcategory of $\mathcal{C}({}_R\mathbf{Mod})$ and colimits are easily seen to commute with the inclusion functor). By question (a) of problem A.10.2 , we a canonical isomorphism, for every $X \in \mathrm{Ob}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}}(S^n, X) = \mathrm{Hom}_R(R, Z^n(X)) \simeq Z^n(X).$$

So it suffices to show that the functor $Z^n : \mathscr{C} \to {}_R\mathbf{Mod}$ commutes with filtrant colimits, which follows immediately from the fact that filtrant colimits are exact on the category ${}_R\mathbf{Mod}$ and from the construction of colimits of complexes.

(ii) This is question (e) of problem A.10.2 .

(iii) This is question (b) of problem A.10.2 .

(iv) This follows from (i)-(iii) and from Definition VI.5.4.1 .

(v) Let is denote by (P) the property "for every $n \in \mathbb{Z}$, the morphism f^n is injective and its cokernel is a free R-module" of morphisms f of $\mathscr{C}$. As (P) is true for morphisms of I, it suffices to check that it is stable by direct sums, pushouts and transfinite composition.

The stability by direct sums is clear.

Let $X : \mathbb{N} \to \mathscr{C}$ be an ω-sequence such that each morphism $i_r : X_r \to X_{r+1}$ satisfies property P, and let $X = \varinjlim_{r \geq 0} X_r$. We want to show that the morphism $i : X_0 \to X$ satisfies property (P). As injective morphisms in ${}_R\mathbf{Mod}$ are stable by composition and filtrant colimits, the morphism $i^n : (X_0)^n \to X^n$ is injective for every $n \in \mathbb{Z}$. It remains to show that $\mathrm{Coker}(i^n)$ is free for every n. Fix $n \in \mathbb{Z}$. We have

$$\mathrm{Coker}(i^n) = \mathrm{Coker}(\varinjlim_{r \geq 0}(i_r \circ \ldots \circ i_0)^n) = \varinjlim_{r \geq 0} \mathrm{Coker}((i_r \circ \ldots \circ i_0)^n).$$

We construct by induction on $r \geq 1$ a family of subsets B_r of $(X_r)^n$ such that the image of B_r in $\mathrm{Coker}((i_{r-1} \circ \ldots \circ i_0)^n)$ is a basis. We take B_1 to be a lift of a basis of $\mathrm{Coker}((i_0)^n)$. If $r \geq 1$ and we have constructed B_r, then we let B'_{r+1} be a lift of a basis of $\mathrm{Coker}(i_r)$, and we take $B_{r+1} = i_r(B_r) \cup B'_{r+1}$. Let $B \subset \mathrm{Coker}(i^n)$ be the image of the (increasing) union of the images of B_r in X^n, for $r \geq 1$. We claim that B is a basis of $\mathrm{Coker}(i^n)$. Indeed, let $x \in X^n$. As $\mathrm{Coker}(i^n) = \varinjlim_{r\geq 0} \mathrm{Coker}((i_r \circ \ldots \circ i_0)^n)$, there exists $r \geq 1$ and $y \in (X_r)^n$ such that x is the image of y in $\mathrm{Coker}(i^n)$. We can write $y \mod \mathrm{Im}((i_{r-1} \circ \ldots \circ i_0)^n) = \sum_{e\in B_r} a_e e$, with $a_e \in R$, which shows that x is in the span of B. Now if we have a finite subset B' of B and an identity $\sum_{e\in B'} a_e e = 0$ with $a_e \in R$, then we can find $r \geq 1$ such that B' is contained in the image of B_r in X^n; then the a_e are the coefficients of a relation of linear dependence among the elements of B_r, so they are all 0. This shows that B is linearly independent.

For the stability by pushouts, as pushouts in $\mathscr{C}$ are calculated degree by degree, it suffices to proved that, if

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

is a cocartesian square in ${}_R\mathbf{Mod}$ and if f is injective, then g is injective and the morphism $v : B \to D$ induces an isomorphism $\mathrm{Coker}\, f \overset{\sim}{\to} \mathrm{Coker}\, g$. The first statement is Corollary II.2.1.16 , and the second statement To prove the second statement, we may assume that $D = (B \oplus C)/i(A)$, where the morphism $i : A \to B \oplus C$ is $\begin{pmatrix} u \\ f \end{pmatrix}$, that v sends $b \in B$ to the class of $(b, 0)$ and g sends $c \in C$ to the class of $(0, c)$. Let $b \in B$. Then $v(b) \in im(g)$ if and only if there exists $c \in C$ such that the classes of $(b, 0)$ and $(0, c)$ in D are equal, that is, if and only if b is the image of an element of A by $f : A \to B$. This shows that $\mathrm{Coker}\, f \to \mathrm{Coker}\, g$ is injective. Now let $d \in D$, and choose $(b, c) \in B \oplus C$ representing it. After translating d by an element of $\mathrm{Im}\, g$, we may assume that $c = 0$, and then $d = v(b)$. This shows that $\mathrm{Coker}\, f \to \mathrm{Coker}\, g$ is surjective.

(b). (i) In the solution of question (b) of problem A.10.2, we showed that a morphism $f : X \to Y$ in $\mathcal{C}({}_R\mathbf{Mod})$ has the right lifting property relatively to $0 \to D^n$ if and only if $f^n : X^n \to Y^n$ is surjective. This immediately implies the desired result.

(ii) Suppose that $f \in W \cap J' - \mathrm{inj}$. By (i), this means that f is a quasi-isomorphism and that f^n is surjective for $n \leq -1$. By the solution of question (d) of problem A.10.2, this implies that f has the right lifting property relatively to $0 \to S^n$ for every $n \leq 0$, i.e. that $Z^n(f)$ is surjective for every $n \leq 0$. By the solution of question (e)(i) of the same problem, we can then deduce that f has the right lifting property relatively to $S^n \to D^{n-1}$ for every $n \leq 0$. So f is in $I' - \mathrm{inj}$.

Conversely, suppose that $f \in I' - \mathrm{inj}$. By the solution of question (d) and the beginning of the solution of question (e)(ii) of problem A.10.2, this implies that $Z^n(f)$ is surjective for every $n \leq 0$. Then the rest of the solution of (e)(ii) of the same problem shows that f^n is surjective for $n \leq -1$ and that $\mathrm{H}^n(f)$ is an isomorphism for $n \leq 0$.

(iii) The morphisms of J' all have source 0, which is small in $\mathscr{C}'$, and we have proved in question (a) that S^n is small in $\mathscr{C}$, hence in $\mathscr{C}'$, for every $n \leq 0$. This fives condition (a) of the theorem. Also, we proved in (ii) that $I' - \mathrm{inj} = W \cap J' - \mathrm{inj}$, which gives conditions (c) and (d). It remains to proved condition (b), i.e. the fact that $J' - \mathrm{cof} \subset W \cap I' - \mathrm{cof}$. As $I' - \mathrm{inj} \subset J' - \mathrm{inj}$, we have $J' - \mathrm{cof} \subset I' - \mathrm{cof}$. It remains to show that $J' - \mathrm{cof} \subset W$. By Corollary VI.5.3.4, every morphism of $J' - \mathrm{cof}$ is a retract of a morphism of $J' - \mathrm{cell}$, so it suffices to show that morphism of $J' - \mathrm{cell}$ are quasi-isomorphisms. As every morphism of J' is a quasi-isomrophism, it suffices to show that quasi-isomorphisms are stable by direct sums, transfinite compositions and pushouts; the first is obvious, the second follows from the exactness of filtrant colimits in ${}_R\mathbf{Mod}$, and the third is a striaghtforward calculation.

□

A.11.2 The model structure on simplicial R-modules

Remember the simplicial category Δ and the category of simplicial sets $\mathbf{sSet}$ from problem PS1.9 and problem PS2.2. As in problem PS1.9 we define morphisms $\delta_0, \delta_1, \dots, \delta_n : [n-1] \to [n]$ in Δ by the condition that δ_i is the unique increasing map $[n-1] \to [n]$ such that $i \notin \mathrm{Im}(\delta_i)$.

(a). Let $\mathscr{A}$ be an additive category. Let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. For $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$, we denote the morphism $X_\bullet(\delta_i^n)$ by $d_i^n : X_n \to X_{n-1}$. The *unnormalized chain complex* of $X_\bullet$ is the complex $C(X_\bullet)$ in $\mathcal{C}^{\leq 0}(\mathscr{A})$ given by: for every $n \geq 0$,

$$C(X_\bullet)^{-n} = X_n$$

and

$$d_{C(X_\bullet)}^{-n} = \sum_{i=0}^{n} (-1)^i d_i^n.$$

(i) Show that $C(X_\bullet)$ is a complex.

From now on, we assume that $\mathscr{A}$ is also pseudo-abelian. We use the notation of problem A.10.1. In particular, we denote by $\mathscr{C}$ the category $\mathrm{kar}((\mathbb{Z}[\Delta])^{\oplus})$, identified to full subcategory of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$, and we extend object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$ to additive functors from $\mathscr{C}$ to $\mathbf{Ab}$. Let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$.

(ii) For every $r \in \mathbb{N}$ and every $n \geq 0$, we consider the following direct summand of $C(X_\bullet)^{-n}$:

$$C_{\leq r}(X_\bullet)^{-n} = \begin{cases} X_\bullet(\mathbb{Z}^{(\Delta_n^{\leq r})}) & \text{if } r \leq n-1 \\ X_\bullet(\mathbb{Z}^{(\Delta_n^{\leq n-1})}) & \text{otherwise.} \end{cases}$$

(With the convention that $\Delta_0^{\leq -1} = \varnothing$.)

Show that this defines a subcomplex $C_{\leq r}(X^\bullet)$ of $C(X^\bullet)$.

(iii) Let $i_r : C_{\leq r}(X^\bullet) \to C_{\leq r+1}(X^\bullet)$ be the obvious inclusion. Show that there exists a morphism $f_r : C_{\leq r+1}(X^\bullet) \to C_{\leq r}(X^\bullet)$ such that $f_r \circ i_r$ is the identity morphism.

(iv) Show that i_r is a homotopy equivalence.

(v) If $\mathscr{A}$ is an abelian category, show that the inclusion $N(X^\bullet) \subset C(X^\bullet)$ is a quasi-isomorphism.

(b). Let R be a ring, and let $\mathscr{C} = \mathrm{Func}(\Delta^{\mathrm{op}}, {}_R\mathbf{Mod})$. Show that there is a model structure on $\mathscr{C}$ for which the weak equivalences are the morphisms $f : X_\bullet \to Y_\bullet$ such that $C(f) : C(X_\bullet) \to C(Y_\bullet)$ is a quasi-isomorphism, and the cofibrations are the morphisms $f : X_\bullet \to Y_\bullet$ such that $N(f)^n : N(X_\bullet)^{-n} \to N(X_\bullet)^{-n}$ is injective with projective cokernel for every $n \geq 0$.

Solution.

(a).

(b). Using the Dold-Kan equivalence $N : \mathscr{C} \to \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$, we can transport the model structure on $\mathcal{C}^{\leq}({}_R\mathbf{Mod})$ (defined in problem A.11.1(b)) to $\mathscr{C}$. The characterization of weak equivalences follows from question (a).

□

A.11.3 A Quillen adjunction

Let k be a commutative ring, let Γ be a group, and let $R = k[\Gamma]$. Consider the categories $\mathscr{C} = \mathcal{C}^-({}_R\mathbf{Mod})$ and $\mathscr{D} = \mathcal{C}^-({}_k\mathbf{Mod})$ with the projective model structures, and the functor $F : \mathscr{C} \to \mathscr{D}$ sending a complex X to the complex $\mathrm{H}_0(\Gamma, X)$.

(a). Show that F has a right adjoint G.

(b). Show that (F, G) is a Quillen adjunction.

Solution.

(a). Consider the k-algebra morphism $k[\Gamma] \to k$ sending every $\gamma \in \Gamma$ to 0. Then we have $\mathrm{H}_0(\Gamma, X) = k \otimes_{k[\Gamma]} X$, for every complex of left $k[\Gamma]$-modules X. Hence, if Y is a complex of left k-modules and if we see Y as a complex of left $k[\Gamma]$-modules using the morphism $k[\Gamma] \to k$ that we just defined, then we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C}({}_k\mathbf{Mod})}(\mathrm{H}_0(\Gamma, X), Y) = \mathrm{Hom}_{\mathcal{C}({}_R\mathbf{Mod})}(X, Y).$$

This shows that F has a right adjoint G, where G is the functor sending

$Y \in \mathrm{Ob}(\mathcal{C}^-({}_k\mathbf{Mod}))$ to Y, seen as a complex of left $k[\Gamma]$-modules using the morphism $k[\Gamma] \to k$.

(b). To prove that (F, G) is a Quillen adjunction, it suffices by Corollary VI.4.2.3 to show that G preserves fibrations and acyclic fibrations. As a morphism in $\mathcal{C}^-({}_k\mathbf{Mod})$ or $\mathcal{C}^-({}_R\mathbf{Mod})$ is a fibration or an acyclic fibration if and only if the underlying morphism of complexes of abelian groups is, this follows immediately from the description of G.

□

A.11.4 Kähler differentials

Let R be a commutative ring. If B is a commutative R-algebra and M is a B-module, a R-linear *derivation* from B to M is a R-linear map $d : B \to M$ such that, for all $b, b' \in B$, we have

$$d(bb') = bd(b') + b'd(b).$$

We denote by $\mathrm{Der}_R(B, M)$ the abelian group of derivations from B to M.

We fix a commutative R-algebra B.

(a). Show that the functor ${}_B\mathbf{Mod} \to \mathbf{Ab}$, $M \longmapsto \mathrm{Hom}_R(B, M)$ is representable and give a pair representing it.

(b). Show that the functor ${}_B\mathbf{Mod} \to \mathbf{Ab}$, $M \longmapsto \mathrm{Der}_R(B, M)$ is representable by a pair $(\Omega^1_{B/R}, d_{\mathrm{univ}})$, where $\Omega^1_{B/R}$ is a B-module (called the module of *Kähler differentials*) and $d_{\mathrm{univ}} : B \to \Omega^1_{B/R}$ is a R-linear derivation. (Hint: The functor $M \longmapsto \mathrm{Der}_R(B, M)$ is a subfunctor of $M \longmapsto \mathrm{Hom}_R(B, M)$, so $\Omega^1_{B/R}$ should be a quotient of the B-module representing the functor of (a).)

(c). If B is the polynomial ring $R[X_i, i \in I]$ (where I is a set), show that $\Omega^1_{B/R}$ is a free B-module on the set I.

Solution.

(a). If N is a R-module and M is a B-module, there is an isomorphism, fonctorial in N and M:

$$\mathrm{Hom}_B(N \otimes_R B, M) \xrightarrow{\sim} \mathrm{Hom}_R(N, M)$$

sending a B-linear map $f : N \otimes_R B \to M$ to the R-linear map $N \to M$, $x \mapsto f(x \otimes 1)$. To get the result, it suffices to apply this to the R-module $N = B$. So a pair representing the functor ${}_B\mathbf{Mod} \to \mathbf{Ab}$, $M \mapsto \mathrm{Hom}_R(B, M)$ is $(B \otimes_R B, u)$, where B acts on the second factor of $B \otimes_R B$ and $u \in \mathrm{Hom}_R(B, B \otimes_R B)$ is defined by $u(b) = b \otimes 1$.

(b). For every B-module M, we have

$$\mathrm{Der}_R(B, M) = \{d \in \mathrm{Hom}_R(B, M) \mid \forall b, b' \in B,\ d(bb') = bd(b') + b'd(b)\}.$$

If $d : B \to M$ is a R-linear map and if $f : B \otimes_R B \to M$ is the corresponding B-linear map, then we have $d(b) = f(b \otimes 1)$ for every $b \in B$, so d is a derivation if and only if, for all $b, b' \in B$,
$$f((bb') \otimes 1) = bf(b' \otimes 1) + b'f(b \otimes 1).$$
By definition of the B-module structure on $B \otimes_R B$ and B-linearity of f, this identity is equivalent to
$$f((bb') \otimes 1) = f(b \otimes b') + f(b' \otimes 1).$$

So the functor $\mathrm{Der}_R(B, \cdot)$ is representable by the pair $(\Omega^1_{B/R}, d_{\mathrm{univ}})$, with $\Omega^1_{B/R} = B \otimes_R B/I$, where I is the B-submodule of $B \otimes_R B$ generated by the set $\{(bb') \otimes 1 - b \otimes b' - b' \otimes 1,\ b, b' \in B\}$, and with d_{univ} equal to the composition of $u : B \to B \otimes_R B$ and of the canonical projection $B \otimes_R B \to \Omega^1_{B/R}$.

(c). It suffices to show that the family $(d_{\mathrm{univ}}(X_i))_{i \in I}$ is a basis of the B-module $\Omega^1_{B/R}$. As $\mathrm{Hom}_B(\Omega^1_{B/R}, M) = \mathrm{Der}_R(B, M)$ for every B-module M, it suffices to prove that, for every B-module M and every family of elements $(t_i)_{i \in I}$ of M such that $t_i = 0$ for all but finitely many $i \in I$, there exists a unique R-linear derivation $d : B \to M$ such that $d(X_i) = t_i$ for every $i \in I$.

So we fix M and the family $(t_i)_{i \in I}$.

Suppose that $d, d' \in \mathrm{Der}_R(B, M)$ are such that $d(X_i) = d'(X_i) = t_i$ for every $i \in I$. We want to show that $d(P) = d'(P)$ for every polynomial $P \in B = R[X_i,\ i \in I]$. As d and d' are R-linear, it suffices to prove it for P a monomial of the form $\prod_{i \in I} X_i^{d_i}$ (with $d_i = 0$ for all but finitely many $i \in I$). In this case, we show the result by induction on $d = \sum_{i \in I} d_i$. If $d = 0$, then $P = 1$. Note that $d(1) = d(1^2) = d(1) + d(1)$, so $d(1) = 0$, and similarly $d'(1) = 0$; so $d(P) = d'(P)$ in this case. Suppose that $d \geq 1$. Then there exists $i_0 \in I$ such that $d_{i_0} \geq 1$, so $p = X_{i_0} Q$, with $Q = X_{i_0}^{i_0 - 1} \prod_{i \in I - \{i_0\}} X_i^{d_i}$. We have $d(Q) = d'(Q)$ by the induction hypothesis, so

$$d(P) = X_{i_0} d(Q) + Q d(X_{i_0}) = X_{i_0} d(Q) + Q t_{i_0} = X_{i_0} d'(Q) + Q d'(X_{i_0}) = d'(P).$$

Finally, we show that there exists $d \in \mathrm{Der}_R(B, M)$ such that $d(X_i) = t_i$ for every $i \in I$. For every $i \in I$, denote by $\frac{\partial}{\partial X_i} : B \to B$ the R-linear map that sends a polynomial to its derivative with respect to X_i. It is easy to see that this is a derivation, so the morphism $B \to R$, $P \mapsto \frac{\partial}{\partial X_i} P(1)$ is also a derivation. Define $d : B \to M$ by

$$d(P) = \sum_{i \in I} \frac{\partial}{\partial X_i} P(1) t_i.$$

(By the hypothesis on the family $(t_i)_{i \in I}$, this is a finite sum.) This is a R-linear derivation, and we have $d(X_i) = t_i$ for every $i \in I$.

$\square$

A.11.5 Abelianization and Kähler differentials

Let $\mathscr{C}$ is a category that has finite products, and denote a final object of $\mathscr{C}$ by $*$. An *abelian group in* $\mathscr{C}$ is a triple (X, m, e), where X is an object of $\mathscr{C}$, and $m : X \times X \to X$ and $e : * \to X$ are morphisms such that, for every object Y of $\mathscr{C}$, the morphisms

$$m_* : \mathrm{Hom}_{\mathscr{C}}(Y, X \times X) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, X) \times \mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

and

$$e_* : \mathrm{Hom}_{\mathscr{C}}(Y, *) = * \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

(where we also denote by $*$ a final object of $\mathbf{Set}$) define the structure of an abelian group on the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$. The morphism m is called the *multiplication morphism* of the group, and the morphism e is called the *unit*.

If $G = (X, m, e)$ and $G' = (X', m', e')$ are two abelian groups in $\mathscr{C}$, a *morphism from G to G′* is a morphism $f : X \to X'$ in $\mathscr{C}$ such that $f \circ e = e'$ and that the following diagram commutes:

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "f\times f"'] & X \arrow[d, "f"] \\ X' \times X' \arrow[r, "m'"'] & X' \end{tikzcd}$$

We denote by $\mathscr{C}_{\mathrm{ab}}$ the category of abelian groups in $\mathscr{C}$.

An *abelianization functor* on $\mathscr{C}$ is a left adjoint to the forgetful functor $\mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$.

(a). Show that $\mathbf{Set}_{\mathrm{ab}} \simeq \mathbf{Ab}$, that $\mathbf{Grp}_{\mathrm{ab}} \simeq \mathbf{Ab}$, that $\mathbf{Top}_{\mathrm{ab}}$ is equivalent to the category of commutative topological groups and that $\mathbf{sSet}_{\mathrm{ab}} \simeq \mathbf{sAb}$.

(b). Show that $\mathbf{Set}$, $\mathbf{Grp}$ and $\mathbf{sSet}$ have abelianization functors, and give formulas for these functors.

Let R be a commutative ring and A be a commutative R-algebra. We denote by $\mathscr{C}$ the slice category $R - \mathbf{CAlg}/A$ (see Definition I.2.2.6).

If M is a A-module, we define an A-algebra structure on $A \oplus M$ by taking the mutliplication given by the formula

$$(a, m)(a', m') = (aa', am' + a'm),$$

for $a, a' \in A$ and $m, m' \in M$. We have a morphism of A-algebras $A \oplus M \to A$ sending (a, m) to a. This gives a functor ${}_A\mathbf{Mod} \to \mathscr{C}$.

If $B \to A$ is an object of $\mathscr{C}$ and M is a A-module, we denote by $\mathrm{Der}_R(B, M)$ the abelian group of R-linear derivations from B to M (where M is seen as a B-module using the morphism $B \to A$).

(c). Show that we have an isomorphism of functors $\mathscr{C} \times {}_A\mathbf{Mod} \to \mathbf{Ab}$:

$$\mathrm{Hom}_{\mathscr{C}}(B, A \oplus M) \simeq \mathrm{Der}_R(B, M).$$

(d). If M is an A-module, show that $A \oplus M$ is an abelian group in $\mathscr{C}$, and give formulas for its multiplication and unit.

(e). Show that the functor ${}_A\mathbf{Mod} \to \mathscr{C}$ sending M to $A \oplus M$ factors through the subcategory $\mathscr{C}_{\mathrm{ab}}$, and that it induces an equivalence of categories ${}_A\mathbf{Mod} \to \mathscr{C}_{\mathrm{ab}}$.

(f). Show that the functor $\mathscr{C} \to {}_A\mathbf{Mod}$ sending $B \to A$ to $A \otimes_B \Omega^1_{B/R}$ is an abelianization functor for $\mathscr{C}$.

Solution.

(a). Let (X, m, e) be an abelian group in a category $\mathscr{C}$. By the Yoneda lemma (Corollary I.3.2.3), the condition on m and e are equivalent to the following conditions:

- m is associative, that is, the following diagram commutes:

$$\begin{tikzcd} X \times X \times X \arrow[r, "m \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times m"'] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "m"'] & X \end{tikzcd}$$

- m is commutative, that is, the following diagram commutes:

$$\begin{tikzcd} X \times X \arrow[r, "\iota"] \arrow[dr, "m"'] & X \times X \arrow[d, "m"] \\ & X \end{tikzcd}$$

  where $\iota : X \times X \to X \times X$ is the morphism exchanging the two factors.

- e is a unit for m, that is, the morphism $m \circ (\mathrm{id}_X \times e) : X \times * \to X$ (resp. $m \circ (e \times \mathrm{id}_X) : * \times X \to X$) is equal to the first (resp. second) projection.

- There exists an inverse, that is, there exists a morphism $i : X \to X$ such that the endomorphisms $m \circ (i \times \mathrm{id}_X)$ and $m \circ (\mathrm{id}_X \times i)$ of X are both equal to the composition $X \to * \xrightarrow{e} X$, where $X \to *$ is the unique morphism from X to the final object $*$.

This shows that $\mathbf{Set}_{\mathrm{ab}} = \mathbf{Ab}$ and that $\mathbf{Top}_{\mathrm{ab}}$ is the category of topological abelian groups. Also, by the definition of morphisms and products in categories of presheaves, if $\mathscr{I}$ is any category and $\mathscr{C}$ is a category having finite products, we have an equivalence $\mathrm{PSh}(\mathscr{I}, \mathscr{C})_{\mathrm{ab}} \simeq \mathrm{PSh}(\mathscr{I}, \mathscr{C}_{\mathrm{ab}})$; applying this to $\mathscr{I} = \Delta$ and $\mathscr{C} = \mathbf{Set}$ gives $\mathbf{sSet}_{\mathrm{ab}} \simeq \mathbf{sAb}$.

Finally, let (G, m, e) be an abelian group in $\mathbf{Grp}$. By question (a) of problem A.2.1, the morphism $m : G \times G \to G$ is equal to the multiplication of G and it is commutative. Conversely, if G is an abelian group, then its multiplication and inverse maps are morphisms of groups, so they make G into an abelian group in $\mathbf{Grp}$. This shows that $\mathbf{Grp} \simeq \mathbf{Ab}$.

(b). An abelianization functor for $\mathscr{C}$ is left adjoint to the forgetful functor $\mathrm{For} : \mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$. If $\mathscr{C} = \mathbf{Set}$, then For is (isomorphic to) the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$, so it has a left adjoint, which is the free $\mathbb{Z}$-module functor. Similarly, if $\mathscr{C} = \mathbf{sSet}$ (or more generally if $\mathscr{C} = \mathrm{PSh}(\mathscr{I}, \mathbf{Set})$ for any category $\mathscr{I}$), then the forgetful functor $\mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$ has a left adjoint, which sends a presheaf $X_\bullet : \Delta \to \mathscr{C}$ to the presheaf $n \mapsto \mathbb{Z}^{(X_n)}$. Finally, if $\mathscr{C} = \mathbf{Grp}$, then For is (isomorphic to) the inclusion $\mathbf{Ab} \subset \mathbf{Grp}$, so its left adjoint is the abelianization functor.

(c). If $u : B \to A$ is an object of $\mathscr{C}$, M is a A-module and $d \in \mathrm{Der}_R(B, M)$, we define a map $f : B \to A \oplus M$ by $f(b) = u(b) + d(b)$. Then f is R-linear because u and d are, we have $f(1) = u(1) = 1$ (we have seen in the proof of (c) of problem **??** that a derivation always sends 1 to 0), and, for $b, b' \in B$, we have

$$f(bb') = u(bb') + bd(b') + b'd(b) = (u(b) + d(b))(u(b') + d(b')) = f(b)f(b')$$

by definition of the multiplication on $A \oplus M$ and of the B-module structure on M. So f is a morphism of R-algebras, and it is obviously compatible with the morphisms from $A \oplus M$ and B to A, so it is a morphism of $\mathscr{C}$. This defines a map $\alpha(B, M) : \mathrm{Der}(B, M) \to \mathrm{Hom}_{\mathscr{C}}(B, A\oplus)$, and this map is clearly functorial in B and M, so we get a morphism of functors $\alpha : \mathrm{Der}(B, \cdot) \to \mathrm{Hom}_{\mathscr{C}}(B, A \oplus (\cdot))$.

To finish the proof, it remains to show that α is an isomorphism of functors. So fix an object $u : B \to A$ of $\mathscr{C}$ and a A-module M again, and let $f : B \to A \oplus M$ be a morphism of $\mathscr{C}$. In particular, the map f must be compatible with the morphisms from B and $A \oplus M$ to A, so, for every $b \in B$, we have $f(b) = u(b) + d(b)$ for some uniquely determined $d(b) \in M$. This defines a map $d : B \to M$, which is R-linear because both f and u are. It remains to show that d is a derivation, but this follows immediately from the fact that f is compatible with multiplication (it's the same calculation as in the previous paragraph).

(d). Note that $\mathscr{C}$ has all finite products: the product of two objects $B \to A$ and $B' \to A$ of $\mathscr{C}$ is their fiber product $B \times_A B'$ over A (with the canonical map to A), and the final object of $\mathscr{C}$ is $A \overset{\mathrm{id}_A}{\to} A$. Also, if M and M' are A-modules, then $(A \oplus M) \times_A (A \oplus M') = A \oplus (M \times M')$.

Let M be a A-module. By question (c), the presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, A \oplus M)$ on $\mathscr{C}$ is a presheaf in abelian groups, so, by the Yoneda lemma I.3.2.3), there exist morphisms $m : (A \oplus (M \times M)) \to A \oplus M$ and $e : A \to A \oplus M$ that make $(A \oplus M, m, e)$ an abelian group in $\mathscr{C}$.

We calculate m and e. The morphism $e : A \to A \oplus M$ corresponds to the zero derivation in $\mathrm{Der}_R(A, M)$ by the isomorphism of (c), so we have $e(a) = a + 0$ for every $a \in A$. Also, as m is a morphism of $\mathscr{C}$, there exists a family of maps $m_a : M \times M \to M$, for $a \in A$, such that, for all $a \in A$ and $x, y \in M$, we have

$$m(a + (x, y)) = a + m_a(x, y).$$

So we have to calculate the family of maps $m_a : M \times M \to M$. Let $u : B \to A$ be an object of $\mathscr{C}$, let $d, d' \in \mathrm{Der}_R(B, M)$, and let $f, f' : B \to A \oplus M$ be the corresponding morphisms of $\mathscr{C}$. Then $m \circ (f \times f') : B \to A \oplus M$ is the morphisms of $\mathscr{C}$ corresponding to $d + d' \in \mathrm{Der}_R(B, M)$. So, for every $b \in B$, we have

$$m(f(b), f'(b)) = m(u(b)+(d(b), d'(b))) = u(b)+m_{u(b)}(d(b), d'(b)) = u(b)+d(b)+d'(b),$$

that is,

$$m_{u(b)}(d(b), d'(b)) = d(b) + d'(b).$$

Now let $x, y \in M$ and $a \in A$. By question (c) of problem **??**, if we take $B = R[T]$, then there exist $d, d' \in \mathrm{Der}_R(B, M)$ such that $d(T) = x$ and $d'(T) = y$. We make B an object of $\mathscr{C}$ by using the morphism of R-algebras $u : B \to A$ sending T to a. Applying the previous identity to these derivations (and taking $b = T$), we get $m_a(x, y) = x + y$. So $m_a : M \times M \to M$ is given by the addition of M for every $a \in A$.

(e). By question (d), if $f : M \to M'$ is a R-linear map, the the morphism of R-algebras $A \oplus M \to A \oplus M'$, $a + x \mapsto a + f(x)$ is also compatible with the abelian group structures on $A \oplus M$ and $A \oplus M'$. This shows that the functor ${}_A\mathbf{Mod} \to \mathscr{C}$, $M \mapsto A \oplus M$ factors through a functor $\Phi : {}_A\mathbf{Mod} \to \mathscr{C}_{\mathrm{ab}}$. We want to show that Φ is an equivalence of categories, so we prove that it is fully faithful and essentially surjective.

First, the functor Φ is clearly faithful. We prove that it is full. Let M and M' be A-modules, and let $g : A \oplus M \to A \oplus M'$ be a morphism of $\mathscr{C}_{\mathrm{ab}}$. As g is a morphism of R-algebras over A, there exists a family of maps $g_a : M \to M'$, for $a \in A$, such that, for all $a \in A$ and $x \in M$, we have

$$g(a + x) = a + g_a(x).$$

As g is compatible the abelian group structures of $A \oplus M$ and $A \oplus M'$, we have, for all $a \in A$ and $x, y \in M$:

- $g(a + 0) = a$, that is, $g_a(0) = 0$;
- $g(a + (x + y)) = a + (g_a(x) + g_a(y))$, that is, $g_a(x + y) = g_a(x) + g_a(y)$.

Let $a \in A$ and $x \in M$. As $a + x = (a + 0) + (0 + x)$ and g is additive, we get $a + g_a(x) = a + (g_a(0) + g_0(x)) = a + g_0(x)$, hence $g_a(x) = g_0(x)$. Now using the fact that $(a + 0)(1 + x) = a + ax$ and that g is multiplicative, we get $(a + 0)(1 + g_1(x)) = a + ag_1(x) = a + g_a(ax)$; as $g_1 = g_0 = g_a$, this implies that $g_0(ax) = ag_0(x)$. So we have found a A-linear map $g_0 : M \to M'$ such that $g(a + x) = a + g_0(x)$ for all $a \in A$ and $x \in M$. In other words, we have $g = \Phi(g_0)$.

We finally show that Φ is fully faithful. So let $(u : B \to A, m, e)$ be an abelian group in $\mathscr{C}$. In particular, the map $e : A \to B$ is a morphism of R-algebras such that $u \circ e = \mathrm{id}_A$, so we can write $B = A \oplus M$ as a R-module with $M = \operatorname{Ker} u$ an ideal and R-submodule of B, and $e : A \to A \oplus \operatorname{Ker}(u)$ is the map $a \mapsto a + 0$. So we have an isomorphism of R-modules $B \times_A B = A \oplus (M \times M)$, and we can see m as a morphism from $A \oplus (M \times M)$ to $A \oplus M$. As e is a unit for m, for every $a \in A$ and every $x \in M$, the morphism m sends the elements $(e(a), a+x)$ and $(a+x, e(a))$ of $B \times_A B$ to $a+x$, or in other words, we have

$$m(a + (0, x)) = m(a + (x, 0)) = a + x.$$

As moreover m is a morphism of abelian groups, we have, for $a \in A$ and $x, y \in M$,

$$m(a + (x, y)) = m((a + (x, 0)) + (0 + (0, y))) = a + (x + y).$$

To finish the proof, it suffices to show that M is a square-zero ideal of B. Let $x, y \in M$. Then the product of the elements $(0, y)$ and $(1 + x, 1)$ of $B \times_A B$ is $(0, y)$, the morphism m sends $(0, y)$ to $0 + y$ and $(1 + x, 1)$ to $1 + x$, and m is morphism of R-algebras, so $(1 + x)(0 + y) = 0 + y$, which implies that $xy = 0$.

(f). By (e), the forgetful functor $\mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$ is isomorphic to the functor ${}_A\mathbf{Mod} \to \mathscr{C}$, $M \mapsto A \oplus M$. Let $B \to A$ is be object of $\mathscr{C}$ and M be a A-module; we can see M as a B-module via the morphism $B \to A$. By (b) of problem **??** and question (c), we have isomorphisms, functorial in B and M:

$$\begin{aligned} \operatorname{Hom}_{{}_A\mathbf{Mod}}(A \otimes_B \Omega^1_{B/R}, M) &\simeq \operatorname{Hom}_{{}_B\mathbf{Mod}}(\Omega^1_{B/R}, M) \\ &\simeq \operatorname{Der}_R(B, M) \\ &\simeq \operatorname{Hom}_{\mathscr{C}}(B, A \oplus M). \end{aligned}$$

This shows that the functor $\mathscr{C} \to {}_A\mathbf{Mod}$, $(B \to A) \mapsto A \otimes_B \Omega^1_{B/R}$ is left adjoint to the functor ${}_A\mathbf{Mod} \to \mathscr{C}$, $M \mapsto A \oplus M$.

□

A.12 Final problem set

For every category $\mathscr{C}$, we write $\mathrm{s}(\mathscr{C}) = \operatorname{Func}(\Delta^{\mathrm{op}}, \mathscr{C})$ for the category of simplicial objects of $\mathscr{C}$. If X is an object of $\mathscr{C}$, we still denote by X the constant simplicial object with value X, that is, the object $X_\bullet$ of $\mathrm{s}(\mathscr{C})$ defined by $X_n = X$ for every $n \in \mathbb{N}$ and $X_\bullet(\alpha) = \mathrm{id}_X$ for every morphism α of Δ.

If $F : \mathscr{C} \to \mathscr{D}$ is a functor, it induces a functor $\mathrm{s}(F) : \mathrm{s}(\mathscr{C}) \to \mathrm{s}(\mathscr{D})$ by composition (an object of $\mathrm{s}(\mathscr{C})$ is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathscr{C}$, and we take $\mathrm{s}(F)(X_\bullet) = F \circ X_\bullet$). We often write F instead of $\mathrm{s}(F)$.

Let R be a commutative ring and A be a commutative R-algebra. We consider the slice category $R-\mathbf{CAlg}/A$ of commutative R-algebras with a morphism to A (see Definition I.2.2.6), we denote by $\mathscr{C}$ the category $\mathrm{s}(R - \mathbf{CAlg}/A)$.

Remember that the category $\mathbf{s}(R - \mathbf{CAlg})$ of simplicial R-algebras has a model structure induced from that of the category $\mathbf{s}({}_R\mathbf{Mod})$ of simplicial R-modules via the forgetful functor and its left adjoint Sym_R (where $\mathrm{Sym}_R : {}_R\mathbf{Mod} \to R - \mathbf{CAlg}$ sends a R-module to the symmetric R-algebra over it), where the model structure on $\mathbf{s}({}_R\mathbf{Mod})$ comes from the projective model structure on $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ via the Dold-Kan equivalence $N : \mathbf{s}({}_R\mathbf{Mod}) \to \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$.

This induces a model structure on $\mathscr{C}$, for which a morphism of $\mathscr{C}$ is a weak equivalence (resp. a fibration, resp. a cofibration) if and only if its image in $\mathbf{s}(R - \mathbf{CAlg})$ is. This is an easy fact and you don't need to prove it (see Proposition 1.1.8 of Hovey's book and the remark following it).

More generally, let $R_\bullet$ be a simplicial commutative ring. The category${}_{R_\bullet}\mathbf{Mod}$ of $R_\bullet$-modules is the category whose objects are simplicial abelian groups $M_\bullet$ such that:

- M_n is a R_n-module for every $n \in \mathbb{N}$;
- for every $\alpha \in \mathrm{Hom}_\Delta([n],[m])$, the morphism $\alpha^* : M_m \to M_n$ is R_m-linear, where R_m acts on M_n via $\alpha^* : R_m \to R_n$;

and whose morphisms are morphisms of simplicial abelian groups $f : M_\bullet \to N_\bullet$ such that f_n is R_n-linear for every $n \in \mathbb{N}$. We define similarly the category $R_\bullet - \mathbf{CAlg}$ of commutative $R_\bullet$-algebras. Note that, if $R_\bullet$ is the constant simplicial ring R, then we have ${}_{R_\bullet}\mathbf{Mod} = \mathbf{s}({}_R\mathbf{Mod})$ and $R_\bullet - \mathbf{CAlg} = \mathbf{s}(R - \mathbf{CAlg})$. These categories have model structures, that are induced via the forgetful functors into $\mathbf{sAb}$ from the model structure on this last category (see Theorem); in particular, a morphism in ${}_{R_\bullet}\mathbf{Mod}$ or $R_\bullet - \mathbf{CAlg}$ is a weak equivalence (resp. a fibration) if and only if the underlying morphism of simplicial abelian groups is a weak equivalence (resp a fibration).

If $B_\bullet$, $B'_\bullet$ and $B''_\bullet$ are objects of $\mathbf{s}(R-\mathbf{CAlg})$, we denote by $B_\bullet \otimes_{B'_\bullet} B''_\bullet$ the simplicial commutative R-algebra $C_\bullet$ defined by $C_n = B_n \otimes_{B'_n} B''_n$ for every $n \in \mathbb{N}$, and $C_\bullet(\alpha) = B_\bullet(\alpha) \otimes B''_\bullet(\alpha)$ for every morphism α of Δ^{op}.

You may admit the following description of cofibrations in $\mathbf{s}(R - \mathbf{CAlg})$:[34]

- A morphism of $\mathbf{s}(R-\mathbf{CAlg})$ is called *free* if it is of the form $A_\bullet \to A_\bullet \otimes_R B_\bullet$, $a \mapsto a \otimes 1$, and if there exists a family of projective R-modules $(P_k)_{k\geq 0}$ such that:
  (a) For every $n \geq 0$, we have
  $$B_n = \bigotimes_{\alpha:[n]\twoheadrightarrow[k]} \mathrm{Sym}_R(P_k),$$
  where the tensor product is over R and we take it over all surjective nondecreasing maps $\alpha : [n] \to [k]$.
  (b) For every surjective nondecreasing maps $f : [n] \to [m]$ and $\alpha_0 : [m] \to [k]$, for every $x \in \mathrm{Sym}_R(P_k)$, the morphism $f^* : B_m \to B_n$ sends a pure tensor

[34] It is not very hard to prove, but a bit tedious. See for example Proposition 4.21 of the paper [4]

$\bigotimes_{\alpha:[m]\twoheadrightarrow[l]} x_\alpha \in B_m$, with $x_\alpha \in \mathrm{Sym}_R(P_k)$ equal to x for $\alpha = \alpha_0$ and to 1 otherwise, to the pure tensor $\bigotimes_{\beta:[n]\twoheadrightarrow[k]} y_\beta \in B_n$, with $y_\beta = x$ if $\beta = \alpha_0 \circ f$ and 1 otherwise.

- A morphism of $\mathbf{s}(R-\mathbf{CAlg})$ is a cofibration if and only if it is a retract of a free morphism.

In problem 5 of problem set 11, we have defined a functor ${}_A\mathbf{Mod} \to R - \mathbf{CAlg}/A$, $M \mapsto A \oplus M$. This induces a functor $\mathbf{s}({}_A\mathbf{Mod}) \to \mathscr{C}$, that we denote by $M_\bullet \mapsto A \oplus M_\bullet$. We denote the functor $\mathbf{s}(\Omega^1_{\cdot/A}) : \mathscr{C} \to \mathbf{s}({}_A\mathbf{Mod})$ by $B_\bullet \mapsto \Omega^1_{B_\bullet/A}$. The following statements are easy generalizations of the results of problem 5 of problem set 11, and you may use them without proving them:

(a) The functor $M_\bullet \mapsto A \oplus M_\bullet$ induces an equivalence of categories from $\mathbf{s}({}_A\mathbf{Mod})$ to $\mathscr{C}_{\mathrm{ab}}$.

(b) Using the equivalence of (a) to identify $\mathscr{C}_{\mathrm{ab}}$ and $\mathbf{s}({}_A\mathbf{Mod})$, the functor $B_\bullet \mapsto A \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}$ is an abelianization functor for $\mathscr{C}$.

In fact, we could generalize further to the case where $R_\bullet$ is a simplicial commutative ring and $A_\bullet$ is a simplicial $R_\bullet$-algebra; then we get that the category of abelian groups in $R_\bullet - \mathbf{CAlg}/A_\bullet$ is equivalent to ${}_{A_\bullet}\mathbf{Mod}$, and the functor $B_\bullet \mapsto A_\bullet \otimes_{B_\bullet} \Omega^1_{B_\bullet/R_\bullet}$ is ab abelianization functor (where $\Omega^1_{B_\bullet/R_\bullet}$ is the $B_\bullet$-module equal to $\Omega^1_{B_n/R_n}$ in degree n).

We denote the left derived functor of $B_\bullet \mapsto A \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}$ by $L\mathbf{Ab} : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathbf{s}({}_A\mathbf{Mod})) \simeq \mathrm{D}^{\leq 0}({}_A\mathbf{Mod})$. (You don't need to prove that the left derived functor exists. This follows immediately from the easy fact that the functor $M_\bullet \mapsto A \oplus M_\bullet$ preserves weak equivalences and fibrations.) By definition of the left derived functor, for every object $B_\bullet$ of $\mathscr{C}$, we have a morphism $L\mathbf{Ab}(B_\bullet) \to A \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}$ in $\mathrm{D}^{\leq 0}({}_A\mathbf{Mod})$.

The cotangent complex of A over R is the simplicial A-module $\mathbb{L}_{A/R} = L\mathbf{Ab}(A)$. In other words, to calculate $\mathbb{L}_{A/R}$, we take a cofibrant replacement $A_\bullet \to A$ of the constant object A of $\mathscr{C}$, and then

$$\mathbb{L}_{A/R} = A \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}.$$

(1). Let M be a R-module. Show that the image of the constant simplicial R-module M by the Dold-Kan equivalence N is the R-module M, seen as a complex concentrated in degree 0. (In other words, our two embeddings of ${}_R\mathbf{Mod}$ into $\mathbf{s}({}_R\mathbf{Mod})$ and $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ are compatible.)

(2). Show that the forgetful functor $\mathbf{s}(R - \mathbf{CAlg}) \to \mathbf{s}({}_R\mathbf{Mod})$ sends cofibrant objects to cofibrant objects and cofibrant resolutions to cofibrant resolutions.

(3). Let $A_\bullet$ be a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$. Show that $\Omega^1_{A_n/R}$ is a projective A_n-module for every $n \in \mathbb{N}$.

(4). If $P_\bullet$ is a simplicial R-module such that P_n is projective for every $n \in \mathbb{N}$, show that $\mathrm{Sym}_R(P_\bullet)$ is a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$.

(5). If P is a projective R-module and $A = \mathrm{Sym}_R(P)$, show that the canonical morphism $\mathbb{L}_{A/R} \to \Omega^1_{A/R}$ is an isomorphism.

(6). Show that the functor $(\cdot) \otimes_R (\cdot) : \mathbf{s}({}_R\mathbf{Mod}) \times \mathbf{s}({}_R\mathbf{Mod}) \to \mathbf{s}({}_R\mathbf{Mod})$ sends weak equivalences between cofibrant objects to weak equivalences. [35] In particular, it has a left derived functor, which we will denote by $(\cdot) \otimes^L_R (\cdot)$. (Hint: Write $f \times g$ as $(f \times \mathrm{id}) \circ (\mathrm{id} \times g)$.)

(7). Show that the functor $(\cdot) \otimes_R (\cdot) : \mathbf{s}(R - \mathbf{CAlg}) \times \mathbf{s}(R - \mathbf{CAlg}) \to \mathbf{s}(R - \mathbf{CAlg})$ sends weak equivalences between cofibrant objects to weak equivalences, and that, if we denote its left derived functor by $(\cdot) \otimes^L_R (\cdot)$, the following diagram commutes up to an isomorphism of functors:

$$\begin{tikzcd}
\mathrm{Ho}(\mathbf{s}(R - \mathbf{CAlg})) \times \mathrm{Ho}(\mathbf{s}(R - \mathbf{CAlg})) \arrow[r, "(\cdot)\otimes^L_R(\cdot)"] \arrow[d] & \mathrm{Ho}(\mathbf{s}(R - \mathbf{CAlg})) \arrow[d] \\
\mathrm{Ho}(\mathbf{s}({}_R\mathbf{Mod})) \times \mathrm{Ho}(\mathbf{s}({}_R\mathbf{Mod})) \arrow[r, "(\cdot)\otimes^L_R(\cdot)"'] & \mathrm{Ho}(\mathbf{s}({}_R\mathbf{Mod}))
\end{tikzcd}$$

where the vertical arrows are forgetful functors.

(8). Derived tensor products are associative and commutative (just like their underived versions), this is not hard and you can use it without proof.

(9). Consider a commutative diagram of commutative rings:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

(i) Show that the maps $\mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$, $d \mapsto d \circ f$, for every B-module M, induce a canonical morphism $B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/S}$.

Let $A_\bullet \to A$ be a cofibrant replacement in $\mathbf{s}(R - \mathbf{CAlg})$, and factor the morphism $S \otimes_R A_\bullet \to S \otimes_R A \to B$ as $S \otimes_R A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} B$, where i is a cofibration and p is an acyclic fibration (in $\mathbf{s}(R - \mathbf{CAlg})$):

$$\begin{tikzcd}
A_\bullet \arrow[r] \arrow[d] & S \otimes_R A_\bullet \arrow[r, "i"] \arrow[d] & B_\bullet \arrow[d, "p"] \\
A \arrow[r] & B \arrow[r, equal] & B
\end{tikzcd}$$

(ii) Show that p is a cofibrant replacement in $\mathbf{s}(S - \mathbf{CAlg})$.

[35] The model structure on a product of model categories is defined in the obvious way, i.e. a product $f \times g$ of morphisms is a fibration (resp. cofibration, resp. weak equivalence) if and only if both f and g are. See Example 1.1.6 of Hovey's book.

(iii) Construct a natural morphism $B \otimes_A^L \mathbb{L}_{A/R} \to \mathbb{L}_{B/S}$. (This morphism is of course independent on the choices, but you can skip the verification.)

(10). If A and B are commutative R-algebras, and if $C = A \otimes_R B$, show the canonical morphism

$$C \otimes_A \Omega^1_{A/R} \oplus C \otimes_B \Omega^1_{B/R} \to \Omega^1_{C/R}$$

is an isomorphism. (You can prove for example that both C-modules represent the same functor.)

(11). Let A and B be commutative R-algebras such that $\mathrm{Tor}_i^R(A,B) = 0$ for every $i \geq 1$. If $A_\bullet \to A$, $B_\bullet \to B$ are cofibrant replacements in $\mathbf{s}(R - \mathbf{CAlg})$, show that $A_\bullet \otimes_R B_\bullet \to A \otimes_R B$ is a cofibrant replacement.

(12). Let A and B be commutative R-algebras such that $\mathrm{Tor}_i^R(A,B) = 0$ for every $i \geq 1$, and let $C = A \otimes_R B$. Show that the canonical morphism

$$C \otimes_A^L \mathbb{L}_{A/R} \oplus C \otimes_B^L \mathbb{L}_{B/R} \to \mathbb{L}_{C/R}$$

is an isomorphism.

(13). If A and S are commutative R-algebras such that $\mathrm{Tor}_i^R(A,S) = 0$ for $i \geq 1$, and if $A_\bullet \to A$ is a cofibrant replacement in $\mathbf{s}(R - \mathbf{CAlg})$, show that $S \otimes_R A_\bullet \to S \otimes_R A$ is a cofibrant replacement in $\mathbf{s}(S - \mathbf{CAlg})$.

(14). Let A and S be commutative R-algebras such that $\mathrm{Tor}_i^R(A,S) = 0$ for $i \geq 1$, and let $B = S \otimes_R A$. Show that the canonical morphism

$$B \otimes_A^L \mathbb{L}_{A/R} \to \mathbb{L}_{B/S}$$

is an isomorphism.

(15). Let $R \to A \xrightarrow{f} B$ be morphisms of commutative rings.

(i) Show that the sequence

$$B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

is exact.

(ii) If there exists a morphism of R-algebras $g : B \to A$ such that $g \circ f = \mathrm{id}_A$, show that the sequence

$$0 \to B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

is exact.

(16). We say that a sequence $M_\bullet \to M'_\bullet \to M''_\bullet$ in $\mathbf{s}({}_R\mathbf{Mod})$ is a *cofiber sequence* if its image by the Dold-Kan equivalence $N : \mathbf{s}({}_R\mathbf{Mod}) \to \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ extends to a distinguished triangle in $K({}_R\mathbf{Mod})$. (In particular, it induces a long exact sequence of cohomology groups.)

Let $R \to A \xrightarrow{f} B$ be morphisms of commutative rings. The goal of this question is to show that
$$B \otimes_A^L \mathbb{L}_{A/R} \to \mathbb{L}_{B/R} \to \mathbb{L}_{B/A}$$
comes from a cofiber sequence in $\mathbf{s}({}_B\mathbf{Mod})$.

Let $A_\bullet \to A$ be a cofibrant replacement in $\mathbf{s}(R-\mathbf{CAlg})$, and factor the morphism $A_\bullet \to A \xrightarrow{f} B$ as $A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{p} B$, where i is a cofibration and p is an acyclic fibration in $\mathbf{s}(R-\mathbf{CAlg})$.

(i) Show that, for every $n \in \mathbb{N}$, the sequence
$$0 \to B \otimes_{A_n} \Omega^1_{A_n/R} \to B \otimes_{B_n} \Omega^1_{B_n/R} \to B \otimes_{B_n} \Omega^1_{B_n/A_n} \to 0$$
is exact.

(ii) Show that the morphism $A \otimes_{A_\bullet} B_\bullet \to B$ is a cofibrant replacement in $\mathbf{s}(A-\mathbf{CAlg})$.

(iii) Show that
$$B \otimes_A^L \mathbb{L}_{A/R} \to \mathbb{L}_{B/R} \to \mathbb{L}_{B/A}$$
comes from a cofiber sequence in $\mathbf{s}({}_B\mathbf{Mod})$.

(17). Let $R \to S \to A$ be morphisms of commutative rings, and suppose that the morphism $A \otimes_R^L S \to A$ is an isomorphism in $\mathrm{Ho}(\mathbf{s}(A-\mathbf{CAlg}))$. (In particular, we have $\mathrm{Tor}_i^R(A,S) = 0$ for $i \geq 1$.) Show that the canonical morphism $\mathbb{L}_{A/R} \to \mathbb{L}_{A/S}$ is an isomorphism in $\mathrm{Ho}(\mathbf{s}({}_A\mathbf{Mod}))$.

(18). Let $R \to A$ be a morphism of commutative rings such that $A \otimes_R^L A \to A$ is an isomorphism. Show that $\mathbb{L}_{A/R} = 0$.

(19). Let $R \to A$ be a morphism of commutative rings such that $\mathrm{Tor}_i^R(A,A) = 0$ for every $i \geq 1$ and that $\mathbb{L}_{A/A\otimes_R A} = 0$. Show that $\mathbb{L}_{A/R} = 0$.

(20). Let $R \to A$ be a morphism of commutative rings such that the morphisms $R \to A$ and $A \otimes_R A \to A$ are both flat. [36] Show that $\mathbb{L}_{A/R} = 0$.

(21). Let R be a commutative ring. Show that, for every R-module M and every $i \in \mathbb{N}$, the endofunctor $\mathrm{Tor}_i^R(M, \cdot)$ of ${}_R\mathbf{Mod}$ commutes with filtrant colimits.

(22). We fix a prime number p. Remember that a commutative ring R of characteristic p (i.e. such that $p \cdot 1_R = 0$) is called *perfect* if the endomorphism $x \mapsto x^p$ of R is an isomorphism. We then denote its inverse by $x \mapsto x^{1/p}$.

For example, if R is perfect and I is a set, then the R-algebra
$$R[X_i^{1/p^\infty},\ i \in I] = \bigcup_{n \geq 1} R[X_i^{1/p^n},\ i \in I]$$

[36] Such a morphism is called *weakly étale*. For example, an étale morphism is weakly étale.

is perfect.

Let $R \to A$ and $R \to B$ be morphisms of commutative rings of characteristic p, with R, A and B perfect.

(i) Show that $A \otimes_R B$ is perfect.

The goal of the rest of this question is to show that $\mathrm{Tor}_i^R(A, B) = 0$ if $i \geq 1$.

(ii) Show that there exists a set I and an ideal $\mathfrak{a}$ of $S = R[X_i^{1/p^\infty}, \ i \in I]$ such that $A \simeq S/\mathfrak{a}$.

(iii) Show that it suffices to prove that $\mathrm{Tor}_i^R(S, B) = 0$ for $i \geq 1$ and that $\mathrm{Tor}_i^S(A, S \otimes_R B) = 0$ for $i \geq 1$.

(iv) Show that $\mathrm{Tor}_i^R(S, B) = 0$ for $i \geq 1$.

By (i)-(iv), we may now assume that the morphism $R \to A$ is surjective. We denote its kernel by $\mathfrak{a}$. Remember that the goal is to prove that $\mathrm{Tor}_i^R(A, B) = 0$ for $i \geq 1$, or, in other words, that the canonical morphism $A \otimes_R^L B \to A \otimes_R B$ is an isomorphism (in $\mathrm{Ho}(\mathbf{s}({}_R\mathbf{Mod}))$).

(v) Show that we may assume that there exist $f_1, \ldots, f_r \in R$ such that $\mathfrak{a} = \bigcup_{n \geq 1}(f_1^{1/p^n}, \ldots, f_r^{1/p^n})$.

(vi) Show that the result follows from the case $r = 1$.

We now assume that there exists $f \in R$ such that $\mathfrak{a} = \bigcup_{n \geq 1}(f^{1/p^n})$.

We write $M_n = R$ and $N_n = B$ for every $n \geq 1$. For every $n \geq 1$, let $u_n : M_n \to M_{n+1}$ (resp. $v_n : N_n \to N_{n+1}$) be the multiplication by $f^{1/p^n - 1/p^{n+1}}$. Let M (resp. N) be the colimit of the functor $\mathbb{N}_{\geq 1} \to {}_R\mathbf{Mod}$ given by $M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \ldots$ (resp. $N_1 \xrightarrow{v_1} N_2 \xrightarrow{v_2} \ldots$), and consider the morphism $\gamma : M \to \mathfrak{a}$ (resp. $\delta : N \to \mathfrak{a}B$) that is given by multiplication by f^{1/p^n} on M_n (res. N_n).

(vii) Show that γ and δ are isomorphisms.

(viii) Show that the canonical morphism $\mathfrak{a} \otimes_R B \to \mathfrak{a}B$ is an isomorphism.

(ix) Show that the canonical morphism $A \otimes_R^L B \to A \otimes_R B$ is an isomorphism.

(23). The goal of this question is to show that, if $A \to B$ is a morphism of commutative rings of characteristic p with B and A perfect, then $\mathbb{L}_{B/A} = 0$.

(i) Consider the canonical morphism $R = B \otimes_A B \to B$. Show that $B \otimes_R^L B \to B$ is an isomorphism (in $\mathrm{Ho}(\mathbf{s}({}_B\mathbf{Mod}))$).

(ii) Deduce that $\mathbb{L}_{B/A} = 0$.

Solution. We first collect some useful results:

(A) *A weak equivalence in $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ is an acyclic fibration if and only if it is surjective in every degree.*

A morphism that is surjective in every degree is clearly a fibration. Conversely, let $f : X \to Y$ be an acyclic fibration. As f is a fibration, we know that f^n is surjective for $n \leq -1$. Let $y \in Y^0$. As $\mathrm{H}^0(f) : \mathrm{Coker}(d_X^{-1}) \to \mathrm{Coker}(d_Y^{-1})$ is an isomorphism, there exist $x \in X^0$ and $y' \in Y^{-1}$ such that $y = f^0(x)+d_Y^{-1}(y')$. As f^{-1} is surjective, there exists $x' \in X^{-1}$ such that $f^{-1}(x') = y'$. Then $y = f^0(x) + d_Y^{-1}(f^{-1}(x')) = f^0(x + d_X^{-1}(x'))$. This shows that f^0 is surjective.

(B) *A morphism $i : X \to Y$ of $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ is a cofibration if and only if i^n is injective with projective cokernel for every $n \in \mathbb{Z}$. In particular, an object of $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ is cofibrant if and only if it is a complex of projective R-modules.*

Suppose that i^n is injective with projective cokernel for every $n \in \mathbb{Z}$. In question (n) of problem A.10.2, we proved that i has the left lifting property relatively to any quasi-isomorphism f of $\mathcal{C}({}_R\mathbf{Mod})$ that is surjective in every degree, and by (A), these morphisms are exactly the acyclic fibrations. So i is a cofibration. Conversely, suppose that i is a cofibration. To show that i^n is injective with projective cokernel for every $n \in \mathbb{Z}$, it suffices, by question (n) of problem A.10.2, to show that i is still a cofibration in $\mathcal{C}^{-}({}_R\mathbf{Mod})$. In other words, we want to show that the forgetful functor $\mathrm{For} : \mathcal{C}^{\leq 0}({}_R\mathbf{Mod}) \to \mathcal{C}^{-}({}_R\mathbf{Mod})$ preserves cofibrations. Note that For has a right adjoint, the truncation function $\tau^{\leq 0}$, and that $\tau^{\leq 0}$ preserves fibrations and weak equivalences. So For preserves cofibrations by Corollary VI.4.2.3.

(C) *An object $A_\bullet$ of $\mathbf{s}({}_R\mathbf{Mod})$ is cofibrant if and only if A_n is a projective R-module for every $n \in \mathbb{N}$.*

Let $X \in \mathrm{Ob}(\mathcal{C}^{\leq 0}({}_R\mathbf{Mod}))$ be the image of $A_\bullet$ by the Dold-Kan equivalence. It suffices to prove that X^n is a projective R-module for every $n \in \mathbb{Z}$ if and only if A_n is a projective R-module for every $n \in \mathbb{N}$. If all the X^{-n} are projective, then so are all the A_n, because A_n is isomorphic to $\bigoplus_{0 \leq k \leq n}(X^{-k})^{\binom{n}{k}}$. Now suppose that all the A_n are projective. By the description of the Dold-Kan equivalence in the solution of question (p) of problem A.10.1, and by question (f) of that same problem, the R-module X^{-n} is a direct summand of the R-module A_n for every $n \in \mathbb{N}$, so it is projective.

(1). We use the formula for the equivalence $N : \mathbf{s}({}_R\mathbf{Mod}) \to \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ given in question (q) of problem A.10.1.

Let M be a R-module, and let $X_\bullet$ be the constant simplicial R-module M. This means that $X_n = M$ for every $n \in \mathbb{N}$ and that $X_\bullet(\alpha) = \mathrm{id}_M$ for every morphism α of Δ. If $n = 0$, then $N(X_\bullet)^{-n} = X_0 = M$. If $n \geq 1$, then $N(X_\bullet)^{-n} = \bigcap_{i=1}^{n} \mathrm{Ker}(\mathrm{id}_M) = 0$. This gives the result.

(2). Let $A_\bullet$ be a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$. We want to show that it is cofibrant as an object of $\mathbf{s}({}_R\mathbf{Mod})$. By (C), it suffices to prove that A_n is a projective R-module for every

$n \in \mathbb{N}$.

As cofibrant objects are stable by retracts (because cofibrations are), we may assume that $A_\bullet$ is free over the initial object R of $\mathbf{s}(R - \mathbf{CAlg})$. In particular, there exists a family of projective R-modules $(P_k)_{k \geq 0}$ such that, for every $n \geq 0$, we have

$$A_n = \bigotimes_{\alpha:[n] \twoheadrightarrow [k]} \mathrm{Sym}_R(P_k).$$

As a (finite) tensor product of projective R-modules is projective, it suffices to show that, for every projective R-module P, the symmetric algebra $\mathrm{Sym}_R(P)$ is projective as a R-module. As $\mathrm{Sym}_R(P) = \bigoplus_{d \geq 0} \mathrm{Sym}^d_R(P)$, it suffices to show that $\mathrm{Sym}^d_R(P)$ is projective for every $d \in \mathbb{N}$. Let Q be another R-module such that $P \oplus Q$ is free, and let $d \in \mathbb{N}$. Then we have

$$\mathrm{Sym}^d_R(P \oplus Q) \simeq \bigoplus_{a+b=d} \mathrm{Sym}^a_R(P) \otimes_R \mathrm{Sym}^b_R(Q),$$

so in particular $\mathrm{Sym}^d_R(P) = \mathrm{Sym}^d_R(P) \otimes_R \mathrm{Sym}^0_R(Q)$ (remember that $\mathrm{Sym}^0_R(M) = R$ for every R-module M) is a direct summand of $\mathrm{Sym}^d_R(P \oplus Q)$. But, as $P \oplus Q$ is a free R-module, so is $\mathrm{Sym}^d_R(P \oplus Q)$, and so $\mathrm{Sym}^d_R(P)$ is a projective R-module.

Now let $f : A_\bullet \to B_\bullet$ be a cofibrant replacement of $B_\bullet$ in $\mathbf{s}(R - \mathbf{CAlg})$. This means that $A_\bullet$ is cofibrant and that f is an acyclic fibration in $\mathbf{s}(R - \mathbf{CAlg})$. We have just proved that $A_\bullet$ is still cofibrant as an object of $\mathbf{s}({}_R\mathbf{Mod})$, and we know that the forgetful functor $\mathbf{s}(R - \mathbf{CAlg}) \to \mathbf{s}({}_R\mathbf{Mod})$ preserves weak equivalences and fibrations by construction of the model structure on $\mathbf{s}(R - \mathbf{CAlg})$. So f is still a cofibrant replacement in $\mathbf{s}({}_R\mathbf{Mod})$.

(3). First we prove the following statement: If A and B are two commutative R-algebras such that A is a retract of B (i.e. if we have two morphisms of R-algebras $A \to B$ and $B \to A$ whose composition is id_A), then we have an isomorphism of A-modules $\Omega^1_{A/R} \simeq A \otimes_B \Omega^1_{B/R}$; in particular, if $\Omega^1_{B/R}$ is a projective B-module, then $\Omega^1_{A/R}$ is a projective A-module. Indeed, applying the result of question (15)(ii) to the sequence $A \to B \to A$, we get an exact sequence

$$0 \to A \otimes_B \Omega^1_{B/A} \to \Omega^1_{A/A} \to \Omega^1_{A/B} \to 0.$$

As $\Omega^1_{A/A} = 0$, this shows that $A \otimes_B \Omega^1_{B/A} = 0$. Now applying the result of question (15)(ii) again, this times to the sequence $R \to B \to A$, we get an exact sequence

$$0 \to B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0.$$

Taking the tensor product by A over B, and using the fact that $A \otimes_B \Omega^1_{B/A} = 0$, we get the desired result.

Now let $A_\bullet$ be a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$. There exists a free object $B_\bullet$ of $\mathbf{s}(R - \mathbf{CAlg})$ such that $A_\bullet$ is a retract of $B_\bullet$ and then, for every $n \in \mathbb{N}$, the R-algebra

A_n is a retract of B_n. By the previous paragraph, it suffices to prove that $\Omega^1_{B_n/R}$ is a projective B_n-module for every $n \in \mathbb{N}$. In other words, we may assume that $A_\bullet$ is free over R. Then there exists a family of projective R-modules $(P_k)_{k\geq 0}$ such that, for every $n \geq 0$, we have

$$A_n = \bigotimes_{\alpha:[n]\twoheadrightarrow[k]} \mathrm{Sym}_R(P_k).$$

Fix $n \in \mathbb{N}$. We want to show that $\Omega^1_{A_n/R}$ is a projective A_n-module. By question (10), it suffices to prove that, if P is a projective R-module and $A = \mathrm{Sym}_R(P)$, then $\Omega^1_{A/R}$ is a projective A-module. Let Q be another R-module such that $P \oplus Q$ is a free R-module. We claim that the R-algebra $\mathrm{Sym}_R(P)$ is a retract of $\mathrm{Sym}_R(P \oplus Q)$; as $\Omega^1_{\mathrm{Sym}_R(P\oplus Q)/R}$ is a free $\mathrm{Sym}_R(P \oplus Q)$-module by question (c) of problem **??**, this finishes the proof by the first paragraph. Remember that, for every R-module M and every commutative R-algebra B, restriction along the map $M = \mathrm{Sym}^1_R(M) \subset \mathrm{Sym}_R(M)$ induces a canonical bijection

$$\mathrm{Hom}_{R-\mathbf{CAlg}}(\mathrm{Sym}_R(M), B) = \mathrm{Hom}_{_R\mathbf{Mod}}(M, B).$$

Let $s : \mathrm{Sym}_R(P) \to \mathrm{Sym}_R(P \oplus Q)$ be the R-algebra morphism corresponding to the R-module morphism $P \to P \oplus Q = \mathrm{Sym}^1_R(P \oplus Q) \subset \mathrm{Sym}_R(P \oplus Q)$, where $P \to P \oplus Q$ is the obvious inclusion, and let $r : \mathrm{Sym}_R(P \oplus Q) \to \mathrm{Sym}_R(P)$ be the R-algebra morphism corresponding to the R-module morphism $P \oplus Q \to P = \mathrm{Sym}^1_R(P) \subset \mathrm{Sym}_R(P)$, where $P \oplus Q \to P$ is the obvious projection. Then $r \circ s : \mathrm{Sym}_R(P) \to \mathrm{Sym}_R(P)$ is equal to id_P on $P = \mathrm{Sym}^1(P)$, so $r \circ s = \mathrm{id}_{\mathrm{Sym}_R(P)}$. This shows that $\mathrm{Sym}_R(P)$ is a retract of $\mathrm{Sym}_R(P \oplus Q)$.

(4). By (C), we know that $P_\bullet$ is a cofibrant object of $\mathbf{s}(_R\mathbf{Mod})$. As $\mathrm{Sym}_R : \mathbf{s}(_R\mathbf{Mod}) \to \mathbf{s}(R-\mathbf{CAlg})$ is left adjoint to the forgetful functor, and as the model structure on $\mathbf{s}(R-\mathbf{CAlg})$ is transported from that of $\mathbf{s}(_R\mathbf{Mod})$ using this adjunction, the functor Sym_R preserves cofibrations. In particular, it sends cofibrant objects to cofibrant objects (note that, as a left adjoint, the functor Sym_R preserves initial objects), and so $\mathrm{Sym}_R(P_\bullet)$ is a cofibrant object of $\mathbf{s}(R-\mathbf{CAlg})$.

(5). By question (4), the constant simplicial R-algebra A is a cofibrant object of $\mathbf{s}(R-\mathbf{CAlg})$, so id_A is a cofibrant replacement of A. So

$$\mathbb{L}_{A/R} = A \otimes_A \Omega^1_{A/R} = \Omega^1_{A/R}.$$

(6). Let u be a morphism of $\mathbf{s}(_R\mathbf{Mod})\times\mathbf{s}(_R\mathbf{Mod})$ that is a weak equivalence between cofibrant objects; we want to show that the image of u by the functor $(\cdot)\times_R(\cdot)$ is a weak equivalence. We can write $u = f\times g$ with f and g morphisms of $\mathbf{s}(_R\mathbf{Mod})$. Then $u = (f\times\mathrm{id})\circ(\mathrm{id}\times g)$, so it suffices to treat the case where f or g is an identity morphism.

In fact, we claim that, if f is a weak equivalence in $\mathbf{s}(_R\mathbf{Mod})$ and if $Z_\bullet$ is a cofibrant object, then $f \otimes_R \mathrm{id}_{Z_\bullet}$ and $\mathrm{id}_{Z_\bullet} \otimes_R f$ are weak equivalences. As tensor products are commutative, it suffices to treat the first case. Write $f : X_\bullet \to Y_\bullet$. Let $h : A \to B$ and C be the images of

$f : X_\bullet \to Y_\bullet$ and $Z_\bullet$ by the Dold-Kan equivalence $N : \mathbf{s}({}_R\mathbf{Mod}) \to \mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$. Then h is a quasi-isomorphism, and C is a complexe of projective R-modules by (C) and (B). Also, by the Eilenberg-Zilberg theorem (see Section 8.5 of Weibel's book [15]) and the definition of the model structure on $\mathbf{s}({}_R\mathbf{Mod})$, the morphism $f \otimes_R \mathrm{id}_{Z_\bullet}$ is a weak equivalence if and only if the morphism $v := \mathrm{Tot}(h \otimes_R \mathrm{id}_C) : \mathrm{Tot}(A \otimes_R C) \to \mathrm{Tot}(B \otimes_R C)$ is a quasi-isomorphism (the double functor obtained by tensoring two simple complexes is defined in Example IV.1.6.3(1)).

To prove that v is a quasi-isomorphism, we use the second spectral sequences ${}^{II}E(A \otimes_R C)$ and ${}^{II}E(B \otimes_R C)$ of the double complexes $A \otimes_R C$ and $B \otimes_R C$, see Theorem IV.4.1.7; we know that these spectral sequences converge to the cohomology of the total complexes of these double complexes, because $A \otimes_R C$ and $B \otimes_R C$ are third quadrant double complexes. By the construction of these spectral sequences in Subsection IV.4.2, the morphism of double complexes v induces compatible morphisms $v_r : {}^{II}E_r(A \otimes_R C) \to {}^{II}E_r(B \otimes_R C)$, for every $r \in \mathbb{N} \cup \{\infty\}$, and v_∞ is also compatible with the morphism $\mathrm{H}^*(\mathrm{Tot}(v))$ that v induces on the cohomology of the total complexes.

For every $q \in \mathbb{Z}$, as the R-module C^q is projective, we have $\mathrm{H}^*(A \otimes_R C^q) = \mathrm{H}^*(A) \otimes_R C^q$, and similarly for $B \otimes_R C^q$; hence the morphisms $v_0^{pq} : {}^{II}E_0^{pq}(A \otimes_R C) \to {}^{II}E_0^{pq}(B \otimes_R C)$ induced by v are isomorphisms. As each page of the spectral sequence is the cohomology of the preceding one, this implies that $v_r : {}^{II}E_r(A \otimes_R C) \to {}^{II}E_r(B \otimes_R C)$ is an isomorphism for every $r \in \mathbb{N}$, hence also for $r = \infty$. So, for every $n \in \mathbb{Z}$, there exist filtrations on $\mathrm{H}^n(\mathrm{Tot}(A \otimes_R C))$ and $\mathrm{H}^n(\mathrm{Tot}(B \otimes_R C))$ such that $\mathrm{H}^n(\mathrm{Tot}(v))$ is compatible with these filtrations and induces isomorphisms on their graded quotients; we see easily that this implies that $\mathrm{H}^n(\mathrm{Tot}(v))$ is also an isomorphism.

Note that it also follows from the Eilenberg-Zilberg theorem that the derived tensor products on $\mathbf{s}({}_R\mathbf{Mod})$ and on $\mathcal{C}^{\leq 0}({}_R\mathbf{Mod})$ (see Example V.4.4.12(1) for the second) correspond to each other by the Dold-Kan equivalence; this implies easily that, to calculate $X_\bullet \otimes_R^L Y_\bullet$, it suffices to take a cofibrant replacement of just one of $X_\bullet$ or $Y_\bullet$. (We could also deduce this from the stronger result we proved above, as in Proposition V.3.2.2).

(7). We denote by For the forgetful functor from $\mathbf{s}(R - \mathbf{CAlg})$ to $\mathbf{s}({}_R\mathbf{Mod})$. We know that a morphism f of $\mathbf{s}(R - \mathbf{CAlg})$ is a weak equivalence if and only if $\mathrm{For}(f)$ is a weak equivalence, and we have seen in question (2) that F sends cofibrant objects to cofibrant objects. So the first statement follows immediately from question (6).

The second statement follows from the construction of the left localization from Theorem VI.4.1.1 and from the fact that For sends cofibrant replacements to cofibrant replacements, which we have also proved un question (2).

(9). (i) Consider the functors $F, G : {}_B\mathbf{Mod} \to \mathbf{Ab}$ defined by $F(M) = \mathrm{Der}_S(B, M)$ and $G(M) = \mathrm{Der}_R(A, M)$, where, in the definition of G, the B-module M is seen as a A-module via the map $f : A \to B$. As the map $R \to B$ factors through S, we have $\mathrm{Der}_S(B, M) \subset \mathrm{Der}_R(B, M)$ for every B-module M, so we have a morphism of functors $\alpha : F \to G$ sending $d \in \mathrm{Der}_S(B,M)$ to $d \circ f \in \mathrm{Der}_R(A,M)$. On the other hand, by the universal property of Kähler differentials, we have isomorphisms, functorial in M:

$$\mathrm{Der}_S(B,M) \simeq \mathrm{Hom}_B(\Omega^1_{B/S}, M)$$

and

$$\mathrm{Der}_R(A,M) \simeq \mathrm{Hom}_A(\Omega^1_{A/R}, M) \simeq \mathrm{Hom}_B(B \otimes_A \Omega^1_{A/R}, M).$$

By the Yoneda lemma, the morphism α comes from a unique morphism of B-modules $B \otimes_A \Omega^1 A/R \to \Omega^1_{B/S}$.

(ii) If $M_1, \dots, M_r$ are R-modules, then we have canonical isomorphisms

$$S \otimes_R (M_1 \otimes_R \dots \otimes_R M_r) \simeq (S \otimes_R M_1) \otimes_S \dots \otimes_S (S \otimes_R M_r)$$

and

$$S \otimes_R \mathrm{Sym}_R(M) \simeq \mathrm{Sym}_S(S \otimes_R M)$$

(these follow for example from the universal properties of the tensor product and of the symmetric algebra). Hence it follows immediately from the definition of free morphisms that the functor $S \otimes_R (\cdot) : \mathbf{s}(R - \mathbf{CAlg}) \to \mathbf{s}(S - \mathbf{CAlg})$ sends free morphisms to free morphisms. As functors preserve retracts, these functor also preserves cofibrations. In particular, the simplicial S-algebra $S \otimes_R A_\bullet$ is cofibrant in $\mathbf{s}(S - \mathbf{CAlg})$. As i is a cofibration, this implies that $B_\bullet$ is also cofibrant in $\mathbf{s}(S - \mathbf{CAlg})$. On the other hand, we know that p is an acyclic fibration in $\mathbf{s}(R - \mathbf{CAlg})$, so it is also an acyclic fibration in $\mathbf{s}(S - \mathbf{CAlg})$; this shows that it is a cofibrant replacement.

(iii) We have $\mathbb{L}_{A/R} = A \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$. As $A_\bullet$ is cofibrant, the A_n-module $\Omega^1_{A_n/R}$ is projective for every $n \in \mathbb{N}$, so the A-module $A \otimes_{A_n} \Omega^1_{A_n/R}$ is projective. By (C), this implies that $A \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$ is a cofibrant object of $\mathbf{s}({}_A\mathbf{Mod})$, and so

$$B \otimes_A^L \mathbb{L}_{A/R} = B \otimes_A A \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} = B \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} = B \otimes_{B_\bullet} B_\bullet \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$$

(see the end of the solution of question (6)). On the other hand, by (ii), we have $\mathbb{L}_{B/S} = B \otimes_{B_\bullet} \Omega^1_{B_\bullet/S}$. By (i), we have a natural morphism $B_\bullet \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} \to \Omega^1_{B_\bullet/S}$. We get the derived morphism by applying the functor $B \otimes_{B_\bullet} (\cdot)$.

(10). Let M be a C-module, which we also see as a A-module (resp. a B-module) using the R-algebra morphism $f : A \to C$, $a \mapsto a \otimes 1$ (resp. $g : B \to C$, $b \mapsto 1 \otimes b$). We have

$$\mathrm{Hom}_C(\Omega^1_{C/R}, M) = \mathrm{Der}_R(C, M),$$

$$\mathrm{Hom}_C(C \otimes_A \Omega^1_{A/R}, M) = \mathrm{Hom}_A(\Omega^1_{A/R}, M) = \mathrm{Der}_R(A, M)$$

and

$$\mathrm{Hom}_C(C \otimes_B \Omega^1_{B/R}, M) = \mathrm{Hom}_B(\Omega^1_{B/R}, M) = \mathrm{Der}_R(B, M),$$

and the morphism $\alpha : \mathrm{Der}_R(C, M) \to \mathrm{Der}_R(A, M) \times \mathrm{Der}_R(B, M)$ induced by $C \otimes_A \Omega^1_{A/R} \oplus C \otimes_B \Omega^1_{B/R} \to \Omega^1_{C/R}$ sends $d \in \mathrm{Der}_R(C, M)$ to $(d \circ f, d \circ g)$. By the Yoneda lemma, it suffices to show that α is an isomorphism. We show that α is injective. Let $d \in \mathrm{Der}_R(C, M)$ such that $\alpha(d) = 0$. Then, for every $a \in A$ and every $b \in B$, we have

$$d(a \otimes b) = (a \otimes 1)d(1 \otimes b) + (1 \otimes b)d(a \otimes 1) = (a \otimes 1)(d \circ g)(b) + (1 \otimes b)(d \circ f)(a) = 0.$$

As every element of C is a finite sum of pure tensors, this implies that $d = 0$. Now we show that α is surjective. Let $d_1 \in \mathrm{Der}_R(A, M)$ and $d_2 \in \mathrm{Der}_R(B, M)$. We want to construct $d \in \mathrm{Der}_R(C, M)$ such that $d \circ f = d_1$ and $d \circ g = d_2$. Consider the map $u : A \times B \to M$ defined by $u(a, b) = (a \otimes 1)d_2(b) + (1 \otimes b)d_1(a)$. Then u is R-bilinear, so it comes from a unique R-linear map $d : C \to M$, and we clearly have $d \circ f = d_1$ and $d \circ g = d_2$. To check that d is a derivation, it suffices to verify that, for $a, a' \in A$ and $b, b' \in B$, we have

$$d((a \otimes b)(a' \otimes b')) = (a \otimes b)d(a' \otimes b') + (a' \otimes b')d(a \otimes b).$$

This follows easily from the definition of d.

(11). We first show that $A_\bullet \otimes_R B_\bullet$ is a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$. If $A_\bullet$ (resp. $B_\bullet$) is a retract of $A'_\bullet$ (resp. $B'_\bullet$), then $A_\bullet \otimes_R B_\bullet$ is a retract of $A'_\bullet \otimes_R B'_\bullet$. So we may assume that $A_\bullet$ and $B_\bullet$ are free over R. Choose families of projective R-modules $(P_k)_{k \geq 0}$ and $(Q_k)_{k \geq 0}$ such that:

(a) For every $n \geq 0$, we have

$$A_n = \bigotimes_{\alpha:[n] \twoheadrightarrow [k]} \mathrm{Sym}_R(P_k)$$

and

$$B_n = \bigotimes_{\alpha:[n] \twoheadrightarrow [k]} \mathrm{Sym}_R(Q_k).$$

(b) For every surjective nondecreasing maps $f : [n] \to [m]$ and $\alpha_0 : [m] \to [k]$, for every $x \in \mathrm{Sym}_R(P_k)$, the morphism $f^* : A_m \to A_n$ sends a pure tensor $\bigotimes_{\alpha:[m] \twoheadrightarrow [l]} x_\alpha \in B_m$, with $x_\alpha \in \mathrm{Sym}_R(P_k)$ equal to x for $\alpha = \alpha_0$ and to 1 otherwise, to the pure tensor $\bigotimes_{\beta:[n] \twoheadrightarrow [k]} y_\beta \in B_n$, with $y_\beta = x$ if $\beta = \alpha_0 \circ f$ and 1 otherwise; and similarly for $f^* : B_m \to B_n$.

Then, for every $n \in \mathbb{N}$, we have

$$A_n \otimes_R B_n = \bigotimes_{\alpha:[n] \twoheadrightarrow [k]} \mathrm{Sym}_R(P_k) \otimes_R \mathrm{Sym}_R(Q_k) = \bigotimes_{\alpha:[n] \twoheadrightarrow [k]} \mathrm{Sym}_R(P_k \otimes Q_k),$$

and the morphisms $f^* : A_m \otimes_R B_m \to A_n \otimes_R B_n$ (for $f \in \mathrm{Hom}_\Delta([n], [m])$ surjective) have a description similar to that of (b), with P_k replaced by $P_k \otimes_R Q_k$. As the R-modules $P_k \otimes_R Q_k$ are projective, this shows that $A_\bullet \otimes_R B_\bullet$ is free over R, hence cofibrant.

It remains to show that the morphism $A_\bullet \otimes_R B_\bullet \to A \otimes_R B$ is an acyclic fibration. For this, we can work in the category $\mathbf{s}({}_R\mathbf{Mod})$, and even, by the end of the proof of question (6), apply the Dold-Kan equivalence to reduce the question to a calculation with complexes. So let X and Y be the images of $A_\bullet$ and $B_\bullet$ by the Dold-Kan equivalence. Then $X, Y \in \mathrm{Ob}(\mathcal{C}^{\leq 0}({}_R\mathbf{Mod}))$ are complexes of projective R-modules, and we have quasi-isomorphisms $X \to A$ and $Y \to B$. The morphism $\mathrm{Tot}(X \otimes_R Y) \to A \otimes_R B$ is clearly surjective in degree ≤ -1, hence a fibration, and we want to show that it is a quasi-isomorphism. But we know that $\mathrm{H}^{-n}(\mathrm{Tot}(X \otimes_R Y)) = \mathrm{Tor}_n^R(A, B)$ for every $n \in \mathbb{N}$, so this follows from the hypothesis on A and B.

(12). Let $A_\bullet \to A$ and $B_\bullet \to B$ be cofibrant replacements, and let $C_\bullet = A_\bullet \otimes_R B_\bullet$. We have seen in the proof of question (9)(iii) that $C \otimes_A^L \mathbb{L}_{A/R}$ (resp. $C \otimes_B^L \mathbb{L}_{B/R}$) is representend by the simplicial C-module $C \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$ (resp. $C \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}$). By question (10), the canonical morphism

$$C \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} \oplus C \otimes_{B_\bullet} \Omega^1_{B_\bullet/R} = C \otimes_{C_\bullet} (C_\bullet \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} \oplus C_\bullet \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}) \to C \otimes_{C_\bullet} \Omega^1_{C_\bullet/R}$$

is an isomorphism. But, by question (11), the morphism $C_\bullet := A_\bullet \otimes_R B_\bullet \to C$ is a cofibrant replacement. So $\mathbb{L}_{C/R} = C \otimes_{C_\bullet} \Omega^1_{C_\bullet/R}$, and we are done.

(13). We want to prove that $S \otimes_R A_\bullet$ is a cofibrant object of $\mathbf{s}({}_S\mathbf{CAlg})$ and that the morphism $S \otimes_R A_\bullet \to S \otimes_R A$ is an acyclic fibration. We can prove the second statement exactly as in the second part of the proof of question (11). As for the first statement, we can assume as in the proof of question (11) that $A_\bullet$ is free over R, and then we deduce that $S \otimes_R A_\bullet$ is free over S because, for every R-module M, we have a canonical isomorphism $S \otimes_R \mathrm{Sym}_R(M) = \mathrm{Sym}_S(S \otimes_R M)$ (as these two S-algebras both represent the functor ${}_S\mathbf{Mod} \to \mathbf{Ab}$, $N \mapsto \mathrm{Hom}_R(M, N)$).

(14). First we show that, if A and S are any commutative R-algebras, and if $B = S \otimes_R A$, then the canonical morphism $B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/S}$ of question (9)(i) is an isomorphism. We have to prove that, for every B-module M (that we also see as an A-module via the map $f : A \to B$, $a \mapsto 1 \otimes a$), the map $\alpha : \mathrm{Der}_S(B, M) \to \mathrm{Der}_R(A, M)$, $d \mapsto d \circ f$ is an isomorphism. Let $d \in \mathrm{Der}_S(B, M)$ such that $d \circ f = 0$. Then, for all $s \in S$ and $a \in A$, we have $d(s \otimes a) = sd(1 \otimes a) = s(d \circ f)(a) = 0$. As every element of B is a finite sum of elements of the form $s \otimes a$, this shows that $d = 0$. Hence α is injective. Now let $d' \in \mathrm{Der}_R(A, M)$. We want to find $d \in \mathrm{Der}_S(B, M)$ such that $d' = d \circ f$. Let $d : B \to M$ be the R-linear map induced by the R-bilinear map $S \times A \to M$, $(s, a) \mapsto sd'(a)$. It is easy to check that d is a S-linear derivation. So α is surjective.

Now we come back to the notation of question (14). Let $A_\bullet \to A$ be a cofibrant replacement, and let $B_\bullet = S \otimes_R A_\bullet$. We have seen in the proof of question (9)(iii) that $B \otimes_A^L \mathbb{L}_{A/R}$ is representend by the simplicial B-module $B \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$. By the previous paragraph, the canonical morphism

$$B \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} = B \otimes_{B_\bullet} B_\bullet \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} \to B \otimes_{B_\bullet} \Omega^1_{B_\bullet/S}$$

is an isomorphism. By question (13), the morphism $B_\bullet \to B$ is a cofibrant replacement, so $\mathbb{L}_{B/S} = B \otimes_{B_\bullet} \Omega^1_{B_\bullet/S}$. This finishes the proof.

(15). (i) By the hint in question (h)(i) of problem A.9.2, it suffices to prove that, for every B-module M (seen as a A-module via $f : A \to B$), the sequence

$$0 \to \mathrm{Hom}_B(\Omega^1_{B/A}, M) \to \mathrm{Hom}_B(\Omega^1_{B/R}, M) \to \mathrm{Hom}_B(B \otimes_A \Omega^1_{A/R}, M)$$

is exact. This sequence is canonically isomorphic to

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M) \to \mathrm{Hom}_A(\Omega^1_{A/R}, M) = \mathrm{Der}_R(A, M),$$

so it suffices to prove that this last sequence is exact. The fact that $\mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M)$ is injective is obvious. If $d \in \mathrm{Der}_R(B, M)$ is actually A-linear, then, for every $a \in A$, we have

$$a \cdot d(1) = d(f(a)) = d(1 \cdot f(a)) = d(f(a)) + a \cdot d(1),$$

so $d(f(a)) = 0$; this shows that $d \circ f = 0$. Finally, let $d \in \mathrm{Der}_R(B, M)$ such that $d \circ f = 0$; we want to show that d is A-linear. Let $a \in A$ and $b \in B$. Then

$$d(f(a)b) = a \cdot d(b) + b \cdot d(f(a)) = a \cdot d(b).$$

(ii) By (i), it suffices to show that the canonical morphism $u : B \otimes_A \Omega^1 A/R \to \Omega^1_{B/R}$ is injective. By question (9)(i), the morphism $g : B \to A$ gives a canonical morphism $v : A \otimes_B \Omega^1_{B/R} \to \Omega^1_{A/R}$, and, by the definition of these two canonical morphisms, the composition

$$\Omega^1_{A/R} = A \otimes_B B \otimes_A \Omega^1_{A/R} \overset{A \otimes_B u}{\to} A \otimes_B \Omega^1_{B/R} \overset{v}{\to} \Omega^1_{A/R}$$

is equal to the identity. Applying the functor $B \otimes_A (\cdot)$, we see that the composition

$$\mathrm{B} \otimes_A \Omega^1_{A/R} \to B \otimes_A A \otimes_B \Omega^1_{B/R} = \Omega^1_{B/R} \to B \otimes_A \Omega^1_{A/R}$$

is equal to the identity. Also, the first morphism is equal to $B \otimes_A A \otimes_B u = u$. So u is injective.

(16). (i) We saw in question (3) that, if $A_\bullet$ is a cofibrant object of $\mathbf{s}(R - \mathbf{CAlg})$, then the A_n-module $\Omega^1_{A_n/R}$ is projective for every $n \in \mathbb{N}$. We need the following generalization: If $A_\bullet \to B_\bullet$ is a cofibration in $\mathbf{s}(R - \mathbf{CAlg})$, then the B_n-module $\Omega^1_{B_n/A_n}$ is projective for every $n \in \mathbb{N}$. Indeed, we can apply the beginning of the solution of (3) to reduce to the case where $A_\bullet \to B_\bullet$ is a free morphism. Let $(P_k)_{k \geq 0}$ be a family of projective R-modules such that $B_n = A_n \otimes_R \bigotimes_{\alpha:[n]\twoheadrightarrow[k]} \mathrm{Sym}_r(P_k)$ for every $n \in \mathbb{N}$. By the beginning of the solution of (14), we have an isomorphism

$$B_n \otimes_{R_n} \Omega^1_{R_n/R} \overset{\sim}{\to} \Omega^1_{B_n/A_n},$$

where $R_n = \bigotimes_{\alpha:[n]\twoheadrightarrow[k]} \mathrm{Sym}_R(P_k)$. We have seen in the solution of (3) that $\Omega^1_{R_n/R}$ is a projective R_n-module, so $\Omega^1_{B_n/A_n}$ is a projective A_n-module.

Now we come back to the notation of question (16). First note that the morphism $A_n \to B_n$ satisfies the hypothesis of question (15)(ii) for every $n \in \mathbb{N}$ (i.e. there exists $B_n \to A_n$ such that the composition $A_n \to B_n \to A_n$ is id_{A_n}); indeed, this property is clear if i is free (because, for every R-module M, there exists a morphism of R-algebras $\mathrm{Sym}_R(M) \to R$ that is equal to id_R on $\mathrm{Sym}^0_R(M) = R$; just take the morphism corresponding to $0 \in \mathrm{Hom}_R(M, R)$), and it is obviously stable by taking retracts. Hence, by question (15)(ii), we get exact sequences

$$0 \to B_n \otimes_{A_n} \Omega^1_{A_n/R} \to \Omega^1_{B_n/R} \to \Omega^1_{B_n/A_n} \to 0.$$

By the preceding paragraph, the B_n-module $\Omega^1_{B_n/A_n}$ is projective, hence flat, so the sequence above stays exact if we apply the functor $B \otimes_{B_n} (\cdot)$. This gives the desired exact sequence.

(ii) First we show that $A \otimes_{A_\bullet} B_\bullet$ is a cofibrant object in $\mathbf{s}({}_A\mathbf{CAlg})$. If the morphism $i : A_\bullet \to B_\bullet$ is a retract of a morphism $A_\bullet \to C_\bullet$, then $A \otimes_{A_\bullet} B_\bullet$ is a retract of $A \otimes_{A_\bullet} C_\bullet$. So we may assume that the morphism i is free, and then it follows immediately from teh description of free morphisms that the morphism $A \to A \otimes_{A_\bullet} B_\bullet$ is also free (for the same family of projective R-modules $(P_k)_{k\geq 0}$).

It remains to show that the morphism $A \otimes_{A_\bullet} B_\bullet \to B$ is an acyclic fibration. I am stuck on the "acyclic" part too.

(iii) By the beginning of the solution of question (14), the canonical morphism

$$(A \otimes_{A_\bullet} B_\bullet) \otimes_{B_\bullet} \Omega^1_{B_\bullet/A_\bullet} \to \Omega^1_{A\otimes_{A_\bullet} B_\bullet/A}$$

is an isomorphism. As $B \otimes_{A\otimes_{A_\bullet} B_\bullet} \Omega^1_{A\otimes_{A_\bullet} B_\bullet/A}$ represents $\mathbb{L}_{B/A}$ by (ii), we deduce that $B \otimes_{B_\bullet} \Omega^1_{B_\bullet/A_\bullet}$ also represents $\mathbb{L}_{B/A}$. On the other hand, the morphism $A_n \to B_n$ satisfies the condition of (i) for every n (that is, there exists a morphism $B_n \to A_n$ such that the composition $A_n \to B_n \to A_n$ is id_{A_n}); this is clear if $i : A_\bullet \to B_\bullet$, and this property is stable by retract. Hence we have an exact sequence (in $\mathbf{s}({}_B\mathbf{Mod})$):

$$0 \to B \otimes_{A_\bullet} \Omega^1_{A_\bullet/R} \to B \otimes_{B_\bullet} \Omega^1_{B_\bullet/R} \to B \otimes_{B_\bullet} \Omega^1_{B_\bullet/A_\bullet} \to 0,$$

which is a cofiber sequence by Proposition V.4.1.3(iv). We have seen in the solution of question 9(iii) that $B \otimes_{A_\bullet} \Omega^1_{A_\bullet/R}$ represents $B \otimes^L_A \mathbb{L}_{A/R}$, and we know that $B \otimes_{B_\bullet} \Omega^1_{B_\bullet/R}$ represents $\mathbb{L}_{B/R}$ because $B_\bullet$ is a cofibrant simplicial R-algebra (as $A_\bullet \to B_\bullet$ is a cofibration) and $B_\bullet \to B$ is a weak equivalence; so we get the desired result.

(17). The hypothesis says that $\mathrm{Tor}^R_i(A, S) = 0$ for $i \geq 1$ and that the canonical morphism $A \otimes_R S \to A$ is an isomorphism. By question (14), the canonical morphism

$$(A \otimes_R S) \otimes^L_A \mathbb{L}_{A/R} = A \otimes^L_A \mathbb{L}_{A/R} = \mathbb{L}_{A/R} \to \mathbb{L}_{A\otimes_R S/S} = \mathbb{L}_{A/S}$$

is an isomorphism.

(18). Applying the result of question (17) with $S = A$, we get an isomorphism $\mathbb{L}_{A/R} \xrightarrow{\sim} \mathbb{L}_{A/A} = 0$.

(19). Let $u, v : A \to A \otimes_R A$ be the two R-algebra maps sending $a \in A$ to $a \otimes 1$ and $1 \otimes a$ respectively. Note that the composition of u or v with the R-algebra map $A \otimes_R A \to A$, $a \otimes b \mapsto ab$ is id_A. By question (14) (applied to $S = A$), the canonical morphism $(A \otimes_R A) \otimes^L_A \mathbb{L}_{A/R} \to \mathbb{L}_{A\otimes_R A/A}$ is an isomorphism, where we use the map $u : A \to A \otimes_R A$ to form the derived tensor product on the left and the map v to form $\mathbb{L}_{A\otimes_R A/A}$. Applying the functor $A \otimes^L_{A\otimes_R A} (\cdot)$, we get an isomorphism $\mathbb{L}_{A/R} \xrightarrow{\sim} A \otimes_{A\otimes_R A} \mathbb{L}_{A\otimes_R A/A}$.

Now consider the morphisms of R-algebras $A \xrightarrow{v} A \otimes_R A \to A$, and applying question (16)(iii). We get a sequence

$$A \otimes_{A\otimes_R A} \mathbb{L}_{A\otimes_R A/A} \to \mathbb{L}_{A/A} \to \mathbb{L}_{A/A\otimes_R A}$$

that comes from a cofiber sequence in $\mathbf{s}({}_A\mathbf{Mod})$. But $\mathbb{L}_{A/A\otimes_R A} = 0$ by assumption, and $\mathbb{L}_{A/A} = 0$ by question (5), so $A \otimes_{A\otimes_R A} \mathbb{L}_{A\otimes_R A/A} = 0$.

(20). Let $S = A \otimes_R A$. Then the morphism $S \to A$ is flat by assumption, and we see easily that the canonical morphism $A \otimes_S A \to A$ is an isomorphism. So, by question (18), we have $\mathbb{L}_{A/S} = 0$. We can now apply question (19) to conclude that $\mathbb{L}_{A/R} = 0$.

(21). Let $P^\bullet \to M$ be a projective resolution of M. This means that $P^\bullet$ is a complex of projective R-modules concentrated in degree ≤ 0 and that the morphism $P^\bullet \to M$ is a quasi-isomorphism. Then, for every $i \in \mathbb{N}$, the functor $\mathrm{Tor}^R_i(M, \cdot)$ is isomorphic to the functor $\mathrm{H}^{-i}(P^\bullet \otimes_R (\cdot))$, so it suffices to show that the second functor commutes with filtrant colimits. We know that, if P is a R-module, then the functor $P \otimes_R (\cdot) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ commutes with all colimits, because it has a right adjoint (see Proposition I.5.4.3); the result follows from this and from the fact that filtrant colimits in ${}_R\mathbf{Mod}$ are exact (Corollary I.5.6.5).

(22). Recall that, if S is a commutative ring of characteristic p, then the map $S \to S$, $x \mapsto x^p$ is a $\mathbb{F}_p$-algebra endomorphism of S, called the (absolute) Frobenius of S.

(i) Consider the map $f : A \times B \to A \otimes_R B$, $(a, b) \mapsto a^{1/p} \otimes b^{1/p}$. Then f is bi-additive and we have $f(ar, b) = f(a, rb)$ for all $a \in A$, $b \in B$ and $r \in R$ (note that the last assertion uses the fact that R is perfect). So f defines a morphism of abelian groups $A \otimes_R B \to A \otimes_R B$ that sends a pure tensor $a \otimes b$ to $a^{1/p} \otimes b^{1/p}$. It is clear that this morphism is an inverse of the Frobenius endomorphism of $A \otimes_R B$.

(ii) For every set E, we have a canonical bijection

$$\mathrm{Hom}_{R-\mathbf{CAlg}}(R[X_i,\ i \in E], A) \xrightarrow{\sim} A^E$$

(sending a morphism of R-algebras $f : R[X_i,\ i \in E] \to A$ to the family $(f(X_i))_{i\in E}$). As A is perfect, every R-algebra morphism $R[X_i,\ i \in E]$ extends uniquely to a

R-algebra morphism $R[X_i^1/p^\infty,\ i\in E]\to A$, and every R-algebra morphism $R[X_i^1/p^\infty,\ i\in E]\to A$ is uniquely determined by the images of the X_i. So we get a bijection

$$\mathrm{Hom}_{R-\mathbf{CAlg}}(R[X_i^{1/p^\infty},\ i\in E],A)\overset{\sim}{\to}A^E.$$

For example, if we take $E=A$, then the family $(a)_{a\in A}$ corresponds to a morphism of R-algebras $f:R[X_a^{1/p^\infty},\ a\in A]\to A$ such that $f(X_a)=a$ for every $a\in A$. In particular, the morphism f is surjective, so we can take $S=R[X_a^{1/p^\infty},\ a\in A]$ and $\mathfrak{a}=\mathrm{Ker}(f)$.

(iii) If C is a commutative ring and C', C'' are commutative C-algebras, then we have $\mathrm{Tor}_n^C(C',C'')=\mathrm{H}^{-n}(C'\otimes_C^L C'')$ for every $n\in\mathbb{N}$ (see Example V.4.4.12(1)). So $\mathrm{Tor}_n^C(C',C'')=0$ for every $n\geq 1$ if and only if the canonical morphism $C'\otimes_C^L C''\to\tau^{\geq 0}(C'\otimes_C^L C'')=C'\otimes_C C''$ is an isomorphism.

Now suppose that $\mathrm{Tor}_i^R(S,B)=0$ for $i\geq 1$ and that $\mathrm{Tor}_i^S(A,S\otimes_R B)=0$ for $i\geq 1$. Then we have canonical isomorphisms $S\otimes_R^L B\to S\otimes_R B$ and $A\otimes_S^L(S\otimes_R B)\to A\otimes_S(S\otimes_R B)$, hence the canonical morphism

$$A\otimes_R^L B\simeq A\otimes_S^L(S\otimes_R^L B)\to A\otimes_S^L(S\otimes_R B)\to A\otimes_S(S\otimes_R B)\simeq A\otimes_R B$$

is an isomrophism. This implies that $\mathrm{Tor}_i^R(A,B)=0$ for every $i\geq 1$.

(iv) For every $N\in\mathbb{N}$, let $S_N=R[X_i^{1/p^N},\ i\in I]\subset S$. Then $S_N\subset S_{N+1}$ for every $N\in\mathbb{N}$ and $S=\bigcup_{N\geq 0}S_N$. So S is a filtrant colimit of the R-algebras S_N, and, by question (20), it suffices to prove that $\mathrm{Tor}_i^R(S_N,B)=0$ for every $N\in\mathbb{N}$ and every $i\geq 1$. But we have isomorphisms of R-algebras $S_N\simeq S_0$, so it suffices to treat the case $N=0$. Also, the R-algebra S_0 is free as a R-module, so the functor $\mathrm{Tor}_i^R(B_0,\cdot)$ is 0 for every $i\geq 1$.

(v) For every finite subset I of $\mathfrak{a}$, let $\mathfrak{a}_I=\bigcup_{n\geq 1}(f^{1/p^n},\ f\in I)$ and $A_I=R/\mathfrak{a}_I$. Then $\mathfrak{a}=\bigcup_{I\subset\mathfrak{a}\text{ finite}}\mathfrak{a}_I$, hence $A=\varinjlim_{I\subset\mathfrak{a}\text{ finite}}A_I$, where the set of finite subset of $\mathfrak{a}$ is ordered by inclusiond and the transition morphisms are the obvious projections. By question (21), to prove that $\mathrm{Tor}_n^R(A,B)=0$ for every $n\geq 1$, it suffices to prove that $\mathrm{Tor}_n^R(A_I,B)=0$ for every $n\geq 1$ and every finite subset I of $\mathfrak{a}$. So we may assume that $\mathfrak{a}$ is equal to one of the $\mathfrak{a}_I$.

(vi) If $f_1,\dots,f_n\in R$, we set

$$(f_1^{1/p^\infty},\dots,f_n^{1/p^\infty})=\bigcup_{m\geq 1}(f_1^{1/p^m},\dots,f_n^{1/p^m}).$$

Suppose that, for every perfect commutative ring R, every $f\in R$, every perfect R-algebra B and every $i\geq 1$, we have $\mathrm{Tor}_i^R(R/(f^{1/p^\infty}),B)=0$. We show by induction on n that, for every $n\geq 1$, every perfect commutative ring R, all $f_1,\dots,f_n\in R$, every perfect R-algebra B and every $i\geq 1$, we have

$\mathrm{Tor}_i^R(R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty}), B) = 0$. We are already assuming that the result holds in the case $n = 1$. Suppose that it holds for some $n \geq 1$, and let R be a perfect commutative ring, B be a perfect R-algebra and $f_1, \dots, f_{n+1} \in R$. Let $A = R/(f_1^{1/p^\infty}, \dots, f_{n+1}^{1/p^\infty})$, $A' = R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty})$, and f be the image of f_{n+1} in A'. Then we have $A = A'/(f^{1/p^\infty})$. The case $n = 1$ and the induction hypothesis imply that the canonical morphisms $A' \otimes_R^L B \to A' \otimes_R B$ and $A \otimes_{A'}^L (A' \otimes_R B) \to A \otimes_{A'} (A' \otimes_R B)$ are isomorphisms. Hence the canonical morphism

$$A \otimes_R^L B \simeq A \otimes_{A'}^L (A' \otimes_R^L B) \to A \otimes_{A'}^L (A' \otimes_R B) \to A \otimes_{A'} (A' \otimes_R B) \simeq A \otimes_R B$$

is an isomorphism, which means that $\mathrm{Tor}_i^R(A, B) = 0$ for every $i \geq 1$.

(vii) Let $n \geq 1$, and let $x \in M$ (resp. $y \in N$) be the image of $1 \in M_n$ (resp $1 \in N_n$). Then $\gamma(x) = f^{1/p^n}$ and $\delta(y) = f^{1/p^n} \cdot 1_B$. This shows that γ and δ are surjective.

Let $x \in M$ such that $\gamma(x) = 0$. We choose $n \geq 1$ and $a \in M_n = A$ such that a represents x. Then $\gamma(x) = f^{1/p^n} a = 0$, so, as A is perfect, we have $f^{1/p^{n+1}} a^{1/p} = 0$. Multiplying by $a^{(p-1)/p}$, we get $f^{1/p^{n+1}} a = 0$, and so $u_n(a) = f^{1/p^n - 1/p^{n+1}} a = 0$. This shows that $x = 0$, hence that γ is injective. A similar proof shows that δ is injective.

(viii) For every $n \geq 1$, let $f_n : M_n \otimes_R B \to N_n$ be the canonical morphism (sending $a \otimes b$ to ab). Then we have commutative diagrams

$$\begin{tikzcd} M_n \otimes_R B \arrow[r, "f_n"] \arrow[d, "u_n \otimes \mathrm{id}_B"'] & N_n \arrow[d, "v_n"] \\ M_{n+1} \otimes_R B \arrow[r, "f_{n+1}"'] & N_{n+1} \end{tikzcd}$$

and

$$\begin{tikzcd} M_n \otimes_R B \arrow[r, "f_n"] \arrow[d, "\gamma"'] & N_n \arrow[d, "\delta"] \\ \mathfrak{a} \otimes_R B \arrow[r, "f_{n+1}"'] & \mathfrak{a}B \end{tikzcd}$$

In other words, the canonical morphism $\mathfrak{a} \otimes_R B \to \mathfrak{a}B$ is the colimit of the morphisms f_n. As each f_n is an isomorphism, we get the result.

(ix) We have seen in (vii) that the R-module $\mathfrak{a}$ is a filtrant colimit of free R-modules. By question (21), this implies that $\mathrm{Tor}_i^R(\mathfrak{a}, B) = 0$ for every $i \geq 1$. Consider the exact sequence of R-modules $0 \to \mathfrak{a} \to R \to A \to 0$. As $\mathrm{Tor}_i^R(R, B) = 0$ for every $i \geq 1$, it induces an exact sequence

$$0 \to \mathrm{Tor}_1^R(A, B) \to \mathfrak{a} \otimes_R B \to R \otimes_R B \to A \otimes_R B \to 0$$

and isomorphisms
$$\mathrm{Tor}^R_{i+1}(A,B) \xrightarrow{\sim} \mathrm{Tor}^R_i(\mathfrak{a},B) = 0$$
for every $i \geq 1$. In particular, we get that $\mathrm{Tor}^R_{i+1}(A,B) = 0$ for every $i \geq 1$. Finally, by (viii), the morphism $\mathfrak{a} \otimes_R B \to R \otimes_R B = B$ is injective, so $\mathrm{Tor}^R_1(A,B) = 0$.

(23). (i) The canonical morphism $B \otimes_R B \to B$ is clearly an isomorphism, so we have to show that $\mathrm{Tor}^R_i(B,B) = 0$ for every $i \geq 1$. But we know that R is perfect by question (22)(i), so this follows from the rest of question (22).

(ii) By (i) and question (18), we have $\mathbb{L}_{B/B\otimes_A B} = 0$. Also, by question (22), we have $\mathrm{Tor}^A_i(B,B) = 0$ for every $i \geq 1$. So the result follows from question (19).

$\square$

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